Destruction of photocount oscillations by thermal noise

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We show that oscillations in photocount statistics will be destroyed by a sufficient admixture of thermal noise. In particular, we derive a "diffusion" equation evolving in temperature (rather than time) that describes the response of the photocount distribution to the admixture of such noise. We also derive an analytic condition for the temperature to guarantee the existence of these oscillations.

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I. INTRODUCTION

One of the most interesting properties of squeezed states is the oscillation in their number distribution [1,2]. Interference in phase space between a narrow annular band—the number state—and a Gaussian Wigner function—a squeezed state—is responsible for this anomalous nonclassical behavior of squeezed states [1]. The regions of overlap between these two Wigner functions identify two well defined zones in phase space that contribute to the probability amplitude. The total probability amplitude is then equal to the sum of these contributions weighted with an appropriate phase. The presence of noise [2–6] or losses [7] can diminish the squeezing and hence destroy the oscillations. One form of noise in particular has been extensively studied: thermal noise [2–6,8].

In this paper we will give a general formalism that describes the destruction of these photocount oscillations, for an arbitrary state, due to the admixture of thermal noise. By "admixture of thermal noise" we shall mean specifically the semigroup mapping of a fiducial state $\hat{\rho}_0$ into

$$\hat{\rho}(\bar{n}) = \int \frac{d^2\beta}{\pi\bar{n}} \exp(-|\beta|^2/\bar{n}) \hat{D}(\beta)\hat{\rho}_0\hat{D}^\dagger(\beta) , \qquad (1.1)$$

where $\hat{D}(\beta)$ is the standard displacement operator and $\bar{n}$ is the mean number of thermal photons "added." This formula is a direct generalization of those given by Refs. [2,9,10]. The physical meaning of this mapping is determined in Sec. IV. We will see that for this type of thermalization we are able to obtain a simple intuitive picture of how the photocount oscillations are destroyed. Further, we obtain not only a simple picture but also a powerful calculational tool to study the destruction of these oscillations.

We begin in Sec. II by recalling the oscillatory behavior of the photocount distribution of a displaced squeezed state, first noted by Schleich and Wheeler [1], and then introduce, in Sec. III, thermal squeezed states [8], which may be considered as the admixture of thermal noise to a squeezed state. Our more general approach to this question of the effect of thermal noise on the photocount distribution is studied in Sec. IV, where we derive a general "diffusion" equation that evolves in temperature (rather than time) and applies to an arbitrary initial state—not just squeezed states. We see that as the temperature of the thermal noise increases the photocount distribution becomes flatter, thus destroying the oscillations. In Sec. V we extend the interference-in-phase-space results to finite temperature (in the sense of admixture with thermal noise) to determine the range of temperatures in which the oscillations of the number distribution persist. The result shows that in the case of finite temperature, the two contributing amplitudes differ from each other, unlike the case at zero temperature in which the two amplitudes are identical, leading to dephasing.

II. SQUEEZED STATES

Squeezed states of the electromagnetic radiation [11] are important from both theoretical and experimental points of view. Because, among other things, the reduction of fluctuations in one quadrature may be used to produce a better signal-to-noise ratio in interferometric measurements [12,13]. Many different but equivalent definitions of squeezed states exist. In this paper we will describe pure squeezed displaced states by the density operator

$$\hat{\rho}_0 = |\alpha,\xi\rangle\langle\alpha,\xi| , \qquad (2.1)$$

where

$$|\alpha,\xi\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle \qquad (2.2)$$

is the squeezed displaced state obtained by displacing the vacuum state followed by the squeezing operation. Here

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \qquad (2.3)$$

is the usual displacement operator and

$$\hat{S}(\xi) = \exp[\tfrac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})] \qquad (2.4)$$

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is the squeezing operator, with the squeezing parameter $\xi = re^{i\phi}$ (throughout this paper we will take $\phi = 0$) [14]. The squeezing operator defines a Bogoliubov transformation of the annihilation and creation operators via

$$\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \hat{a}\cosh r - \hat{a}^\dagger \sinh r,$$
$$\hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r) = \hat{a}^\dagger\cosh r - \hat{a}\sinh r . \quad (2.5)$$

