9

The Structure of Partial Isometries

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Abstract

It is well known that the "quantum logic" approach to the foundations of quantum mechanics is based on the subspace ordering of projectors on a Hilbert space. In this paper, we show that this is a special case of an ordering on partial isometries, introduced by Halmos and McLaughlin. Partial isometries have a natural physical interpretation, however, they are notoriously not closed under composition. In order to take a categorical approach, we demonstrate that the Halmos-McLaughlin partial ordering, together with tools from both categorical logic and inverse categories, allows us to form a category of partial isometries.

This category can reasonably be considered a "categorification" of quantum logic – we therefore compare this category with Abramsky and Coecke's "compact closed categories" approach to foundations and with the "monoidal closed categories" view of categorical logic. This comparison illustrates a fundamental incompatibility between these two distinct approaches to the foundations of quantum mechanics.

9.1 Introduction

As early as 1936, von Neumann and Birkhoff proposed treating projectors on Hilbert space as propositions about quantum systems (Birkhoff and von Neumann 1936), by direct analogy with classical order-theoretic approaches to logic. Boolean lattices arise as the Lindenbaum-Tarski algebras of propositional logics, and as the set of all projectors on a Hilbert space also forms an orthocomplemented lattice, the operations *meet*, *join*, and *complement* were analogously interpreted as the logical connectives *conjunction*, *disjunction*, and *negation*.

However, the lattice of projectors is not a Boolean lattice, so this interpretation requires modifications to the rules of propositional logic (notably the distributive law, $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ fails and is replaced by the weaker condition $A \leq C \Rightarrow A \wedge (A^{\perp} \vee C) = C$). The resulting system of *orthomodular lattices* has become known as *quantum logic*, and a number of authors (Finkelstein

1962; Putnam 1969) have suggested that the nonclassical behavior of quantum systems simply results from the fact that orthomodular lattices, rather than Boolean lattices, provide the "natural logic" of such systems.

This paper does not take a position on this claim, although we do discuss arguments for and against a logical interpretation of orthomodular lattices. Instead, we consider a natural order-theoretic generalization, where the dynamics of quantum systems may also be viewed in order-theoretic terms.

The lattice of projectors on a Hilbert space is an inherently static view of a quantum system. In the usual treatment, the dynamics of a system is interpreted in terms of operations on this lattice – for example, a unitary map induces an automorphism of the orthomodular lattice of projectors. However, a specific subspace may be considered as providing partial information about a quantum system, whereas an automorphism is an inherently global operation. Thus the "static" view provided by quantum logic is based on partial information, whereas the "dynamics" is based on a global view.

In this paper we study combinations of dynamical processes (considered as unitary maps) and measurements (considered as projectors), from both an order-theoretic and category-theoretic viewpoint. To this end, we study *partial isometries*. These generalize both projectors and unitaries in a natural way, and we demonstrate in Corollary 9.3.7 that partial isometries on a finite-dimensional space may be characterized as the composite of a unitary map and a projector (the infinite-dimensional case is, not unexpectedly, more complex). Partial isometries also have a natural partial order introduced by Halmos and McLaughlin (1963), and when restricted to projectors this is exactly the partial order of the orthomodular lattice of "quantum propositions."

Using this partial order, we study partial isometries from a category-theoretic point of view. It is well known (Erdelyi 1968) that the composite of two partial isometries is not, in general, a partial isometry. However, there is a natural associative composition (based on the conjunction of quantum logic and closely related to the treatment of partial isometries in inverse semigroup theory) that allows us to define a category of partial isometries. The resulting category is shown to be an *inverse category*. Inverse categories also have a natural partial order on their hom-sets – this is exactly the Halmos-McLaughlin partial order and hence, when restricted to projectors, the orthomodular lattice ordering of quantum logic.

Given this "categorification" of quantum logic, a natural question is then: *what are the similarities and differences between the resulting categorical structures, and Abramsky and Coecke's "categorical foundations" program for quantum mechanics (Abramsky and Coecke 2004)?*

We demonstrate that superficially there is a good case to be made for agreement between the two categorical approaches, but despite this, detailed calculations shows that they are *incompatible*. However, the reason for this incompatibility is somewhat unsatisfactory; the categorical foundations approach is based on compact closure, as an abstract version of postselected teleportation. However, a treatment of this cannot be given in the category of partial isometries, due to its inability to express postselection. We further demonstrate that no possible agreement can be found; as an application of general categorical principles we show that the category of partial isometries is not closed, never mind compact closed. From a categorical logic point of view, this is the statement that this categorification of quantum logic cannot be made to fit within the "monoidal closed categories" approach to categorical logic.

9.2 The Order Theory of Projectors and "Quantum Logic"

The order theory of projectors on Hilbert space is the foundation of the "quantum logic" of Birkhoff/von Neumann (Birkhoff and von Neumann 1936). The partial order on projectors is defined as follows:

Definition 9.2.1 (The lattice of projectors on a Hilbert space). *Let $E, F : H \to H$ be projectors (i.e., self-adjoint idempotents) on a Hilbert space H. We say that E is* **below** *F, written $E \leq F$ when $EF = E$. Note that this implies $EF = FE$. It is straightforward to check that $\leq$ is a partial order, and the set of all projectors on H forms a lattice, with top element the identity map $\top = 1_H$ and bottom element the zero map $\bot = 0_H$.*

The meet and join of this lattice may be given explicitly:

Proposition 9.2.2 (Partial orders and meets on projectors). *Let E, F be projectors on some Hilbert space, corresponding to the subspaces H_E, H_F respectively. Then*

(i) *The* join *$E \vee F$ is defined by*

$$E \vee F = Inf\{G : E \leq G \text{ and } F \leq G\}$$

and is simply the projector onto the smallest subspace containing both H_E and H_F.

(ii) *The* meet *$E \wedge F$ is defined by*

$$E \wedge F = Sup\{G : G \leq E \text{ and } G \leq F\}$$

and is the projector onto the largest subspace contained within both H_E and H_F. It may be given given explicitly by $E \wedge F = lim_{N \to \infty}(EF)^N = lim_{n \to \infty}(FE)^N$.

Proof. We refer to Jauch (1986) for these results, and Shimony (1970) for a proposed physical interpretation of the meet operation. □

We illustrate the preceding characterization of the meet, with two one-dimensional subspaces P, Q, in Figure 9.1.

Lattices of projectors on Hilbert space are shown in Piron (1976) to be *orthomodular lattices*, defined as follows:

Definition 9.2.3 (Orthomodular lattices, orthocomplemented lattices). *Let $L, \leq$ be a complete lattice. We say that it is* **orthocomplemented** *when there exists an*

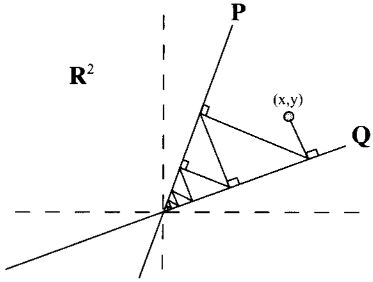


Figure 9.1. The meet of two projectors, in $\mathbb{R}^2$

involution $(\,)^\perp : L \to L$ *that satisfies*

(i) $A \vee A^\perp = \top$.
(ii) $A \wedge A^\perp = \perp$.
(iii) $A \leq B$ *if and only if* $B^\perp \leq A^\perp$.

An orthocomplemented lattice L is called **orthomodular** *if it satisfies the additional condition*

(iv) $A \leq B$ *implies* $B = A \vee (A^\perp \wedge B)$

The canonical example of an orthomodular lattice is the lattice of projectors on a Hilbert space (in fact, we refer to Gleason, 1957, for a stronger statement). The orthocomplement of the projector onto a subspace $S \leq H$ is simply the projector onto the orthogonal complement $S^\perp = \{h \in H : \langle s|h\rangle = 0\ ,\ \forall\, s \in S\}$.

Orthomodular lattices as logics. Following a long-established tradition in order theory, "quantum logic" treats the elements of a lattice of projectors as propositions, and *meet* and *join* as *conjunction* and *disjunction* respectively (although note the failure of the distributive law described in Section 9.1). The involution is interpreted as negation and satisfies analogues of de Morgan's laws $(P \wedge Q)^\perp = P^\perp \vee Q^\perp$.

At this point, readers familiar with mathematical logic will naturally wonder about *implication*. By analogy with order-theoretic approaches to classical logic, there are five distinct candidates for an implication. The most commonly studied is the *Sasaki hook*, which may be defined in terms of conjunction, disjunction, and negation as $P \stackrel{S}{\to} Q = P^\perp \vee (P \wedge Q)$. As we will discuss in Section 9.7.1, it is controversial whether this should be accepted as a genuine implication. We refer to Hellman (1981) for properties of the Sasaki hook and a strong defense of this connective as a form of implication.

