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RELATIVISTIC QUANTUM INFORMATION PROCESSING WITH BOSONIC AND FERMIONIC INTERFEROMETERS

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We derive the relativistic transformation laws for the annihilation operators of the scalar field, the massive spin-1 vector field, the electromagnetic field and the spinor field. The technique developed here involves straightforward mathematical techniques based on fundamental quantum field theory, and is applicable to the study of entanglement in arbitrary coordinate transformations. In particular, it predicts particle creation for non-inertial motion. Furthermore, we present a unified description of relativistic transformations and multi-particle interferometry with bosons and fermions, which encompasses linear optical quantum computing.

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In his last contribution to the quantum archive, called *Quantum Information and General Relativity*,[1] Asher Peres wrote that "when I was a young man, my thesis adviser was Nathan Rosen, and the subject was the existence of gravitational radiation in general relativity. Only much later, I seriously learnt quantum mechanics, and still much later information theory. I now want to return to my roots and try to combine all these subjects together." It therefore seems appropriate to honor the memory of Asher Peres with a paper that describes aspects of quantum information theory for observers in a (general) relativistic setting.

1. Introduction

As a fundamental physical theory, relativistic quantum information theory (RQIT) has widespread applications, ranging from practical tools for describing moving observers in quantum communication protocols, to black hole thermodynamics. For example, it is hoped that RQIT will play a central role in clock synchronization and (optical) quantum communication between a ground station and a relativistically moving satellite, as well as the resolution of the black hole information paradox.

So far, work on RQIT has focused predominantly on relativistic transformations of single-particle wave functions. Early work by Czachor defined the relativistic spin operator for a relativistic description of the violation of Bell inequalities.[2] Then, Peres *et al.* showed that the reduced density matrix for the spin of an electron is not a Lorentz invariant scalar,[3] indicating that there is a spin-momentum interaction in Lorentz transformations. Subsequently, Alsing and Milburn determined the transformation properties of entangled particles in momentum eigenstates,[4] and it was shown by Gingrich and Adami that spin and polarization entanglement between two Gaussian wave packets is transferred to momentum entanglement under Lorentz boosts.[5,6] Also, relativistic effects on polarization was studied by Peres and Terno.[7] A review of the important issues in relativistic quantum information theory is given by Peres *et al.*[8]

Many of these results have been obtained using Wigner's *little group* formalism.[4–6,9] Since the little group is constructed from a standard momentum four-vector that is invariant under Lorentz transformations, this formalism breaks down for *arbitrary* coordinate transformations. In addition, it is not always straightforward to find the representations of the little group. For a wider application of relativistic results, "plug-and-play" transformation rules for the annihilation operator seem more appropriate.

These transformation rules are essential for the description of (multi-particle) quantum interferometry. In particular, we are interested in a relativistic extension of linear optical quantum computing. In this paper, we present a general technique for deriving the annihilation operator of the scalar field, the spin-1 vector field, the electromagnetic field, and the spin-$\frac{1}{2}$ Dirac field for arbitrary coordinate transformations, and we give the explicit result for Lorentz transformations. We then give a unified description of the resulting Bogoliubov transformations and multi-particle quantum interferometry. This will lead to a relativistic formulation of linear optical quantum computing.[10]

There are several advantages to our technique: (i) The transformation laws are relatively easy to obtain by integration, rather than finding group representations. (ii) The resulting transformations are completely fundamental. (iii) Our technique is applicable to any coordinate transformation (including in the presence of curvature), and is not restricted to Lorentz boosts. In particular, it predicts particle creation for non-inertial motion. (iv) Substituting the transformed annihilation operator into specific expressions of the state of a quantum field automatically yields the correct transformed state. This way, it is straightforward to describe boosted wave packets, and it allows us to study the transformation of *entanglement in arbitrary coordinate systems.*

In the next section, we present a relativistic paradox, to put the issue on edge. In Sec. 3, we give the field transformations for Lorentz-boosted quantum fields. In Sec. 4, we solve the paradox, and sketch relativistic multi-particle quantum interferometry and relativity. In principle, this unification includes arbitrary particle creation associated with non-inertial observers and observers on curved spacetime. Finally, we present our conclusions in Sec. 5.