Choosing units where $\hbar = 1$ and the quadrature operators $\hat{x}$ and $\hat{p}$ are dimensionless, we have $(\Delta x)^2 = e^{-2r}/2$ and $(\Delta p)^2 = e^{2r}/2$ for the squeezed displaced states of Eq. (2.2). Since $(\Delta x)(\Delta p) = \frac{1}{2}$ they are minimum uncertainty states. The mean number of photons in these squeezed displaced states of Eq. (2.2) is

$$\langle \hat{n} \rangle = |\alpha|^2 e^{-2r} + \sinh^2 r \quad (2.6)$$

and the photon number (photocount) distribution (at zero temperature)

$$P_m(0) = |\langle m|\alpha,\xi\rangle|^2 \quad (2.7)$$

has been calculated in Ref. [15]. Using our notation in terms of r ($\phi = 0$) and real α, the result is

$$P_m(0) = \frac{\tanh^m r}{2^m m! \cosh r} \exp\left(\frac{-\alpha^2 e^{-r}}{\cosh r}\right) H_m^2\left(\frac{\alpha}{\sqrt{\sinh 2r}}\right) . \quad (2.8)$$

Here H_m denotes the Hermite polynomial of order m. In particular, if we take $r = 0$, then Eq. (2.8) reduces to the familiar Poisson distribution for the photocount statistics associated with a coherent state. By contrast, for the squeezed vacuum case with $\alpha = 0$ we have $P_{2m+1}(0) = 0$ and $P_{2m}(0) \neq 0$, which is the case where the photocount distribution has the tightest possible oscillations. We plot the function $P_m(0)$ [Eq. (2.8)] in Fig. 1 for values of $\alpha = 7\sqrt{21}$ and $r = \ln(21)/2 \simeq 1.52$ and find that this function oscillates with respect to m. These numbers were chosen to match the parameters defined by Schleich *et al.* [1,14,16,17]. In the limit of strong squeezing the photon number distribution $P_m(0)$ shows rapid oscillations. Mathematically, the Hermite polynomials are the origin of the oscillations. Moreover, Schleich and Wheeler [1] show that these oscillations result from interference in phase space. In addition, they gave a condition relating the squeezing and coherent parameters, which ensure the existence of oscillation. The condition in our notation is [16]

$$e^{3r} \gtrsim \frac{9\pi\sqrt{2}}{4}\alpha e^{-r} ; \quad (2.9)$$

for $\alpha = 7\sqrt{21}$ this gives $r \gtrsim 1.44$ [if one scrutinizes histograms of $P_m(0)$, one sees that this condition is conservative and that oscillations still occur for somewhat lower r]. We now study the effect of thermal noise on these oscillations. It is our intuitive expectation that as we raise the temperature the oscillations will disappear. Motivated by this consideration, we study the influence of thermal noise on the photocount distribution. We start, however, with a discussion on thermal squeezed states.

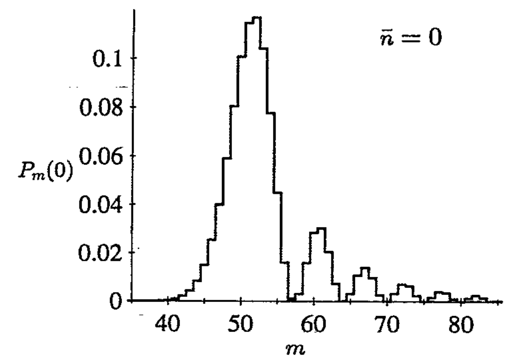

FIG. 1. Plot of the Schleich-Wheeler oscillations for a pure squeezed displaced state (i.e., zero temperature): Histogram of the photocount distribution $P_m(\bar{n} = 0)$ of Eq. (2.8) for a squeezed displaced vacuum with $\alpha = 7\sqrt{21}$ and $r = \ln(21)/2$.