Quantum logic and foundations of quantum mechanics. We emphasize that we are not taking a position on "quantum logic" as a foundation for quantum mechanics. Rather, we consider a natural order-theoretic generalization that introduces a notion of partial dynamics as an intrinsic part of this order theory, in order to study quantum logic from a categorical viewpoint. For an introduction to quantum mechanics via quantum logic, we refer to Hughes (1989) for an excellent exposition. We also refer to Griffiths (2002) for the related, but conflicting, "consistent histories" approach to foundations, and to Vickers (1989) and Abramsky (1991) for connections between order theory, logic, and computation in the classical world.

9.3 Partial Isometries

As stated in the introduction, the objective of this paper is to introduce a notion of partial dynamics to the lattice of projectors on Hilbert spaces. We do this by considering both the order theory and category theory of partial isometries. We first present the definitions and various simple properties.

The following definitions are taken from Halmos and McLaughlin (1963).

Definition 9.3.1 (Partial isometries, initial and final subspaces, isometries). *Let $L : H_1 \rightarrow H_2$ be a linear map of Hilbert spaces, and denote its adjoint by $L^* : H_2 \rightarrow H_1$. Then L is a* **partial isometry** *when $L^*L : H_1 \rightarrow H_1$ is a projector, and hence (or equivalently) $LL^* : H_2 \rightarrow H_2$ is a projector.*

The projectors $E_L = L^*L : H_1 \rightarrow H_1$ and $F_L = LL^* : H_2 \rightarrow H_2$ are called the **initial** *and* **final projectors** *of L, and the corresponding subspaces are the* **initial** *and* **final subspaces**. *When the initial subspace is the whole of H_1, then L is called an* **isometry**. *There is no standard terminology for the adjoint of an isometry, where the final subspace is the whole of H_2.*

Given a partial isometry L, it is immediate that L^* is also a partial isometry and the initial projector of L is the final projector of L^*. Both unitary maps and projectors are trivially partial isometries. The initial and final projectors of a unitary map are the global identities on its source and target space, and a projector is its own initial and final projector.

9.3.1 Basic Properties

We establish some basic algebraic results on partial isometries:

Proposition 9.3.2 (Standard results on partial isometries). *Given partial isometries $L : H_1 \rightarrow H_2$, $M : H_2 \rightarrow H_3$, $N : K_1 \rightarrow K_2$, then:*

- (i) *L is a unitary map between its initial and final subspaces.*
- (ii) *$LL^*L = L$ and $L^*LL^* = L^*$.*
- (iii) *When the initial and final projectors of $L : H_1 \rightarrow H_2$ are the global identities of H_1 and H_2 respectively, then L is a unitary map.*
- (iv) *$ML : H_1 \rightarrow H_3$ is a partial isometry exactly when the initial projector of M commutes with the final projector of L, so $E_M F_L = F_L E_M$.*
- (v) *$L \oplus N : H_1 \oplus K_1 \rightarrow H_2 \oplus K_2$ is a partial isometry.*
- (vi) *$L \otimes N : H_1 \otimes K_1 \rightarrow H_2 \otimes K_2$ is a partial isometry.*

Proof. Results (i)–(iv) are taken from Erdelyi (1968), and results (v) and (vi) are a simple consequence of linearity. Note that (ii) states that the adjoint $(\ )^*$

is a *generalized inverse* (in the sense of semigroup theory; Lawson, 1998) on partial isometries. However, the set of all partial isometries on a Hilbert space H is neither a regular nor an inverse semigroup, since by (iv), it is not closed under composition. □

Corollary 9.3.3. *The composite of a unitary and a partial isometry is always a partial isometry.*

Proof. The initial and final projectors of a unitary map $U : H \to K$ are the global identities on H, K respectively. The result then follows trivially from (iv) of Proposition 9.3.2. □

9.3.2 The Halmos-McLaughlin Partial Order

We now show that the partial order on projectors given in Section 9.2 is a special case of the partial order on partial isometries given by Halmos and McLaughlin (1963):

Definition 9.3.4 (The Halmos-McLaughlin partial order). *The partial order $\leqq$ on partial isometries is defined in Halmos and McLaughlin (1963) by*

$$L \leqq K \Leftrightarrow L = KE_L$$

or equivalently,

$$L \leqq K \Leftrightarrow L = F_L K$$

i.e., K, when restricted to the initial subspace of L, or corestricted to the final subspace of L, is exactly L. It is then immediate that $\leqq$, when restricted to projectors, is exactly the orthomodular lattice partial ordering of Definition 9.2.1. We refer to $\leqq$ as the **Halmos-McLaughlin partial order**, *or* **HML partial order**.

The projectors on a space H may be characterized as *partial isometries beneath the identity* 1_H. From the physical interpretation as composites of unitaries and projectors, we also have a particular interest in partial isometries that are beneath unitary maps:

Definition 9.3.5 (Physical partial isometries). *Given a partial isometry $L : H \to H$ satisfying $L \leqq U$ for some unitary map $U : H \to H$, we refer to L as a* **physical partial isometry**.

Proposition 9.3.6. *Let $L : H \to H$ be a partial isometry. When the codimension of the initial subspace is equal to the codimension of the final subspace, then L is a physical partial isometry.*

Proof. Denote the initial subspace of $L : H \to H$ by S, and the terminal subspace by T, so $S \oplus S^\perp = H = T \oplus T^\perp$. By definition of partial isometries $dim(S) = dim(T)$, and the condition on the codimensions gives that $dim(S^\perp) = dim(T^\perp)$, and so $S^\perp \cong T^\perp$. Given a (not necessarily unique) unitary $L' : S^\perp \to T^\perp$ exhibiting this isomorphism, we may construct a unitary

$U = L + \begin{pmatrix} 0 & 0 \\ 0 & L' \end{pmatrix} : S \oplus S^{\perp} \to T \oplus T^{\perp}$, and it is immediate from the definition that $L \leqq U$. □

Corollary 9.3.7. *All partial isometries on finite-dimensional spaces are physical isometries.*

Proof. The condition on codimensions from Proposition 9.3.6 is trivially satisfied for partial isometries between finite-dimensional spaces. Counterexamples on infinite-dimensional spaces include the Cuntz-Krieger algebras of Cuntz and Krieger (1980) and the Shift operator of Guyker (1976). □

Interpretation. Note that the definition of a *physical partial isometry* is restricted to the case where the source and target space are the same. In this case, a physical partial isometry is simply one that may be "completed" to a unitary map.

In general, we may often give a physical interpretation to partial isometries where the source and target space differ; as a simple example, for a norm-1 vector $\phi \in H$, the bra and ket operators $\langle\phi| : H \to \mathbb{C}$ and $|\phi\rangle : \mathbb{C} \to H$ are both partial isometries. These simple examples become important when considering the structure of the category of partial isometries, in Section 9.8.

9.4 The Interaction of Partial Isometries and Projectors

We now introduce a useful technique for dealing with the interaction of partial isometries and projectors. This is strongly based on a technique from inverse semigroup theory (see, for example, Howie, 1995), although the partial isometries on a space H do not form a semigroup.

Proposition 9.4.1 ("Pushing a projector through a partial isometry"). *Let $L : H_1 \to H_2$ be a partial isometry, and let $G : H_1 \to H_1$ and $D : H_2 \to H_2$ be projectors satisfying $G \leq E_L$ and $D \leq F_L$. Then:*

(i) *There exists a unique projector $G' \leq F_L$ such that $LG = G'L$.*
(ii) *There exists a unique projector $D' \leq E_L$ such that $DL = LD'$.*

Proof.

(i) Define $G' : H_2 \to H_2$ by $G' = LGL^*$. Then $(G')^* = (LGL^*)^* = LGL^*$, so G' is self-adjoint. Similarly,

$$G'G' = LGL^*LGL^* = LGE_LGL^* = LGGL^* = LGL^* = G'$$

so G' is idempotent, and hence it is a projector. Now note

$$G'L = LGL^*L = LGE = LEG = LL^*LG = LG$$

as required. To show that $G' \leq L_F$, note that

$$G'F = G'LL^* = LGL^*LL^* = LGEL^*$$
$$= LEL^* = LL^*LL^* = F^2 = F.$$

Uniqueness follows since L is a unitary when restricted to its initial / final subspaces.