2. The Twin-Photon Paradox

In order to describe relativistic multi-particle quantum interferometry, let us perform the following *gedanken experiment*: Two single-photon plane waves with momenta k_1 and k_2, and polarization $j \in \{H, V\}$ meet at a 50:50 beam splitter. In the outgoing modes of the beam splitter we place two ideal particle detectors, which tell us with perfect fidelity how many photons there are in that mode. It is well known that a 50:50 beam splitter causes two identical photons to "pair off" into the outgoing modes. In other words, we will never find any coincidence counts between the two detectors. This is the so-called Hong-Ou–Mandel effect.[11] Mathematically, the transformation of the incoming modes would be

$$\begin{aligned} \hat{a}_j(k_1) &\to \frac{1}{\sqrt{2}}[\hat{a}_j(k_1) + \hat{a}_j(k_2)], \\ \hat{a}_j(k_2) &\to \frac{1}{\sqrt{2}}[-\hat{a}_j(k_1) + \hat{a}_j(k_2)]. \end{aligned} \tag{1}$$

The input state $\hat{a}_j^\dagger(k_1)\hat{a}_j^\dagger(k_2)|0\rangle$ is then transformed into $\frac{1}{2}[\hat{a}_j^{\dagger 2}(k_2) - \hat{a}_j^{\dagger 2}(k_1)]|0\rangle$, where $|0\rangle$ denotes the vacuum.

An observer in a boosted frame of reference will see quite a different physical process taking place: to him, the two waves do not necessarily have the same frequency and polarization. As a result, the two photons are not identical, and there will be coincidence counts in the detectors. However, photon counting yields numbers, which are Lorentz invariant. Consequently, both observers should obtain exactly the same detector statistics. Thus, we arrive at a contradiction.

A similar paradox can be constructed for fermions. Here, the exclusion principle forbids identical particles from occupying the same quantum state, resulting in the absence of two-fermion states in either output mode of the fermionic beam splitter. Again, to a boosted observer the fermions have different wavelengths and spin, leading to different detector statistics. In order to resolve this paradox, we explicitly calculate the transformation rules for the annihilation operator of the quantum fields. Furthermore, both the twin-photon and the twin-electron paradox are examples of two-particle quantum interferometry.

3. Field Transformations

3.1. *Scalar fields*

The scalar quantum field $\phi(x)$ obeys the Klein–Gordon equation $(\partial_\mu \partial^\mu - m^2)\phi = 0$, where m is the mass of the field, and Greek indices always denote components of a four-vector. It can be expanded in terms of mode functions f:

$$\phi(x) = \int \frac{d\mathbf{k}}{2k_0} \left[\hat{a}(k) f_k(x) + \hat{a}^\dagger(k) f_k^*(x) \right],$$

where k is the four-momentum, $\mathbf{k}$ is the three-vector component of k, and k_0 is the energy component. The annihilation and creation operators associated with mode k are $\hat{a}(k)$ and $\hat{a}^\dagger(k)$, respectively. The annihilation operator is extracted using the time-independent inner product[12]:

$$\hat{a}(k) = i \int d^3x f_k^*(x) \overleftrightarrow{\partial}_0 \phi(x) \equiv (f_k, \phi), \tag{2}$$

where $a(t)\overleftrightarrow{\partial}_0 b(t) = a(t)\partial_0 b(t) - [\partial_0 a(t)]b(t)$. Typically, we choose the plane-wave expansion $f_k(x) = [(2\pi)^3 2k_0]^{-\frac{1}{2}} e^{ikx}$, where $kx \equiv k_\mu x^\mu$.

Alice and Bob are two observers that occupy two different reference frames. Alice describes the field $\phi(x)$ in terms of her coordinates x, whereas Bob describes the field $\phi(x')$ in terms of his coordinates x'. The two coordinate systems are connected by an invertible transformation. In this paper, we will restrict ourselves to Lorentz transformations Λ such that $x' = \Lambda(x - \ell)$, with ℓ an arbitrary translation. However, our results also apply to arbitrary coordinate transformations corresponding to non-inertial relative motion. In addition, Bob uses his own definition of the annihilation and creation operators $\hat{a}'(k')$ and $\hat{a}'^\dagger(k')$, and (plane wave) mode functions $g(x')$. The question is now what are the transformation rules that relate $\hat{a}(k)$ and $\hat{a}'(k')$. To this end, we can extract Bob's annihilation operator:

$$\hat{a}'(k') = i \int d^3x' f_{k'}^*(x') \overleftrightarrow{\partial}_{0'} \phi(x'). \tag{3}$$