III. THERMAL SQUEEZED STATES

In what way does thermal noise affect a quantum state? Naturally the answer will depend on how this noise is introduced. For instance, we might combine a thermal state of light and our quantum state at the two input ports of a beam splitter; by choosing the temperature of the thermal state and the parameters of the beam splitter we can produce varying amounts of thermalization. Alternately, we might use the fact that any individual mode in nondegenerate parametric down-conversion looks thermal if the input to that mode is vacuum. This produces thermalized vacuum. We could consider generalizing this by using some other quantum state as the input to this mode in which case the output would, in some sense, be the thermalized version of the input state.

If the quantum state is initially a squeezed state, then essentially all methods of thermalization lead to Gaussian Wigner functions and so can all be mapped into each other under suitable choice of the parameters involved. Rather than investigating this mapping here we will use one of several mathematical definitions of a thermal squeezed state (TSS) [2,18,19], which can itself be mapped into one of several laboratory procedures. We take the definition of Vourdas and Weiner [2] of a TSS as one having a density matrix of the form

$$\hat{\rho} = \int \frac{d^2\beta}{\pi\bar{n}} \exp(-|\beta|^2/\bar{n})\, \hat{D}(\beta)|\alpha,r\rangle\langle\alpha,r|\hat{D}^\dagger(\beta) , \quad (3.1)$$

where $\bar{n}$ is the mean number of thermal photons

$$\bar{n} = \frac{1}{\exp(\beta\omega) - 1} \quad (3.2)$$

at frequency ω (recall $\hbar = 1$) and inverse temperature $\beta = 1/k_B T$ (with k_B the Boltzmann constant). Intuitively, this formula smears out the state (in this case a squeezed displaced state) according to the spread of the Glauber-Sudarshan P function for the thermal state $\exp(-|\beta|^2/\bar{n})/\pi\bar{n}$.

The representation of thermalization by this formula was first suggested by Glauber [9] when he used it to describe thermalized vacuum (ordinary thermal states) and later by Lachs [10] as a way of describing a thermalized coherent state. Indeed, if we take the particular case in which $r = 0$ and $\alpha = 0$, then Eq. (3.1) is nothing but the density matrix of the thermal state

$$\int \frac{d^2\beta}{\pi\bar{n}} e^{-|\beta|^2/\bar{n}} \hat{D}(\beta)|0\rangle\langle 0|\hat{D}^\dagger(\beta) = \frac{1}{1+\bar{n}} \sum_{m=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^m |m\rangle\langle m| \,. \quad (3.3)$$

Another variation that may be used to make a thermal squeezed state is to generate a displaced squeezed thermal state. Such a state has been studied in detail by Marian and Marian [3]. Because we wish to use the photocount statistics calculated by these authors we need to give the mapping between a displaced squeezed thermal state and the TSS of Eq. (3.1); this is done in Appendix A in terms of the characteristic functions

$$\chi_{\rm TSS} = \langle \exp(\lambda \hat{a}^\dagger - \lambda^* \hat{a}) \rangle \quad (3.4)$$

for these states.

We shall use the calculations of Marian and Marian as a test-bed of our diffusion equation approach of Sec. IV. Marian and Marian were able to calculate the photocount distribution for an arbitrary Gaussian Wigner function. By contrast, we are interested in a much more general problem: that of studying the thermalization of a generic state via the semi-group mapping of Eq. (1.1). We do this in Sec. IV.

In Fig. 2 we plot $P_m(\bar{n})$ using the calculations performed by Marian and Marian [their Eq. (5.2)] and our translation in Appendix A for (a) $\bar{n} = 0.01$, (b) $\bar{n} = 0.03$, and (c) $\bar{n} = 0.06$ for the case already studied in Fig. 1 [i.e., $\alpha = 7\sqrt{21}$ and $r = \ln(21)/2$]. We see that for sufficiently small noise the distribution still has oscillations. However, as we raise the temperature the distribution goes flat. Therefore, the photon number distribution of TSS is very sensitive to the thermal noise. One of our aims is to find an analytic criterion on $\bar{n}$ that ensures the existence of oscillations.