(ii) Defining $D' : H_1 \to H_1$ by $D' = L^*DL$, this result follows by symmetry. □

This "pushing a projector through a partial isometry" operation is order-preserving, as shown:

Lemma 9.4.2. *Let $L : H_1 \to H_2$ be a partial isometry, and let $P, Q : H_2 \to H_2$ be projectors below F_L. Then the unique projectors $P', Q' \leq E_L$ satisfying*

$$PL = LP', \quad QL = LQ'$$

satisfy $P \leq Q \Leftrightarrow P' \leq Q'$.

Proof. ($\Rightarrow$) Assume $P \leq Q$, so $PQ = QP = P$. By construction $P' = L^*PL$ and $Q' = L^*QL$ so

$$P'Q' = L^*PLL^*QL = L^*LL^*PQL = L^*PQL = L^*PL = P'$$

and hence $P' \leq Q'$.
($\Leftarrow$) This proof is almost identical to ($\Rightarrow$) previously shown. □

The technique of "pushing a projector through a partial isometry" allows us to establish a connection between partial isometries and block matrices, as follows:

Proposition 9.4.3. *Let $U : H \to K$ be a unitary map, let $H = H_1 \oplus \ldots \oplus H_a$ and $K = K_1 \oplus \ldots \oplus K_b$ be direct sum decompositions of the source and target space. We may then write U as a $(b \times a)$ block matrix*

$$U = \begin{pmatrix} U_{11} & \ldots & U_{1a} \\ \vdots & \ddots & \vdots \\ U_{b1} & \ldots & U_{ba} \end{pmatrix}$$

where $U_{ij} : H_j \to K_i$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$.

For fixed i, j, the submatrix

$$B_{ij} = \begin{pmatrix} \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & U_{ij} & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix}$$

is given by $B_{ij} = QUP$ where $Q = \mathbf{0} \oplus 1_{K_i} \oplus \mathbf{0} : K \to K$ and $P = \mathbf{0} \oplus 1_{H_j} \oplus \mathbf{0} : H \to H$. This is then a partial isometry exactly when

$$Q'P = PQ' \text{ or equivalently, } QP' = P'Q$$

where $Q' = U^{-1}QU$ and $P' = UPU^{-1}$ are given by "passing the projector P (resp. Q) through the unitary U," as in Proposition 9.4.1.

Proof. Given $B_{ij} = QUP : H \to K$, then since U is unitary, $F_U = 1_K$, and $Q \leq F_U$. Therefore, by Proposition 9.4.1, $QU = UQ'$, and this is a partial isometry with initial projector Q'. Hence, by Proposition 9.3.2, $QUP = UQ'P$ is a partial isometry exactly when $Q'P = PQ'$, as required. The equivalent condition $QP' = PQ$ follows either algebraically (by conjugation by U), or by duality. □

Interpretation. We may consider a quantum computation that consists of a finite series of operations – either unitary maps or measurements[1] on certain subspaces. A natural question would be whether this is equivalent to a series of unitaries (and by composition, a single unitary map) followed by a measurement. However, a unitary followed by a projector is a partial isometry, whereas a series of unitaries and projectors is not, in general. Proposition 9.4.3 gives conditions for a projector, followed by a unitary, followed by a projector to be a partial isometry (*i.e.*, equivalent to a unitary followed by a projector), and this is easily generalized to a series of unitaries and measurements.

9.5 A Category of Partial Isometries

From Proposition 9.3.2, the class of partial isometries is *not* closed under the usual composition of linear maps. However, with a modified composition (that we demonstrate is equivalent to a construction of Lawson, 1998), partial isometries not only are closed under composition but form an *inverse category*.

Definition 9.5.1 (Inverse categories, generalized inverses). *Inverse categories are the natural extension of inverse monoids to the many-object case (Lawson 1998). A category* **C** *is* **inverse** *when for every arrow* $f \in \mathbf{C}(X, Y)$, *there exists a unique* **generalized inverse** $f^{-1} \in \mathbf{C}(Y, X)$ *satisfying*

$$ff^{-1}f = f \quad \text{and} \quad f^{-1}ff^{-1} = f^{-1}.$$

We emphasize that this axiom does not imply $f^{-1}f = 1_X$ *or* $ff^{-1} = 1_Y$. *Generalized inverses that satisfy these additional conditions are called* **left** *and* **right global inverses** *respectively.*

Interpretation. The usual computer science interpretation of inverse categories is strictly stronger than simply requiring reversibility. Rather, inverse categories are used (as in Hines 1997; Haghverdi 2000; Abramsky et al. 2002; Hines and Scott 2009) to model resource-sensitive systems, where copying and deleting are either forbidden, or strictly controlled (such as the *Linear Logic* of Girard, 1987). When modeling quantum information, we would also expect similar structures, due to the no-cloning and no-deleting theorems (Wootters and Zurek 1982; Pati and Braunstein 2000).

In order to define a category of partial isometries, we give a binary operation on partial isometries that we then prove is the composition in a category:

[1] For simplicity, we assume that these measurements are *postselected* – if the desired measurement outcome is not observed, we abandon the experiment and start again.

Definition 9.5.2. *Given partial isometries $L : H_1 \to H_2$ and $M : H_2 \to H_3$, we define*

$$M \circ L = lim_{n\to\infty} [(ML)(ML)^*]^n (ML) = lim_{n\to\infty}(ML) [(ML)^*(ML)]^n$$

For readers familiar with J.-Y. Girard's Geometry of Interaction system, (Girard 1989, 1988), this definition is clearly motivated by the *execution formula.*[2] However, it has an even closer connection with the conjunction in quantum logic:

Lemma 9.5.3. *Given partial isometries $M : H_2 \to H_3$, $L : H_1 \to H_2$, as before, then $M \circ L = M(E_M \wedge F_L)L : H_1 \to H_3$ where E_M and F_L are the final and initial projectors of M and L respectively. Hence $M \circ L$ exists for arbitrary partial isometries $L : H_1 \to H_2$ and $M : H_2 \to H_3$, and is a partial isometry.*

Also, when $E_M F_L = F_L E_M$, then $M \circ L$ is simply ML, the usual composition of M and L as linear maps.

Proof. From Proposition 9.2.2,

$$E_M \wedge F_L = lim_{n\to\infty}(E_M F_L)^n = lim_{n\to\infty}(F_L E_M)^n.$$

For arbitrary $n \geq 1$, $[(ML)(ML)^*]^n (ML) = (MLL^*M^*)^n ML$, and rebracketing gives

$$[(ML)(ML)^*]^n (ML) = M(LL^*M^*M)^n L = M(F_L E_M)^n L.$$

Hence

$$\begin{aligned} lim_{n\to\infty} [(ML)(ML)^*]^n (ML) &= lim_{n\to\infty} M(F_L E_M)^n L \\ &= M(E_M \wedge F_L)L. \end{aligned}$$

As $E_M \wedge F_L$ is a projector that commutes with both E_M and F_L, it follows from Proposition 9.3.2 that:

(i) $(E_M \wedge F_L)L : H_1 \to H_2$ is a partial isometry,
(ii) $M(E_M \wedge F_L) : H_1 \to H_2$ is a partial isometry,
(iii) and hence $M(E_M \wedge F_L)L : H_1 \to H_3$ is a partial isometry.

Finally, when $E_M F_L = F_L E_M$, then $E_M \wedge F_L = E_M F_L = F_L E_M$, and so $M \circ L = ML$. □

Theorem 9.5.4. *Partial isometries, with the composition given previously, form an inverse category.*

[2] As a historical note, the original Geometry of Interaction system was presented in terms of partial isometries acting on infinite-dimensional Hilbert spaces. However – as noted by many authors (Abramsky 1996; Abramsky et al. 2002; Haghverdi 2000; Hines 1997) – the partial isometries used were of a very special form. Precisely, all the action takes place within the image of Barr's injective functor (Barr 1992) from partial injections into Hilbert spaces, $l_2 : \mathbf{pInj}^{op} \to \mathbf{Hilb}$. Thus, although partial isometries were used, all initial and all final projectors commute. This is clearly a very restricted case, and in that sense the framework of Hilbert spaces and partial isometries can be considered as "useless superstructure."

Proof. It is shown in Lawson (1998) that the set of all partial isometries acting on a single space together with this composition is an inverse monoid. The extension to the many-object case is immediate. □

Notation. We denote the category of partial isometries with the foregoing composition by **pIsom**. By contrast, we denote the category of continuous linear maps on Hilbert spaces (with the usual composition) by **Hilb**. We use the notation $\circ$ for the composition in **pIsom** and simply use concatenation to denote the composition in **Hilb**. In both cases, we have a particular interest in the case where we restrict to finite-dimensional spaces. We denote these restrictions by $\mathbf{Hilb_{FD}}$ and $\mathbf{pIsom}_{FD}$.