Alternatively, Alice may describe the field ϕ in terms of her coordinates $x(x')$. When we substitute this into Eq. (3) and use $k' = \Lambda k$, we obtain

$$\hat{a}'(\Lambda k) = \hat{a}(k) e^{-i\Lambda k \ell}, \tag{4}$$

with a similar expression for the creation operators. It is clear that the bosonic commutation relations still hold for $\hat{a}'$ and $\hat{a}'^\dagger$. The state of (multi-particle) wave packets can be expressed in terms of a function f of creation operators $\hat{a}^\dagger$ acting on a vacuum defined by $\hat{a}|0\rangle = 0$. Similarly, the transformed annihilation operator defines a vacuum state $\hat{a}'|0'\rangle = 0$. The state then transforms as $\mathsf{f}(\hat{a}^\dagger)|0\rangle \to \mathsf{f}(\hat{a}'^\dagger)|0'\rangle$. For Lorentz transformations, the vacuum states of Alice and Bob are identical. Other transformations, however, change the vacuum state. Alice and Bob then no longer agree upon the number of particles in the experiment.

In general, since quantum states can be expressed in terms of a function of creation operators $\hat{a}^\dagger$ acting on the vacuum $|0\rangle$, substituting this transformation rule $\hat{a}'^{\dagger}$ on the vacuum $|0'\rangle$ will immediately yield the correctly transformed quantum state.

3.2. *Spin-1 massive boson fields*

The simplest extension to the Klein–Gordon field is the spin-1 degree of freedom, yielding a vector field $V^\mu(x)$ with mass m:

$$V^\mu(x) = \int \frac{d\mathbf{k}}{2k_0} \sum_{j=-1}^{1} \left[\frac{\epsilon_j^\mu \hat{a}_j(k) e^{ikx}}{\sqrt{(2\pi)^3 2k_0}} + \text{H.c.} \right],$$

where ϵ_j^μ is the four-vector associated with the j-component of the field, and H.c. stands for Hermitian conjugate. The field obeys the Lorentz gauge $\partial_\mu V^\mu = k_\mu \epsilon_j^\mu = 0$, which, for a particle at rest, suggests the representation $k = (m, 0, 0, 0)$, $\epsilon_1 = (0, 1, 0, 0)$, $\epsilon_0 = (0, 0, 1, 0)$ and $\epsilon_{-1} = (0, 0, 0, 1)$. The relativistic transformation of the vector field is given by

$$\Lambda^\mu{}_\nu V^\nu(x) = \int \frac{d\mathbf{k}}{2k_0} \sum_j \Lambda^\mu{}_\nu \frac{\epsilon_j^\nu \hat{a}_j(k) e^{ikx}}{\sqrt{(2\pi)^3 2k_0}} + \text{H.c.}$$

Extracting the annihilation operator using $f_{k,j}^\mu(x) = [(2\pi)^3 2k_0]^{\frac{1}{2}} \epsilon_j^\mu e^{ikx}$ then yields

$$\hat{a}'_j(\Lambda k) = \sum_{l=-1}^{1} \epsilon_{\mu,j}^* \Lambda^\mu{}_\nu \epsilon_l^\nu \hat{a}_l(k) e^{-i\Lambda k \ell}. \tag{5}$$

Lorentz transformations do not leave three-volumes invariant, and we need to renormalize the transformation to make it unitary. The boosted annihilation operator then becomes

$$\hat{a}'_j(\Lambda k) = -m^2 \sum_{l=-1}^{1} \frac{\epsilon_{\mu,j}^* \Lambda^\mu{}_\nu \epsilon_l^\nu}{k_\mu \Lambda^\mu{}_\nu k^\nu} \hat{a}_l(k) e^{-i\Lambda k \ell}, \tag{6}$$

which obeys the bosonic commutation relations. Here, we observe a boost-dependent change in spin.