In Sec. V we will return to the TSS and explain the loss of oscillations in terms of the interference-in-phase-space picture. To do that we shall need the Wigner function corresponding to the density operator Eq. (3.1). This is easily calculated by using the properties of Gaussian convolutions to be

$$W_{\rm TSS}(\bar{n}) = \frac{1}{\pi\sqrt{(2\bar{n}+e^{2r})(2\bar{n}+e^{-2r})}} \times \exp\left(-\frac{(x-\sqrt{2}\alpha e^{-r})^2}{2\bar{n}+e^{-2r}} - \frac{p^2}{2\bar{n}+e^{2r}}\right) . \quad (3.5)$$

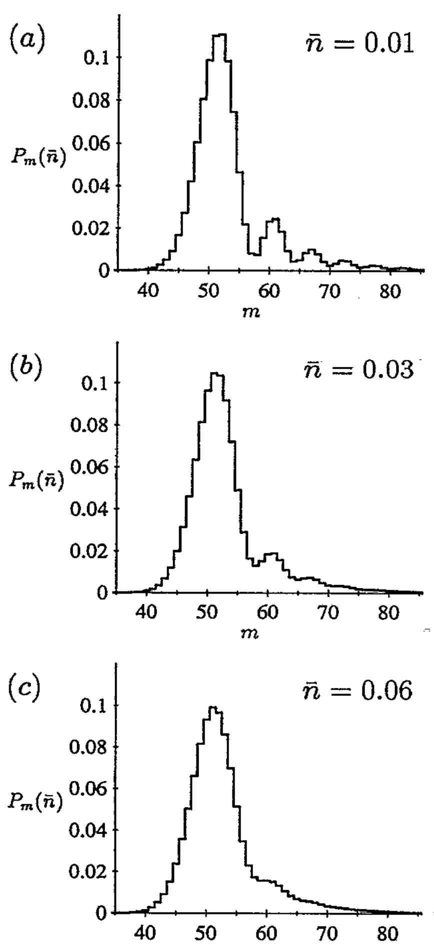

FIG. 2. Histograms of the photocount distribution $P_m(\bar{n})$ at finite temperature. These plots show the Schleich-Wheeler oscillations for the parameters used in Fig. 1 [i.e., $\alpha = 7\sqrt{21}$ and $r = \ln(21)/2$ for (a) $\bar{n} = 0.01$, (b) $\bar{n} = 0.03$, and (c) $\bar{n} = 0.06$. (These plots have been calculated using the analytic results of Ref. [3] [their Eq. (5.2)] and our Appendix A.) Note that our condition (5.9) predicts in this case that oscillations are guaranteed for $\bar{n} \lesssim 0.006$.

IV. THE DIFFUSION EQUATION

In this section we investigate state "thermalization" as described by the formula of Eq. (1.1); as pointed out in the Introduction, this represents a generalization of the forms suggested by Refs. [2,9,10]. We will derive a physical model that yields this semigroup evolution on the state and could be used in the laboratory as a way of thermalizing states according to this specific formula. In addition, we will obtain a diffusion-like equation describing the "evolution" of the photocount distribution as the mean number $\bar{n}$ of thermal photons increases.

To this end we start by expanding $\hat{\rho}(\bar{n}+\delta\bar{n})$ to lowest order in $\delta\bar{n}$. Changing variables so that $\beta \to \sqrt{\bar{n}+\delta\bar{n}}\beta$ and expanding the square root to first order we get

$$\hat{\rho}(\bar{n}+\delta\bar{n}) \simeq \int \frac{d^2\beta}{\pi} e^{-|\beta|^2} \hat{D}\left(\frac{1}{2}\frac{\delta\bar{n}}{\sqrt{\bar{n}}}\beta\right) \hat{D}(\sqrt{\bar{n}}\beta)\hat{\rho}_0 \times \hat{D}^\dagger(\sqrt{\bar{n}}\beta)\hat{D}^\dagger\left(\frac{1}{2}\frac{\delta\bar{n}}{\sqrt{\bar{n}}}\beta\right) \quad (4.1)$$