Not only is the category **pIsom** closed under the composition $\circ$, but there is a very strong sense in which the composition $M \circ L$ can be thought of as a supremum within the HML partial ordering, as follows:

Proposition 9.5.5. *Let $M : H_2 \to H_3$ and $L : H_1 \to H_2$ be partial isometries, and let $P : H_2 \to H_2$ be a projector such that*

$$MP : H_2 \to H_3 \quad \text{and} \quad PL : H_1 \to H_2 \quad \text{and} \quad MPL : H_1 \to H_3$$

are all partial isometries. Then $MPL \leqq M \circ L$, where $\leqq$ is the Halmos-McLaughlin partial order of Definition 9.3.4.

Proof. Since MP and PL are partial isometries, from Proposition 9.3.2, $PE_M = E_M P$ and $PF_L = F_L P$. Hence $Q = E_M P F_L$ is a projector satisfying $MPL = MQL$. We now work with this projector Q. By construction, $Q \leq E_M$ and $Q \leq F_L$, so by definition $Q \leq (E_M \wedge F_L)$. Now consider the unique projectors $Q', R : H_1 \to H_1$ satisfying $MQL = MLQ'$ and $M(E_M \wedge F_L)L = MLR$ given as in Proposition 9.4.1. From Lemma 9.4.2, we deduce $Q' \leq R$, so $Q'R = Q'$, and so $MLRQ' = MLQ'$. However, by definition of the Halmos-McLaughlin partial order,

$$MPL = MQL = MLQ' \leqq MLR = M(E_M \wedge F_L)L$$

as required. □

It is then straightforward to write down the initial and final projectors of $M \circ L$. These are closely related to the notion of "pushing a projector through a partial isometry" as given in Proposition 9.4.1.

Corollary 9.5.6. *Let $L : H_1 \to H_2$ and $M : H_2 \to H_3$ be as before. Then*

(i) *The initial projector of $M \circ L$ is the unique projector $P \leq E_L$ satisfying $(E_M \wedge F_L)L = LP$, as in Proposition 9.4.1.*
(ii) *The final projector of $M \circ L$ is the unique projector $Q \leq F_M$ satisfying $QM = M(E_M \wedge F_L)$, as in Proposition 9.4.1.*

Proof. The initial projector P of $M(E_M \wedge F_L)L$ may be given explicitly by

$$P = L^*(E_M \wedge F_L)M^*M(E_M \wedge F_L)L = L^*(E_M \wedge F_L)E_M(E_M \wedge F_L)L.$$

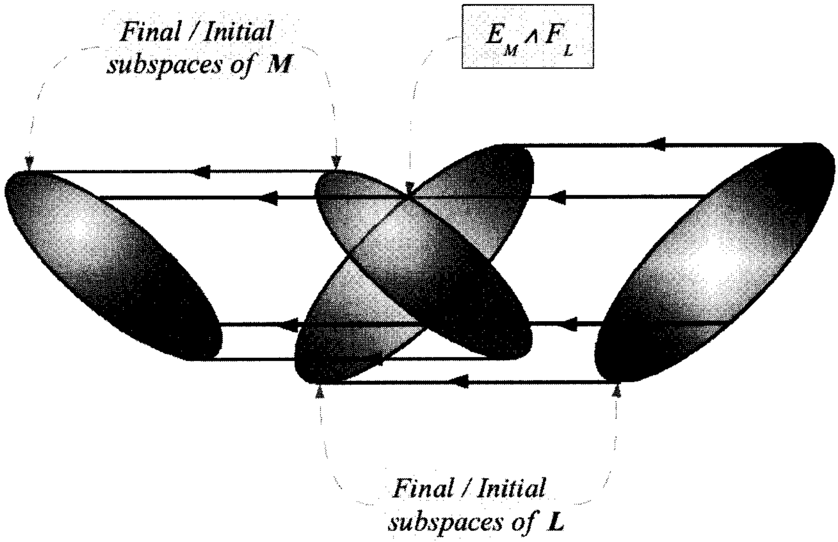


Figure 9.2. The composition $M \circ L$ of partial isometries

However, $(E_M \wedge F_L) \leq E_M$ and $(E_M \wedge F_L)^2 = E_M \wedge F_L$, so $P = L^*(E_M \wedge F_L)L$. The final projector Q may similarly be shown to be $Q = M(E_M \wedge F_L)M^*$.

Results (i) and (ii) then follow by comparing these explicit forms with Proposition 9.4.1. □

9.6 The Inverse Structure of pIsom

From the characterization of the composition given in Proposition 9.5.3 and the inverse category-theoretic result of Theorem 9.5.4, there is a simple graphical representation of the composition $\circ$, as shown in Figure 9.2.

We now consider the consequences of the inverse structure on partial isometries. An immediate corollary is the commutativity of projectors, as follows:

Corollary 9.6.1.

(i) *The idempotents of* **pIsom** *are exactly the projectors of* **Hilb**.
(ii) *All projectors (at the same object) commute, so* $E \circ F = F \circ E$.
(iii) *Given projectors* $E, F : H \to H$, *then* $E \circ F = E \wedge F$.

Proof.

(i) Consider an idempotent $L \circ L = L$ in **pIsom**. Then $L = L(E_L \wedge F_L)L$, and hence $E_L \leq F_L$ and $F_L \leq E_L$. Therefore, $E_L = F_L$, and so $L \circ L = LL$, and L is thus idempotent in the category **Hilb**. Hence, as L is a partial isometry it is Hermitian, and so L is a projector.
(ii) The commutativity of idempotents follows from the uniqueness of generalized inverses, as standard result of inverse semigroup theory (Munn and Penrose 1955).
(iii) This is immediate from the definition of the composition $\circ$. □

Commutativity of projectors. The commutativity of projectors in **pIsom** follows from the existence and uniqueness of generalized inverses, and therefore is an essential feature of this category. We note the strong distinction with the "matrix mechanics" formulation of quantum mechanics and the behavior of projectors in the category **Hilb**, where the *noncommutativity of projectors* captures the "nonclassical" behavior of observations (Mehra and Rechenberg 1982).

9.6.1 Partial Orders and Inverse Structures

So far, we have seen that the Halmos-McLaughlin partial order generalizes the "quantum logic" ordering of projectors on Hilbert space (Definition 9.3.4) and have used the conjunction operation of quantum logic to define a composition on partial isometries (Definition 9.5.2 and Lemma 9.5.3). This composition is then exactly the calculation of a supremum within the HML partial order (Proposition 9.5.5) and allows us to define an inverse category of partial isometries (Theorem 9.5.4).

However, every inverse semigroup has a partial order defined on its elements, as in Lawson (1998), and this has an immediate generalization to the hom-sets of inverse categories:

Definition 9.6.2 (The natural partial order of an inverse category). *Let S be an inverse semigroup. The* **natural partial order** $\trianglelefteq$ *on S is defined by*

$$s \trianglelefteq t \;\Leftrightarrow\; \exists\, e^2 = e \;\; s = te.$$

We refer to Lawson (1998) for proof that this is indeed a partial order and its properties.

The generalization of this notion to inverse categories is immediate. Let $\mathcal{C}$ be an inverse category, and consider $f, g \in \mathcal{C}(X, Y)$. We extend the preceding definition in the obvious way, so

$$f \trianglelefteq g \;\Leftrightarrow\; \exists\, e^2 = e \in \mathcal{C}(X, X) \;\; f = ge$$

and this makes the hom-set $\mathcal{C}(X, Y)$ into a partial order.

Theorem 9.6.3. *The Halmos-McLaughlin partial order $\leqq$ is exactly the natural partial order $\trianglelefteq$ on the inverse category* **pIsom**.

Proof. Consider $L, M \in$ **pIsom**(H, K).

- ($\Rightarrow$) Assume $L \leqq K$. Then by definition, $L = ME_L$. Hence, as ME_L is a partial isometry, $ME_L = M(E_M \wedge E_L) = M \circ E_L$, so $L \trianglelefteq M$.
- ($\Leftarrow$) Assume $L \trianglelefteq M$, so there exists some $G^2 = G \in$ **pIsom**(H, H) such that $L = M \circ G$. Therefore, $L \circ G = M \circ G \circ G = M \circ G = L$, and the initial subspace of L is a subspace of the initial subspace of M. Hence $L \leqq M$, as required. □

The natural partial order and the generalized inverse. Although there is a close connection between the generalized inverse and the natural partial order of an inverse category, we emphasize that the generalized inverse is *not* an orthocomplement. In particular, given $f \trianglelefteq g \in \mathcal{C}(X, Y)$, then $f^{-1} \trianglelefteq g^{-1} \in \mathcal{C}(Y, X)$. Not only is the generalized inverse not an orthocomplement, but $\mathbf{pIsom}(H, H)$ cannot in general be orthocomplemented since in general $L \ntrianglelefteq 1_H$, so we do not even have a lattice structure.[3]

We refer to Lawson (1998) for a comprehensive list of properties of $\trianglelefteq$ from a semigroup-theoretic point of view.