3.3. *Gauge fields*

In order to find the proper Bogoliubov transformations for massless spin-1 fields, it is clear from Eq. (6) that (contrary to scalar fields) we cannot take the limit $m \to 0$. Massless fields with spin, such as the quantized electromagnetic field, have an extra gauge freedom that we need to take into account. Here, we consider the vector potential of the electromagnetic field:

$$A^\mu(x) = \int \frac{d\mathbf{k}}{2k_0} \sum_j \left[\frac{\epsilon_j^\mu \hat{a}_j(k) e^{ikx}}{\sqrt{(2\pi)^3 2k_0}} + \text{H.c.} \right],$$

where j indicates two orthogonal polarizations. In addition to the Lorentz gauge, it has to obey a second gauge relation, usually the Coulomb gauge $\nabla \cdot \mathbf{A} = \mathbf{k} \cdot \boldsymbol{\epsilon}_j = 0$. In this gauge, there is no longitudinal polarization. Since $k_\mu \epsilon_j^\mu$ is an invariant scalar, Lorentz transformations will keep the field in the Lorentz gauge. However, this is not true for the Coulomb gauge, and since this is the gauge that is typically used in the description of multi-particle interferometry, we need to take this change into account in our calculation.

The gauge freedom means that we can add the derivatives of two massless Klein–Gordon scalar fields ϕ_j to the vector potential in order to change the gauge:

$$A^\mu \to A^\mu + \sum_j \alpha_j \partial^\mu \phi_j . \tag{7}$$

The addition of such a gauge term does not change the observable outcomes, since all physical observables depend only on derivatives of A, and $\partial_\mu \partial^\mu \phi = 0$.

The relativistic transformation of the vector potential is given by

$$\Lambda^\mu{}_\nu A^\nu(x) = \int \frac{d\mathbf{k}}{2k_0} \sum_j \Lambda^\mu{}_\nu \frac{\epsilon_j^\nu \hat{a}_j(k) e^{ikx}}{\sqrt{(2\pi)^3 2k_0}} + \text{H.c.}$$

By changing the coordinates $x = \Lambda^{-1} x'$ and changing the integration variable $k = \Lambda^{-1} k'$, we find

$$\Lambda^\mu{}_\nu A^\nu(x') = \int \frac{d\mathbf{k}'}{2k_0} \sum_\lambda \Lambda^\mu_\nu \frac{\epsilon_j^\nu \hat{a}_j(\Lambda^{-1}k') e^{ik'x'}}{\sqrt{(2\pi)^3 2k_0'}} + \text{H.c.}$$

The Lorentz condition is still satisfied, as is easily checked. In order to fix the Coulomb gauge, we need to add the terms in Eq. (7) and choose the α_j's appropriately. We then have

$$\Lambda^\mu{}_\nu A^\nu(x') = \int \frac{d\mathbf{k}'}{2k_0} \sum_j \frac{\tilde{\epsilon}_j^\mu \hat{a}_j(\Lambda^{-1}k') e^{ik'x'}}{\sqrt{(2\pi)^3 2k_0'}} + \text{H.c.},$$

where

$$\tilde{\epsilon}_j^\mu = \Lambda^\mu{}_\nu \epsilon_j^\nu + i\alpha_j k^\mu . \tag{8}$$

We have to choose α_j such that $\mathbf{k}' \cdot \tilde{\boldsymbol{\epsilon}}_j = 0$.

We can again extract the annihilation operator of this field, using the time-independent inner product

$$\hat{a}_j'(k') = i \int d^3x' f^*_{k',j,\mu}(x') \overleftrightarrow{\partial}_{0'} \Lambda^\mu{}_\nu A^\nu(x'), \tag{9}$$

with $f_{k,j,\mu}(x) = \epsilon_{j,\mu}(k)[(2\pi)^3 k_0]^{\frac{1}{2}} e^{ikx}$. This leads to the following Bogoliubov transformation for polarized light:

$$\hat{a}_j'(\Lambda k) = \sum_l \epsilon^*_{j,\mu} \cdot \tilde{\epsilon}_l^\mu \hat{a}_l(k) e^{-i\Lambda k\ell} . \tag{10}$$

We will now evaluate $\tilde{\epsilon}_j^\mu$. A Lorentz transformation Λ can be written as a combination of a pure boost L and two spatial rotations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\Lambda = \mathcal{R}_2 L \mathcal{R}_1$.