(here we used the fact that the extraneous phase factor obtained by writing a displacement operator of a sum as a product of displacement operators cancels itself in this symmetric form). Note that, although this equation appears to be ill defined for $\bar{n} \to 0$, we will see shortly that this limit gives a finite result. Next we expand the displacement operators to first order in $\delta\bar{n}$ to obtain

$$\hat{\rho}(\bar{n}+\delta\bar{n}) \simeq \hat{\rho}(\bar{n}) + \frac{1}{2}\frac{\delta\bar{n}}{\sqrt{\bar{n}}}\int\frac{d^2\beta}{\pi}e^{-|\beta|^2} \times\left[\beta\hat{a}^\dagger - \beta^*\hat{a}, \hat{D}(\sqrt{\bar{n}}\beta)\hat{\rho}_0\hat{D}^\dagger(\sqrt{\bar{n}}\beta)\right] . \quad (4.2)$$

Now noting the identities

$$\beta e^{-|\beta|^2} = -\frac{\partial}{\partial\beta^*}e^{-|\beta|^2},$$
$$\frac{\partial}{\partial\beta^*}\hat{D}(\beta)\hat{\rho}_0\hat{D}^\dagger(\beta) = -\left[\hat{a}, \hat{D}(\beta)\hat{\rho}_0\hat{D}^\dagger(\beta)\right] \quad (4.3)$$

and using them to integrate Eq. (4.2) by parts, we obtain

$$\hat{\rho}(\bar{n}+\delta\bar{n}) \simeq \hat{\rho}(\bar{n}) - \frac{\delta\bar{n}}{2}[\hat{a}^\dagger, [\hat{a}, \hat{\rho}(\bar{n})]] - \frac{\delta\bar{n}}{2}[\hat{a}, [\hat{a}^\dagger, \hat{\rho}(\bar{n})]] . \quad (4.4)$$

Writing this as a differential equation for $\hat{\rho} = \hat{\rho}(\bar{n})$ we find

$$\frac{d\hat{\rho}}{d\bar{n}} = \frac{1}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) +\frac{1}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) , \quad (4.5)$$

which may be compared with the standard single-photon master equation

$$\frac{d\hat{\rho}}{dt} = \frac{\gamma}{2}\bar{n}_R\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) +\frac{\gamma}{2}(\bar{n}_R+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) , \quad (4.6)$$

where γ is the energy damping constant and $\bar{n}_R$ is the mean number of thermal reservoir photons. We see that in the limit of high reservoir temperature $\bar{n}_R \gg \max(1,\bar{n})$, the replacement $dt\gamma\bar{n}_R \to d\bar{n}$ maps Eq. (4.6) onto Eq. (4.5). This mapping thus gives a laboratory implementation of the thermalization rule of Eq. (1.1) in terms of an implementation of the real master equation (4.6).

How could this master equation be implemented? We will give one concrete model. Consider a single beam splitter with evolution operator

$$\hat{U} = \exp[\theta(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})] ; \quad (4.7)$$

then $\hat{U}\hat{a}\hat{U}^\dagger = \hat{a}\cos\theta + \hat{b}\sin\theta$ so $\sin\theta$ is the amplitude reflection coefficient. Suppose we identify mode $\hat{a}$ with our state $\hat{\rho}$ and mode $\hat{b}$ with a thermal state $\hat{\rho}_T(\bar{n}_R)$ with a mean number $\bar{n}_R$ of thermal photons. Assuming that the reflection coefficient is small ($\theta \ll 1$), then expanding $\hat{U}\hat{\rho}\hat{\rho}_T(\bar{n}_R)\hat{U}^\dagger$ to second order in θ, and tracing out mode $\hat{b}$ yields exactly Eq. (4.6) if we take $\Delta t\gamma = \theta^2$. Thus, assuming $\bar{n}_R \gg \max(1,\bar{n})$, we retrieve Eq. (4.5) by taking $R\bar{n}_R = \Delta\bar{n}$, where $R \equiv \theta^2$ is exactly the energy reflection coefficient for the beam splitter and $\Delta\bar{n}$ is the number of additional thermal photons added in this step of thermalization. That is, we have described one step in iterating Eq. (4.5) by taking

$$R = \frac{\Delta\bar{n}}{\bar{n}_R} . \quad (4.8)$$

Iterating this procedure $\bar{n}/\Delta\bar{n}$ times yields the semigroup evolution of Eq. (1.1). This procedure is illustrated schematically in Fig. 3.