9.7 Category-Theoretic Structures and "Quantum Logic"

Now that we have established the structure of the category of partial isometries, we make an explicit comparison with the categorical structures used in the "categorical foundations" program for quantum mechanics. We base this comparison on the logical interpretations in both cases.

We first summarize – without taking a position ourselves – various criticisms of the Birkhoff–von Neumann quantum logic program. This demonstrates that the controversial features (the treatment of the tensor product, and interpretation of implication) are exactly the elements taken as primitive in categorical logic generally.

We then give a very brief summary of the categorical approach to logic, via *closed categories*, and describe the connections between a particular form of categorical closure called *compact closure* and the quantum properties of entanglement and teleportation, which is the basis of the categorical foundations program (Abramsky and Coecke 2004).

Finally, we consider whether the category of partial isometries is compact closed, or indeed closed at all. This is in order to establish either a close connection or an irreconcilable difference between the logical foundations in terms of orthomodular lattices and the logical foundations in terms of compact closed categories.

9.7.1 Criticisms of Birkhoff/von Neumann's Logical Interpretations

As a general principle, the meet and the join of a lattice have a natural intuitive interpretation as conjunction and disjunction. This is the basis of interpretation of projectors as propositions in Birkhoff–von Neumann quantum logic. However, the treatment of implication is more controversial.

Implication in an orthomodular lattice is often defined in terms of the Sasaki hook, $P \overset{S}{\rightarrow} Q = P^{\perp} \vee (P \wedge Q)$, as an analogue of how implication may be

[3] This raises the natural question of what sort of order-theoretic structures are involved in the study of partial isometries. We briefly discuss this in Section 9.10. However, the emphasis here is on the categorical properties.

defined in classical order-theoretic terms. The intuitive interpretation as a statement about "quantum propositions" is not straightforward – we refer to Hellman (1981), Smets (2001), and Hardegree (1979) for various proposals. More seriously, several authors (Holdsworth and Hooker 1983; Hellman 1981) have questioned whether it should be considered as an implication at all. This criticism is often based on the failure of the "deduction theorem" of classical logic: $(a \wedge b) \leq c$ iff $a \leq (b \to c)$. This property is equivalent to distributivity in a lattice, and distributivity is exactly the point at which quantum logic diverges from classical logic – no connective on the lattice of projectors can satisfy this property (Malinowski 1990). Moreover, as described in Hellman (1981), failure of the deduction theorem has serious implications, among which is the failure of *transitivity* (that is, $(A \Rightarrow B)$ and $(B \Rightarrow C)$ together imply $A \Rightarrow C$).

Another criticism of the orthomodular lattices approach is that the tensor product has no natural, purely order-theoretic, definition. Although we may certainly form the tensor product of two Hilbert spaces H, K and interpret the lattice projections in the resulting space $H \otimes K$ as propositions about some compound system, there is no natural way of taking the lattice of projectors of H, and the lattice of projectors of K, and producing the lattice of projectors of $H \otimes K$. In order to use the tensor product (and hence reason about compound systems), we need to step outside lattice theory, use Hilbert space operations, and interpret the resulting lattice of projectors. This is a drawback in the program of providing foundations for quantum mechanics purely in lattice-theoretic, or logical, terms.

We now consider an alternative approach to logic generally, where implication is primitive and the conjunction (interpreted as a categorical tensor) may be defined in terms of its relationship to the implication.

"Even the crows on the roofs caw about the nature of conditionals."
– Callimachus, Librarian of Alexandria, 300 B.C. (Kneale and Kneale 1971).

9.7.2 Logic, Category-Theoretically

In the order-theoretic approach to logic, implication is defined in terms of meet and join, interpreted as conjunction and disjunction. In stark contrast (although see the end of this section), many approaches to categorical logic take the notion of implication or deduction as primitive. For example, in a categorical setting for intuitionistic logic (such as Lambek and Scott, 1986), an arrow $f : A \to B$ is treated, as per the Brouwer-Heyting-Kolmogorov interpretation (Sørensen and Urzyczyn 1998), as a proof of proposition B from the assumption of proposition A.

The closure property of logic (that a proof of B from the assumption of A is equivalent to a proof of $A \Rightarrow B$) is taken as primitive and modeled by *categorical closure* – a hom-set of arrows $\mathcal{C}(A, B)$ is itself an object of $\mathcal{C}$, denoted $[A \to B] \in Ob(\mathcal{C})$, and the operation $[\_ \to \_] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is a functor (the *internal hom* functor) that satisfies various natural coherence conditions (LaPlaza 1977).

This notion of closure describes the properties of implication, with no additional assumptions or connectives. It models logics where the only connective is an implication (such as Mints, 1981), and under the Curry-Howard isomorphism (Sørensen and Urzyczyn 1998), it is used to model purely applicative structures such as combinatory logic – see Lambek and Scott (1986) for a good overview. However, it is more common to consider *monoidal closed*, rather than simply *closed* categories. In a monoidal closed category, the internal hom $[\_ \to \_] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is related to the *monoidal tensor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by the adjunction

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, [A \to C]).$$

The natural interpretation of the tensor in this setting is thus conjunction, and (up to commutativity of conjunction) there is the immediate logical interpretation of

$$(A \wedge B) \Rightarrow C \quad \text{is equivalent to} \quad B \Rightarrow (A \Rightarrow C).$$

Finally, we observe that the adjunction between the monoidal tensor and the internal hom often means that one may be defined in terms of the other. In Mac Lane (1998), the fact that Abelian groups form a *closed* category is used along with the *adjoint functor theorem* to give a tensor product, and hence a *monoidal closed* category of Abelian groups. A similar technique is used to demonstrate the existence of a tensor product for partial commutative monoids in Hines and Scott (2009). In LaPlaza (1977), it is also shown that closed categories may always be embedded into monoidal closed categories.

Thus, from a categorical point of view, there is a strong connection between the two distinct controversial areas of quantum logic – the *tensor product* and the *implication*.

Relating order-theoretic and categorical approaches to logic. Although we have presented order theory and category theory as two competing, opposed approaches to logic, the reality is more subtle. Relating order-theoretic and category-theoretic approaches has been a very fruitful area of study and has shed light on order theory, category theory, logic and theoretical computing. We refer to Lawvere (1973), Abramsky (1991), and Lambek and Scott (1986) for several examples of a large field.

9.7.3 Compact Closed Categories

Compact closed categories are symmetric monoidal closed categories where the categorical closure is of a particularly well-behaved form. We refer to Kelly and Laplaza (1980) for details on compact closed categories, including a coherence theorem; Abramsky (1996) and Joyal et al. (1996) for their construction from traced monoidal categories; Abramsky (1996), Hines (1997), Abramsky et al. (2002), and Hines (2004) for logical and computational interpretations; and Hines (1999) for one-object, or untyped, compact closed categories.

Definition 9.7.1 (Compact closed categories, dual on objects, dual on arrows). *Let* $\mathbf{C}, \otimes$ *be a symmetric monoidal category, with unit object* I. *The category* $\mathbf{C}$

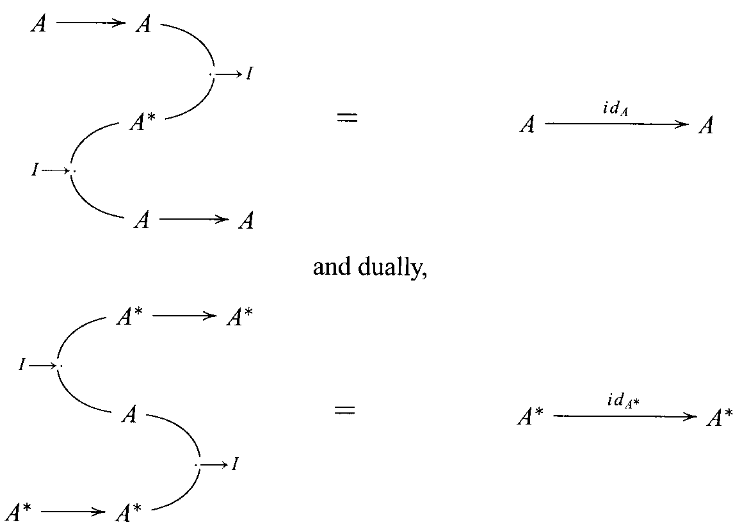


Figure 9.3. Axioms for compact closure

is called **compact closed** *when, for every object* $A \in Ob(\mathbf{C})$*, there exists a* **dual object** A^**, together with distinguished arrows*

- *The unit arrow* $\epsilon_A : A \otimes A^* \to I$
- *The counit arrow* $\eta_A : I \to A^* \otimes A$

that satisfy

$$(\epsilon_A \otimes 1_A)(1_A \otimes \eta_A) = 1_A \quad \textit{and dually,} \quad (1_{A^*} \otimes \epsilon_A)(\eta_A \otimes 1_{A^*}) = 1_{A^*}$$

Using the diagrammatic notation introduced in Joyal and Street (1991) and Joyal et al. (1996), this may be drawn as in Figure 9.3.