Note that $\mathcal{R}_1$ and $\mathcal{R}_2$ are 4×4 matrices of the form $1 \oplus R$, with R a 3×3 rotation matrix and 1 the one-dimensional unit matrix. Using Eq. (8) with $\Lambda = L$, we find after some algebra that the polarization of the electromagnetic field in the Coulomb gauge is *not affected by pure Lorentz boosts*. Consequently, with $\mathcal{R} \equiv \mathcal{R}_2\mathcal{R}_1$ we can write $\tilde{\epsilon}_j^\mu = \mathcal{R}^\mu_\nu \epsilon_j^\nu + i\alpha_j k^\mu$, or $\tilde{\boldsymbol{\epsilon}}_j = R\boldsymbol{\epsilon}_j^\nu + i\alpha_j \mathbf{k}$. In addition, pure space rotations leave the field in both the Coulomb and the Lorentz gauge (i.e. $\alpha_j = 0$). The transformation law for the annihilation operator or the electromagnetic field then becomes

$$\begin{aligned} \hat{a}'_j(\Lambda k) &= \sum_l \epsilon_j \cdot \mathcal{R}\epsilon_l \hat{a}_l(k) e^{-i\Lambda k \ell} \\ &\equiv \sum_l U_{jl} \hat{a}_l(k) e^{-i\Lambda k\ell}, \end{aligned} \tag{11}$$

with U_{jl} the 2×2 unitary matrix associated with the overall spatial rotation R.

3.4. *Spinor fields*

The spinor field $\psi(x)$ obeys the Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$, where m is the mass of the fermion. We use the gamma matrices γ^μ in the standard representation such that $\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$. The plane-wave solutions to the Dirac equation can then be written as

$$\psi(x) = \int \frac{d\mathbf{k}}{2k_0} \sum_{j=1,2} \frac{u_j(k)\hat{b}_j(k)e^{ikx}}{\sqrt{(2\pi)^3 2k_0}} + \frac{v_j(k)\hat{d}_j^\dagger(k)e^{-ikx}}{\sqrt{(2\pi)^3 2k_0}}. \tag{12}$$

Here, $\hat{b}_j$ and $\hat{d}_j^\dagger$ are the annihilation and creation operator of the fermion and the anti-fermion in spin state $j \in \{1, 2\}$, respectively. The spinors $u_j(k)$ and $v_j(k)$ are four-dimensional vectors with $\bar{u}_j \equiv (\gamma^0 u_j)^\dagger = u_j^\dagger \gamma^0$. The annihilation operator $\hat{b}_j(k)$ is extracted using the following time-independent inner product:

$$\hat{b}_j(k) = i \int d^3x \frac{\bar{u}_j(k)e^{-ikx}}{\sqrt{(2\pi)^3 2k_0}} \overleftrightarrow{\partial}_0 \psi(x). \tag{13}$$

Again, we write the spinor field in terms of Bob's coordinates and substitute this into the time-independent inner product. The transformed annihilation operator thus becomes

$$\begin{aligned} \hat{b}'_j(\Lambda k) &\propto \sum_l \bar{u}_j(\Lambda k) u_l(k) \hat{b}_l(k) e^{-i\Lambda k\ell} \\ &\equiv \sum_l D_{jl} \hat{b}_l(k) e^{-i\Lambda k\ell}, \end{aligned} \tag{14}$$

where D_{jl} is a 2×2 unitary matrix up to a non-unit determinant. This apparent loss of unitarity ($\det D \neq 1$) is again due to the fact that the spatial integral in Eq. (13) is not Lorentz invariant (three-volumes are not preserved), and we have to normalize D such that the determinant becomes one. Using $\det D = \frac{1}{2}[1 - k_\nu \Lambda^\nu{}_\mu k^\mu]$, the Bogoliubov transformation for the spinor annihilation operator becomes

$$\hat{b}'_j(\Lambda k) = \sum_l \frac{2\bar{u}_j(\Lambda k) u_l(k)}{1 - k_\nu \Lambda^\nu{}_\mu k^\mu} \hat{b}_l(k) e^{-i\Lambda k \ell}. \tag{15}$$

The Bogoliubov transformation for $\hat{d}'_j(k')$ can be derived along the same lines. Furthermore, it is easily verified that the fermionic anti-commutation relations are unaltered by Lorentz transformations.