Having given Eq. (1.1) a simple form as a differential equation, we now derive the effect of thermalization on the photocount distribution. Taking the matrix element

$$P_m(\bar{n}) \equiv \langle m|\hat{\rho}(\bar{n})|m\rangle , \quad (4.9)$$

we find

$$\frac{dP_m(\bar{n})}{d\bar{n}} = (m+1)P_{m+1}(\bar{n}) -(2m+1)P_m(\bar{n}) + mP_{m-1}(\bar{n}) . \quad (4.10)$$

This differential equation for the photocount statistics may be interpreted as a discrete diffusion equation. Indeed, taking

$$P_m - P_{m-1} \simeq \frac{\partial P_m}{\partial m} \quad (4.11)$$

as an approximation to a continuous derivative, we find the diffusion equation

$$\frac{dP_m(\bar{n})}{d\bar{n}} \simeq \frac{\partial}{\partial m}\left(m\frac{\partial}{\partial m}P_m(\bar{n})\right) . \quad (4.12)$$

It is intuitively clear now that the thermalization de-

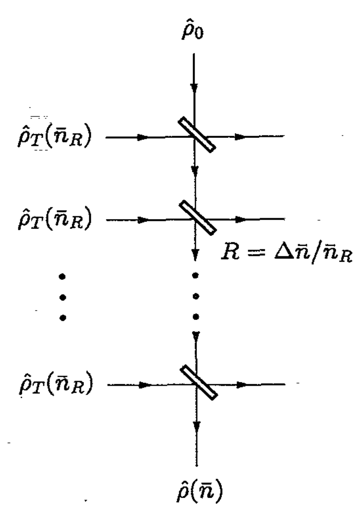

FIG. 3. Schematic implemetation of Eq. (1.1). A fiducial state $\hat{\rho}_0$ passes $\bar{n}/\Delta\bar{n}$ stages. Each stage consists of a single beam splitter with energy reflection coefficient $R = \Delta\bar{n}/\bar{n}_R$, which allows in some thermal fluctuations from a thermal state $\hat{\rho}_T(\bar{n}_R)$ having a mean number $\bar{n}_R$ of thermal photons.

scribed by Eq. (1.1) will destroy any oscillations in the photocount distribution if the diffusion process is carried along far enough, i.e., if the mean number of thermal photons $\bar{n}$ is taken high enough.

Finally, in Fig. 4 we show the calculations of the thermalization of the squeezed displaced state with $\alpha = 7\sqrt{21}$ and $r = \ln(21)/2$, but now based on iterating Eq. (4.10). We see that there is good agreement with the analytic expressions for $\bar{n} = 0.01$, but that a slight but noticable discrepancy appears for larger temperatures. It is believed that these discrepancies are due to small numerical inaccuracies in the iteration procedure.