The dual operation on objects $(\_)^*$*, together with the unit and counit arrows, may be used to define the* **dual on arrows**. *Given* $f \in C(A, B)$*, then* $f^* \in C(B^*, A^*)$ *is defined by*

$$f^* = (1_{A^*} \otimes \epsilon_B)(1_{A^*} \otimes f \otimes 1_{B^*})(\eta_A \otimes 1_{B^*}) : B^* \to A^*.$$

Diagrammatically, this is as shown in Figure 9.4.

The internal hom is particularly easy to define, given by $[X \to Y] = X^* \otimes Y \in Ob(C)$. The details of the adjunction are then straightforward (although note that the symmetry isomorphism $A \otimes B \cong B \otimes A$, or similar condition, is required).

9.7.4 The Categorical Foundations Program

In the categorical foundations of quantum mechanics (Abramsky and Coecke 2004; Abramsky and Duncan 2006), the *teleportation protocol* (Bennett et al. 1993), and the notion of *teleportation as computation* Brassard et al. (1998);

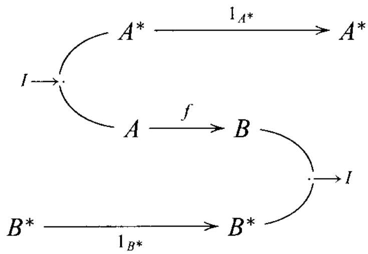


Figure 9.4. The dual operation on arrows

Braunstein et al. (2000) are taken as primitive. For physical interpretations, we refer to Bennett et al. (1993) for the standard description, and Braunstein (1996) for an alternative setting without irreversible measurements.

The interpretation is based on a compact closed category (with additional structure), where the dual operation is the identity on objects, so $H^* = H$. The axioms for compact closure shown in Figure 9.3 reduce to the axiom illustrated in Figure 9.5.

The monoidal tensor is interpreted as the formation of a composite system (as with the usual tensor product of Hilbert space), and the trivial object I is a one-dimensional space, so $I \otimes H \cong H \cong H \otimes I$.

The counit arrow $\eta_H : I \to H \otimes H$ is interpreted as the creation of a maximally entangled Bell state $|\mathcal{B}ell\rangle$ and its dual $\epsilon_H : H \otimes H \to I$ is a (postselected) measurement against the same maximally entangled state. The requirement that the composite shown on the l.h.s. of Figure 9.5 is exactly the identity gives the teleportation protocol (postselected) of Bennett et al. (1993).

In the concrete monoidal category $(\mathbf{Hilb}_{FD}, \otimes)$ of finite-dimensional Hilbert spaces, consider an n-dimensional space H, with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. The maximally entangled Bell state is given by $\mathcal{B}ell = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j$, so the corresponding unit and counit maps are $\langle \mathcal{B}ell| : H \to \mathbb{C}$ and $|\mathcal{B}ell\rangle : \mathbb{C} \to H$. Direct calculation gives that this is the identity, scaled by the appropriate renormalization required for the postselection of measurement outcome.

9.7.5 Logical Interpretations

Although the foundations for quantum mechanics of Section 9.7.4 are presented categorically, rather than logically, the fact that a monoidal closed (in fact, compact closed) category is key to this system leaves it open to a logical interpretation. An explicit logical interpretation is given in Duncan (2004), Abramsky and Duncan (2006), and Duncan (2007) by analogy with the connectives and structure of linear logic.

The application of compact closure to logical systems arose from analyses of Girard's Geometry of Interaction system (Girard 1989, 1988), a representation of linear logic (Girard 1987). We refer to Blute et al. (2000) for a good historical

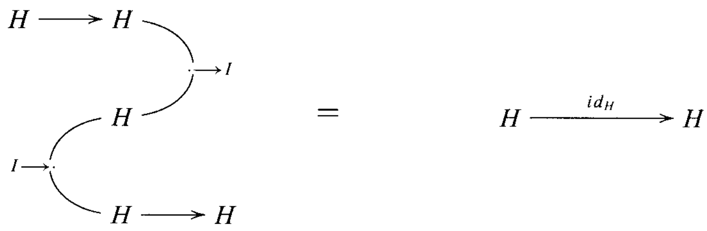


Figure 9.5. Compact closure with self-dual objects

overview. The categorical interpretation of Linear Logic as a whole is in terms of $*$-autonomous categories. However, the Geometry of Interaction system gave a representation of a restricted fragment (the *multiplicatives*). It was also degenerate in many ways, including the identification of the conjunction and disjunction (a case is made in Abramsky et al., 2002, that the correct interpretation of the geometry of interaction system is as a *combinatory logic*).

The logical interpretation of the categorical analysis of Section 9.7.4 is even more degenerate, in that all objects are *self-dual*. In $*$-autonomous and compact closed categories, the dual $*$ is interpreted as the logical negation. However, there remains a monoidal tensor and (via the monoidal closure) an internal hom functor that satisfy the required properties for a logical system.

We refer to Duncan (2004) and Abramsky and Duncan (2006) for more details.

9.8 Compact Closure, Teleportation, and Partial Isometries

The category of partial isometries can reasonably be considered as a categorification of von Neumann–Birkhoff quantum logic. However, from Section 9.6.1 and the interpretation of Lemma 9.4.2, we do not expect to be able to describe all quantum operations within the category of partial isometries.

In order to consider whether the category of partial isometries, as a categorification of von Neumann–Birkhoff quantum logic, fits into the general "categorical foundations" framework, we consider whether a (postselected version of) the teleportation protocol (Bennett et al. 1993) can be expressed in the category $(\mathbf{pIsom}_{FD}, \otimes)$.

9.8.1 Is $(\mathbf{pIsom}_{FD}, \otimes)$ *Compact Closed?*

Before answering the preceding question, we need to explain why (apart from wishful thinking) we might think that $(\mathbf{pIsom}_{FD}, \otimes)$ should be compact closed – at least, in the finite-dimensional case. A suggestive, but incorrect, train of thought is as follows:

Nontheorem 9.8.1. *The category* $(\mathbf{pIsom}_{FD}, \otimes)$ *is compact closed, with self-dual objects.*

Nonproof. Consider the defining identity of compact closure (in the self-dual case):

$$\lambda(\epsilon_A \otimes 1_A)(1_A \otimes \eta_A)\rho^{-1} = 1_A.$$

If, as in the categorical foundations described in Section 9.7.4, we interpret the counit η_H and unit ϵ_H as the bra and ket $|\mathcal{B}ell\rangle : \mathbb{C} \to H \otimes H$ and $\langle \mathcal{B}ell| : H \otimes H \to \mathbb{C}$, we observe that these are partial isometries. Hence, as partial isometries are closed under the tensor product, $(\epsilon_A \otimes 1_A)$ and $(1_A \otimes \eta_A)$ are both partial isometries. The identity isomorphisms $\rho^{-1} : H \cong H \otimes \mathbb{C}$ and $\lambda : \mathbb{C} \otimes H \cong H$ are trivially unitary, and so (by Corollary 9.3.3) the linear maps

$$\mathcal{T}ele = \lambda(\epsilon_A \otimes 1_A) \quad \text{and} \quad \mathcal{P}ort = (1_A \otimes \eta_A)\rho^{-1}$$

are both partial isometries.

Based on the categorical foundations, we wish to claim that the composite $\mathcal{T}ele\mathcal{P}ort : H \to H$ is the identity map (again, a partial isometry). From Theorem 9.3.2, the composite KL of two partial isometries is itself a partial isometry exactly when the final projector of L commutes with the initial projector of K, in which case (by Lemma 9.5.3) $K \circ L = KL$.