The spinors obey the orthogonality relations $\bar{u}_j(k) u_l(k) = -\bar{v}_j(k) v_l(k) = \delta_{jl}$, and

$$\begin{aligned} u_1(k) &= \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \dfrac{k_z}{E+m} & \dfrac{k_x + ik_y}{E+m} \end{pmatrix}^T, \\ u_2(k) &= \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 0 & 1 & \dfrac{k_x - ik_y}{E+m} & \dfrac{-k_z}{E+m} \end{pmatrix}^T. \end{aligned} \tag{16}$$

Here, $E = \sqrt{\mathbf{k}^2 + m^2}$. In this representation it is straightforward to calculate the matrix elements $\bar{u}_j(\Lambda k) u_l(k)$.

4. Bosonic and Fermionic Multi-Particle Interferometry

As was argued in the introduction, two identical single-particle plane waves incident on a beam splitter will result in bunching (bosons) or anti-bunching (fermions) in particle detectors in the outgoing modes of the beam splitter. In the boosted reference frame, these waves will in general no longer be identical, and as a result one would expect deviations in the statistics of the detectors. The resolution of this paradox, both for photons and electrons, will naturally lead to a unified description of multi-particle boson and fermion interferometry.

4.1. *Twin-photon paradox*

First, consider the twin-photon paradox. The transformation law in Eq. (11) indicates that there is no polarization rotation associated with pure boosts. Therefore, let the boosted annihilation operator be given by $\hat{a}'(\Lambda k) = \hat{a}(k)$, where we have chosen $\ell = 0$. The boosted single-photon waves can then be written as $\hat{a}^\dagger(\Lambda k_1)\hat{a}^\dagger(\Lambda k_2)|0\rangle$. It is clear that the frequency components of Λk_1 and Λk_2 will generally be different. However, in the boosted frame, the beam splitter action will also change: we can write the interaction Hamiltonian of the beam splitter at rest with incoming modes k_1 and k_2 as the bilinear form

$$H = \frac{\pi i}{2} \left[\hat{a}^\dagger(k_1)\hat{a}(k_2) - \hat{a}(k_1)\hat{a}^\dagger(k_2) \right]. \tag{17}$$

In the boosted frame, this Hamiltonian will become

$$H' = \frac{\pi i}{2} [\hat{a}'^\dagger(\Lambda k_1)\hat{a}'(\Lambda k_2) - \hat{a}'(\Lambda k_1)\hat{a}'^\dagger(\Lambda k_2)].$$

The linear Bogoliubov transformation corresponding to the beam splitter action in the boosted frame is then given by $\exp(iH')\hat{a}'(\Lambda k_i)\exp(-iH')$. Using the Baker–Campbell–Hausdorff relation

$$e^{\lambda A}Be^{-\lambda A} = B + \lambda[A,B] + \frac{\lambda^2}{2}[A,[A,B]] + \cdots,$$

we find that the boosted beam splitter transformation becomes

$$\begin{aligned}\hat{a}(\Lambda k_1) &\to \frac{1}{\sqrt{2}}(\hat{a}'(\Lambda k_1) - \hat{a}'(\Lambda k_2)),\\ \hat{a}(\Lambda k_2) &\to \frac{1}{\sqrt{2}}(\hat{a}'(\Lambda k_1) + \hat{a}'(\Lambda k_2)).\end{aligned} \tag{18}$$

In other words, the boosted beam splitter induces an interaction between different frequencies of the incoming field. The beam splitter thus defines a preferred frame of reference in which identical photons exhibit the Hong-Ou–Mandel effect. This resolves the twin-photon paradox.

4.2. *Twin-electron paradox*

The fermionic beam splitter is in many ways the dual of the bosonic case: when two identical fermions enter the two input ports of a fermionic beam splitter, the exclusion principle dictates that they will never leave the same output port. In the boosted frame, however, both the wavelength and the spins of the fermions change. Naively, this again results in a way of distinguishing the particles, with a change in detector statistics as a result.

The resolution of this paradox is similar to that of the twin-photon paradox: we need to transform the interaction Hamiltonian of the fermionic beam splitter. In addition, we need to show that the spin rotation does not actually render the particles distinguishable. Due to the fermionic anti-commutation relation $\{\hat{b}_j(k_1), \hat{b}_l(k_2)\} = 0$, a 50:50 electron beam splitter transforms the input state $\hat{b}_j^\dagger(k_1)\hat{b}_j^\dagger(k_2)|0\rangle$ into itself. In the boosted frame, the transformation becomes

$$\begin{aligned}\hat{b}'^\dagger_j(\Lambda k_1)\hat{b}'^\dagger_j(\Lambda k_2) \to \sum_l D_{jl}^2 \hat{b}'^\dagger_l(\Lambda k_1)\hat{b}'^\dagger_l(\Lambda k_2) + \frac{1}{2}\sum_{l\neq m} D_{jl}D_{jm}\hat{b}'^\dagger_l(\Lambda k_1)\hat{b}'^\dagger_m(\Lambda k_1)\\ - \frac{1}{2}\sum_{l\neq m} D_{jl}D_{jm}\hat{b}'^\dagger_l(\Lambda k_2)\hat{b}'^\dagger_m(\Lambda k_2).\end{aligned}$$

The fermionic anti-commutation relations render the last two sums zero. In the boosted frame, the state is then

$$|\psi\rangle = \sum_l D_{jl}^2 \hat{b}'^\dagger_l(\Lambda k_1)\hat{b}'^\dagger_l(\Lambda k_2)|0\rangle. \tag{19}$$

In other words, the fermions always occupy different spatial modes, and they have the same (boosted) spin. Note that it is not the details of D that resolve the paradox, but the canonical commutation relations. This mechanism is indicative of other massive quantum fields as well: the paradox for massive bosons is resolved by invoking the bosonic commutation relations.

4.3. *Non-inertial movement and quantum interferometry*

It should be stressed that the resolution of the twin-particle paradox, as sketched above, is valid only for inertial observers. In other words, when Alice and Bob are in non-inertial motion with respect to each other, they generally cannot agree upon a shared vacuum state. As a consequence, Alice and Bob no longer agree on the number of particles that are involved in the experiment. In physical terms, when Alice prepares the experiment to demonstrate the Hong-Ou–Mandel effect, Bob will see a beam splitter that emits thermal radiation. The energy of this radiation will be supplied by the mechanism that drives the beam splitter away from inertial motion (according to Bob).

4.3.1. *Bosons*

Nevertheless, the transformation properties of the field operators allow a Hamiltonian formulation of multi-particle quantum interferometry for non-inertial observers. We will sketch a proof that the Hamiltonian of a linear interferometer (including squeezing) is properly transformed by substituting the Bogoliubov transformation of the creation and annihilation operators. We use the fact that the Hamiltonian of a free (scalar) field ϕ can be written as

$$H = \int d^3x T_{00} = \frac{1}{2}(i\partial_0\phi, \phi),$$

with T_{00} the Hamiltonian density component of the stress tensor T, and

$$\phi = \int \frac{d\mathbf{k}}{2k_0}(f_k\hat{a}_k + f_k^*\hat{a}_k^\dagger) = \int \frac{d\mathbf{k}}{2k_0}(g_k\hat{b}_k + g_k^*\hat{b}_k^\dagger).$$

Here, f_k and g_k are orthonormal mode functions corresponding to ladder operators $\hat{a}_k \equiv \hat{a}(k)$ and $\hat{b}_k \equiv \hat{b}(k)$, respectively, and $(f_j, f_k) = -(f_j^*, f_k^*) = \delta_{jk}$ and $(f_j^*, f_k) = 0$. Furthermore, we can write the annihilation operator in terms of the inner products of the mode functions and the transformed creation and annihilation operators:

$$\hat{a}_k = \int \frac{d\mathbf{k}}{2k_0}[(g_j, f_k)\hat{b}_j - (f_k^*, g_j)\hat{b}_j^\dagger]. \tag{20}$$

Substituting this transformation rule into $H(\hat{a}, \hat{a}^\dagger)$ and using the completeness relation

$$(\phi_1, \phi_2) = \int dk[(\phi_1, f_k)(f_k, \phi_2) - (\phi_1, f_k^*)(f_k^*, \phi_2)]$$

yields the transformed Hamiltonian $H(\hat{b}, \hat{b}^\dagger)$.