We thus wish to conclude that the initial projector of $\mathcal{T}ele$ commutes with the final projector of $\mathcal{P}ort$, and so

$$\mathcal{T}ele \circ \mathcal{P}ort = \mathcal{T}ele\mathcal{P}ort = 1_H.$$

End of Nonproof

To see that this reasoning is incorrect, we explicitly exhibit the final projector of $\mathcal{P}ort$ and the initial projector of $\mathcal{T}ele$ and show that these do *not* commute.

Lemma 9.8.2. *Consider the partial isometries $\mathcal{T}ele$ and $\mathcal{P}ort$, defined in Nontheorem 9.8.1. Then the initial and final projectors $E_{\mathcal{T}ele}$ and $F_{\mathcal{P}ort}$ do not commute, so $\mathcal{T}ele \circ \mathcal{P}ort \neq \mathcal{T}ele\mathcal{P}ort$.*

Proof. Consider a complex N-dimensional space H with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$. The maximally entangled Bell state is given by $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{e}_i \otimes \mathbf{e}_i$, so the unit and counit maps are

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} |\mathbf{e}_j \mathbf{e}_j\rangle \quad \text{and} \quad \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \langle \mathbf{e}_k \mathbf{e}_k|$$

respectively. From this, the partial isometries $\mathcal{T}ele : H \otimes H \otimes H \to H$ and $\mathcal{P}ort : H \to H \otimes H \otimes H$ may be given explicitly by

$$\mathcal{T}ele = \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} |\mathbf{e}_a\rangle\langle \mathbf{e}_b \mathbf{e}_b \mathbf{e}_a| \quad \text{and} \quad \mathcal{P}ort = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle\langle \mathbf{e}_i|.$$

Thus, the final projector of $\mathcal{P}ort$ may be given by

$$F_{\mathcal{P}ort} = \left(\frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle\langle \mathbf{e}_i| \right) \left(\frac{1}{\sqrt{N}} \sum_{k,l=1}^{N} |\mathbf{e}_k \mathbf{e}_l \mathbf{e}_l\rangle\langle \mathbf{e}_k| \right)^*$$

$$= \frac{1}{N} \sum_{i,j,k,l=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle \delta_{ik} \langle \mathbf{e}_k \mathbf{e}_l \mathbf{e}_l| = \frac{1}{N} \sum_{i,j,l=1}^{N} |\mathbf{e}_i \mathbf{e}_j \mathbf{e}_j\rangle\langle \mathbf{e}_i \mathbf{e}_l \mathbf{e}_l|.$$

Similarly, the initial projector of $\mathcal{T}ele$ is given by

$$E_{\mathcal{T}ele} = \left(\frac{1}{\sqrt{N}} \sum_{c,d=1}^{N} |\mathbf{e}_d\mathbf{e}_d\mathbf{e}_c\rangle\langle\mathbf{e}_c|\right)\left(\frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} |\mathbf{e}_a\rangle\langle\mathbf{e}_b\mathbf{e}_b\mathbf{e}_a|\right)$$

$$= \frac{1}{N} \sum_{a,b,c,d=1}^{N} |\mathbf{e}_d\mathbf{e}_d\mathbf{e}_c\rangle\delta_{ac}\langle\mathbf{e}_b\mathbf{e}_b\mathbf{e}_a| = \frac{1}{N} \sum_{a,b,d=1}^{N} |\mathbf{e}_d\mathbf{e}_d\mathbf{e}_a\rangle\langle\mathbf{e}_b\mathbf{e}_b\mathbf{e}_a|.$$

Direct calculation verifies that $E_{\mathcal{T}ele}F_{\mathcal{P}ort} \neq F_{\mathcal{P}ort}E_{\mathcal{T}ele}$ and so by Proposition 9.3.2, $\mathcal{T}ele \circ \mathcal{P}ort \neq \mathcal{T}ele\mathcal{P}ort$. □

What, then, has gone wrong in the reasoning in Nontheorem 9.8.1? Our claim is that the renormalization, or implicit postselection, in Section 9.7.4 is incompatible with a study of teleportation via partial isometries, as follows:

Theorem 9.8.3. *Consider the partial isometries $\mathcal{T}ele : H \to H \otimes H \otimes H$ and $\mathcal{P}ort : H \otimes H \otimes H \to H$ of Non-Theorem 9.8.1. Their composite, as linear maps, is $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} 1_H$ and therefore their composite in the category* $\mathbf{pIsom}_{FD}$ *is $\mathcal{T}ele \circ \mathcal{P}ort = 0_H$.*

Proof. By direct calculation,

$$\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} \sum_{a,b,i,j=1}^{N} |\mathbf{e}_a\rangle\langle\mathbf{e}_b\mathbf{e}_b\mathbf{e}_a|\mathbf{e}_i\mathbf{e}_j\mathbf{e}_j\rangle\langle\mathbf{e}_i|$$

$$= \frac{1}{N} \sum_{a,b,i,j=1}^{N} |\mathbf{e}_a\rangle\delta_{bi}\delta_{bj}\delta_{aj}\langle\mathbf{e}_i| = \frac{1}{N}\sum_{a=1}^{N} |\mathbf{e}_a\rangle\langle\mathbf{e}_a| = \frac{1}{N} Id_H$$

Using the original definition of composition in **pIsom**,

$$\mathcal{T}ele \circ \mathcal{P}ort = lim_{K\to\infty} [(\mathcal{T}ele\mathcal{P}ort)(\mathcal{T}ele\mathcal{P}ort)^*]^K\, \mathcal{T}ele\mathcal{P}ort$$

As $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} Id_H$ is self-adjoint,

$$\mathcal{T}ele \circ \mathcal{P}ort = lim_{K\to\infty}\left[\frac{1}{N^2} Id_H\right]^K \frac{1}{N} Id_H = 0_H$$

□

Why teleportation cannot be expressed in ($\mathbf{pIsom}_{FD}$, ⊗). A direct calculation has given that, as a straightforward composite of linear maps, $\mathcal{T}ele\mathcal{P}ort = \frac{1}{N} Id_H$. This extraneous factor of $\frac{1}{N}$ occurs simply because the probability of observing this particular experimental outcome (*i.e.*, the Bell state $\mathcal{B}ell = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \mathbf{e}_j \otimes \mathbf{e}_j$) is exactly $\frac{1}{N}$.

In the categorical foundations approach, postselection of measurement outcomes is used to eliminate this scaling factor, and hence give compact closure. In **pIsom**, we cannot simply introduce a scaling factor of N at some point to account for postselection; all partial isometries (except the zero map) have operator norm of 1, and such a "scaled" unit/counit pair would no longer be members of the category.

9.8.2 *Modifying* $\mathbf{pIsom}_{\mathbf{FD}}$*: Rays and Renormalization*

Given the foregoing interpretation of why teleportation cannot be expressed in $(\mathbf{pIsom}_{FD}, \otimes)$, a natural question is to wonder whether this may be "fixed" by either introducing a sufficient number of scalars (categorically, endomorphisms of the unit object; Abransky, 2005) or simply working with rays (*i.e.*, one-dimensional subspaces) rather than norm-1 states.

Partial isometries and rays. We may eliminate any notion of magnitude by working with a system based on rays, so partial isometries are characterized by their actions on one-dimensional subspaces rather than on vectors. The notion of a projector, as a self-adjoint idempotent, remains, and so we may still define partial isometries in this weaker setting. The lattice ordering of projectors also remains (albeit without Shimony's explicit form for the meet, Proposition 9.2.2), as does the Halmos-McLaughlin partial order. However, having eliminated any notion of magnitude, the composition of Definition 9.5.2 is inapplicable, and we must use the lattice-theoretic characterization of composition given in Lemma 9.5.3, as $M \circ L = M(E_M \wedge F_L)L$.

When applied to the preceding analysis of teleportation, working with a magnitude-free system based on rays may be seen to change nothing: regardless of scaling, the initial space of $\mathcal{P}ort$ and the final space of $\mathcal{T}ele$ are both one-dimensional subspaces, and their meet is the zero-dimensional subspace, as shown in Figure 9.1.

Partial isometries and abstract scalars. Alternatively, let us expand the category of partial isometries to include some natural family of abstract scalars[4] $\{s_H : H \to H\}_{H \in Ob(\mathbf{pIsom})}$, satisfying $P \circ s_H = s_K \circ P$, for all $P \in \mathbf{pIsom}(H, K)$. We further assume that $(\mathbb{R}^+, \times) \cong \{s_H\} \subseteq \mathbf{pIsom}_{\mathbf{FD}}(H, H)$, so we have a representation of the multiplicative structure of the positive reals.