In order to find the relativistic extension to linear multi-particle quantum interferometry, we generalize this result to any N-mode bilinear form of the interferometer's interaction Hamiltonian

$$H = \frac{1}{2} \sum_{j,k=1}^{N} (\hat{a}_j^\dagger A_{jk} \hat{a}_k^\dagger + 2\hat{a}_j^\dagger B_{jk} \hat{a}_k + \hat{a}_j A_{jk} \hat{a}_k), \tag{21}$$

where A and B are symmetric $N \times N$ Hermitian matrices. They give a complete description of N-mode multi-particle interferometry, including multi-mode squeezing (particle creation). Such Hamiltonians yield linear Bogoliubov transformations of the annihilation operators. It follows from Eq. (3) that any *arbitrary coordinate transformation* of the quantum fields (including non-inertial transformations) also gives a Bogoliubov transformation of the annihilation and creation operators, and as a result preserves the structure of this Hamiltonian. So far, we have derived the relativistic generalization of multi-particle interferometry for scalar fields. Optical interferometry requires that we take into account the extra gauge freedom of the electromagnetic field. General coordinate transformations will induce a non-trivial polarization rotation. It is easy to see, however, that the resulting Bogoliubov transformations allow for a similar relativistic extension.

There is, however, one subtlety: when non-inertial observers are involved, the parties in question typically do not have access to all the modes in the Hamiltonian. Some modes will be causally separated by (effective) event horizons. Consequently, the field amplitudes of these modes have to be traced out, and the observers will find that the output of the interferometer is in a mixed state.

4.3.2. *Fermions*

We can generalize the *fermionic* beam splitter to arbitrary N-mode fermionic interferometers in a similar way. To include the spin-$\frac{1}{2}$ degree of freedom, we assume M spatial modes, such that $N = 2M$. Furthermore, each mode can be occupied by both electrons and anti-electrons (denoted by b and d, respectively). From the expansion of the spinor field in Eq. (12), we see that general Bogoliubov transformations have the form

$$\begin{aligned} \hat{b}_j &\to \sum_k U_{jk} \hat{b}_k + V_{jk} \hat{d}_k^\dagger, \\ \hat{d}_j^\dagger &\to \sum_k W_{jk} \hat{d}_k^\dagger + Z_{jk} \hat{b}_k, \end{aligned} \tag{22}$$

with U, V, W, and Z determined by the coordinate transformation. The bilinear interaction Hamiltonian of the free spinor field must be closed under these transformations. To this end, we define the $2N$-tuple of creation and annihilation operators $\vec{s} \equiv (\hat{b}_1, \ldots, \hat{b}_N, \hat{d}_1^\dagger, \ldots, \hat{d}_N^\dagger)$. The Hamiltonian for multi-particle fermion interferometry is then given by

$$H = \frac{1}{2} \sum_{j,k=1}^{2N} (\hat{s}_j^\dagger \mathcal{A}_{jk} \hat{s}_k^\dagger + 2\hat{s}_j^\dagger \mathcal{B}_{jk} \hat{s}_k + \hat{s}_j \mathcal{A}_{jk} \hat{s}_k), \tag{23}$$

with

$$\mathcal{A} \equiv \begin{pmatrix} A_1 & C \\ -C & A_2 \end{pmatrix} \quad \text{and} \quad \mathcal{B} \equiv \begin{pmatrix} B_1 & D \\ -D & B_2 \end{pmatrix}, \tag{24}$$

where $A_{1,2}$, $B_{1,2}$, C and D are anti-symmetric $N \times N$ Hermitian matrices. This constitutes a unified theory of relativistic multi-particle fermionic interferometry.

5. Conclusions

In conclusion, we have derived the explicit form of the annihilation operator of some common quantum fields under relativistic transformations. Our results are completely general and offer a straightforward and fundamental way to calculate relativistic effects in bosonic and fermionic interferometers, thus establishing a unified theory of relativistic multi-particle quantum interferometry and linear optical quantum computing. With this theory, we can describe the behavior of multi-particle wave packets rather than momentum eigenstates in a direct manner. Most notably, our technique can be applied to arbitrary coordinate transformations, and can be used to study entanglement and relativistic quantum information theory in arbitrary coordinate systems.

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