Although such an extension of $\mathbf{pIsom}_{\mathbf{FD}}$ is possible, the problem is that the foregoing naturality condition relates to the composition $\circ$ of $\mathbf{pIsom}$, rather than the usual scalar multiplication of a linear map in **Hilb**. Therefore, such "abstract scalars" *cannot* coincide with the usual "scalars" (*i.e.*, the underlying field) of $\mathbf{Hilb}_{FD}$. Given partial isometries $L \in \mathbf{pIsom}_{FD}(H, K)$ and $M \in \mathbf{pIsom}_{FD}(K, H)$ such that $M \circ L = 0_H \in \mathbf{pIsom}_{FD}(H, H)$, it is often easy to find scalars $r_1, r_2 \in \mathbb{C}$ such that

$$r_2 M \circ r_1 L = lim_{n\to\infty} [(r_2 M r_1 L)(r_2 M r_1 L)^*]^n (r_2 M r_1 L) \neq 0_H.$$

[4] In order to consider a wide range of cases, including those that may seem categorically unnatural, we do not assume that these are defined in terms of endomorphisms of the unit object as in Abramsky (2005).

However, given arbitrary abstract scalars s_2, s_1, the naturality condition immediately implies that

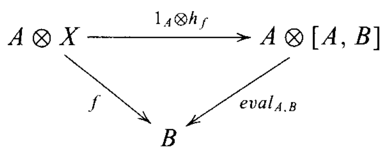
$$(s_2 \circ M) \circ (s_1 \circ L) = (s_2 \circ s_1) \circ (M \circ L) = s_2 \circ s_1 \circ 0_H = 0_H.$$

9.9 Is $(\mathbf{pIsom}_{FD}, \otimes)$ Closed?

So far, we have *not* proved that $(\mathbf{pIsom}_{FD}, \otimes)$ is not compact closed. Rather we have shown that the usual way of demonstrating compact closure (as post-selected teleportation) does not hold in $(\mathbf{pIsom}_{FD}, \otimes)$. We now demonstrate that $(\mathbf{pIsom}_{FD}, \otimes)$ cannot be monoidal closed at all (and therefore is certainly not *compact* closed).

To demonstrate that $(\mathbf{pIsom}_{FD}, \otimes)$ is not monoidal closed, we use an alternative characterization of monoidal closure, from Mac Lane (1998), equivalent to that of Section 9.7.2. This proof may be thought of as a more abstractly categorical version of the calculations of Section 9.8.

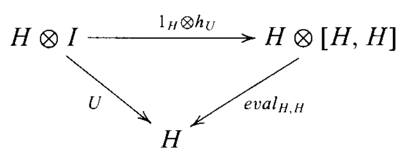
Definition 9.9.1 (Monoidal closed categories, evaluation maps). *Let* $\mathbf{C}, \otimes$ *be a symmetric monoidal category. It is* **monoidal closed** *when, for all objects* $A, B \in Ob(\mathbf{C})$*, there exists an* **internal hom** *object* $[A, B]$ *together with an* **evaluation map** $eval_{A,B} \in \mathbf{C}(A \otimes [A, B], B)$ *such that, for all* $f \in \mathbf{C}(A \otimes X, B)$*, there exists unique* $h_f \in \mathbf{C}(X, [A, B])$ *making the following diagram commute:*



Using this alternative characterization of monoidal closed categories, we demonstrate that no such evaluation map can exist within the category of finite-dimensional partial isometries.

Theorem 9.9.2. $(\mathbf{pIsom}_{FD}, \otimes)$ *is not monoidal closed.*

Proof. Let us assume that $(\mathbf{pIsom}_{FD}, \otimes)$ is monoidal closed, so the diagram of Definition 9.9.1 commutes. Let us now consider the special case where X is the identity object for the monoidal structure and f is some unitary $U : H \to H$.



(For simplicity of notation, we elide the left and right units isomorphism for the monoidal structure.)

As this diagram commutes, we may deduce that the final projector of the map $eval_{H,H} \in \mathbf{pIsom}_{FD}(H \otimes [H, H], H)$ is the whole of H. Thus, the initial subspace of $eval_{H,H}$ is a subspace of $H \otimes [H, H]$ isomorphic to H itself, and so there exists some embedding $embed : H \hookrightarrow H \otimes [H, H]$ such that, for all $U : H \to H$, there exists some unique $k_U : H \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} H \otimes I & \xrightarrow{\quad k_U \quad} & H \\ {\scriptstyle U}\big\downarrow & \searrow^{1_H \otimes h_U} & \big\downarrow{\scriptstyle embed} \\ H & \xleftarrow[\quad eval_{H,H} \quad]{} & H \otimes [H, H] \end{array}$$

In the diagram, both $embed \in \mathbf{pIsom}_{FD}(H, H \otimes [H, H])$ and $eval \in \mathbf{pIsom}_{FD}(H \otimes [H, H])$ are independent of $U \in \mathbf{pIsom}_{FD}(H, H)$. Therefore, the subspace of h_U must be some subspace of $H \otimes [H, H]$ isomorphic to H. By the commutativity of the upper right triangle of this diagram, and the fact that k_U must itself be unitary, the final subspace of h_U is also independent of U.

This then implies that, as h_U has a one-dimensional initial and final subspace, for any pair of unitary maps $U, U' : H \to H$ completing the diagram will demonstrate that they differ by at most some complex phase factor. As this is certainly not the case, we deduce that no such evaluation map may exist, and therefore $(\mathbf{pIsom}_{FD}, \otimes)$ is not monoidal closed. □

The adjoint functor theorem, and orthomodularity. The usual method of proving or disproving that a complete category is monoidal closed is via an application of Freyd's *adjoint functor theorem* (Mac Lane 1998) to the description of categorical closure as an adjunction. It is sometimes claimed that this theorem can be used to demonstrate that no closed structure can be built upon orthomodular lattices.

However, it is also possible to use the adjoint functor theorem to demonstrate that the category of Hilbert spaces and linear maps is not closed. Recall that it is only the category of *finite-dimensional* Hilbert spaces that is compact closed (Abramsky et al. 1998). When working in the finite-dimensional case, the resulting categories (whether based on linear maps, partial isometries, or orthomodular lattice morphisms), are no longer complete categories – and therefore the adjoint functor theorem is inapplicable.

9.9.1 Global and Local Closure

In this chapter, we have studied and emphasized the "global" form of closure provided by monoidal closed categories. This is in order to compare the "categorical foundations" program, which is based on categorical closure, with the "quantum logic program," based on order theory.

An alternative, which sits more easily with the order-theoretic approach of Birkhoff-von Neumann quantum logic, is to consider local Galois adjunctions, as in Coecke et al. (2001) and Faure et al. (1995). Such an approach, while not leading to the global form of closure associated with categorical logic, may lead to natural interpretations for quantum logic connectives – the Sasaki hook in particular is amenable to such a treatment (Coecke and Smets 2001).

The authors of this chapter are not, therefore, taking a position on whether or not "quantum logic" is a valid logical system.

9.10 Future Directions

We have established that, far from being the "monstrosities" of Girard (1989), partial isometries do in fact have a rich categorical structure, closely related to Birkhoff/von Neumann quantum logic. This categorical structure has also let us see that there does not appear to be a natural overlap between the "categorical foundations" and "orthomodular lattices" approaches to foundations of quantum mechanics. Although this may be unsurprising to researchers in both fields, it is hoped that this paper has also provided a new perspective on why these foundational approaches appear to be incompatible.

There remain several unanswered questions. One of the most pressing is: what sort of partially ordered structure is provided by the Halmos-McLaughlin partial order? Although we have concentrated here on the category theory of partial isometries, we will demonstrate in a future paper that (**pIsom**(H, K), $\leq$) is in fact a complete partial order (CPO), where the down-closure of each maximal element is an orthomodular lattice. Whether or not it has more structure (*e.g.*, whether it is a Scott domain) is at present an open question.

There also remains to consider the monoidal structure of **pIsom**, with respect to the direct sum, rather than the tensor product. This is closely related to both the partial ordering, and the inverse category structure. It is demonstrated in Hines (2007) that all inverse categories satisfying very light requirements (including **pIsom**) have a categorical trace. However, this is a trace on the direct sum structure, rather than the tensor product structure, and so does not have the usual natural interpretation in terms of composite systems. There is also the question of whether the usual partial summation on hom-sets of inverse categories coincides with the partial summation provided by the linear structure.

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