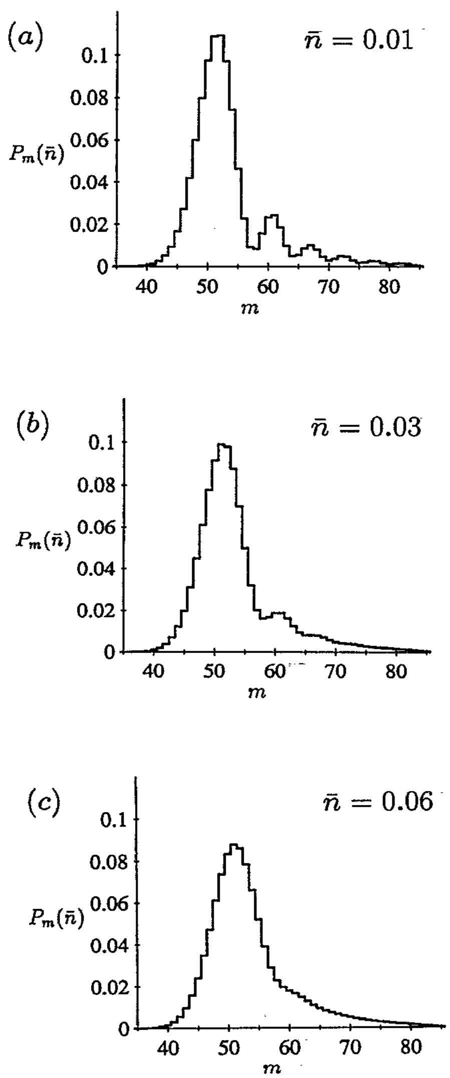

FIG. 4. Histograms of the photocount distribution $P_m(\bar{n})$ at finite temperature. These plots show the Schleich-Wheeler oscillations for the parameters used in Fig. 1 [i.e., $\alpha = 7\sqrt{21}$ and $r = \ln(21)/2$] for (a) $\bar{n} = 0.01$, (b) $\bar{n} = 0.03$, and (c) $\bar{n} = 0.06$. These plots, in contrast to Fig. 2, were calculated by iterating Eq. (4.10) of our paper. We see that the agreement with the analytic expressions is excellent for $\bar{n} = 0.01$ and that a slight but noticeable discrepancy appears as we continue to iterate this equation.

V. THERMAL DESTRUCTION OF INTERFERENCE IN PHASE SPACE

In this section we apply the principle of interference in phase space to explain the oscillations exhibited by the photon number distribution of a TSS. (We shall closely follow the derivation of Ref. [17] and so we only present a schematic of our calculation here.) In the phase space picture, the number state m can be represented as a circular Bohr-Sommerfeld band of area of 2π (in units of $\hbar$) with inner radius $\sqrt{2m}$ and outer radius $\sqrt{2(m+1)}$. The Bohr-Sommerfeld trajectory

$$\tfrac{1}{2}{p_m}^2 + \tfrac{1}{2}x^2 = m + \tfrac{1}{2} \tag{5.1}$$

runs in the middle of the band. This band intersects the thermal squeezed states representative—the Gaussian Wigner function—in two zones. The total amplitude is the sum, with due regard to phase, of the two amplitudes. In this approach the probability of finding m photons in a thermal squeezed state is exactly

$$P_m(\bar{n}) = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, W_m(x,p) W_{\rm TSS}(x,p;\bar{n}) \,, \tag{5.2}$$

where $W_{\rm TSS}$ denotes the Wigner function for a TSS and W_m represents the Wigner function for the harmonic oscillator in its mth state

$$W_m(x,p) = \frac{(-1)^m}{\pi} e^{-x^2-p^2} L_m(2x^2 + 2p^2) \,; \tag{5.3}$$

here L_m is the mth Laguerre polynomial. The probability $P_m(\bar{n})$ is thus given by the overlap in phase space between the distribution $W_{\rm TSS}$ and W_m. To find the area of overlap we consider the case of strong squeezing. In phase space, the Wigner function is represented by an ellipse with height $2\sqrt{\bar{n} + e^{2r}/2}$ and width $2\sqrt{\bar{n} + e^{-2r}/2}$ centered on the positive x axis at $x = \sqrt{2}\alpha e^{-r}$ (for real α). We decompose the phase space integration into two parts

$$P_m(\bar{n}) = 2P_m^{(1)}(\bar{n}) + P_m^{(2)}(\bar{n}) \,, \tag{5.4}$$

where these integrals are given by

$$P_m^{(1)}(\bar{n}) = 2\pi \int_{-\infty}^{\infty} dx \int_{\bar{p}_m(x)}^{\infty} dp\, W_m(x,p) W_{\rm TSS}(x,p;\bar{n}), \tag{5.5}$$

$$P_m^{(2)}(\bar{n}) = 2\pi \int_{-\infty}^{\infty} dx \int_{-\bar{p}_m(x)}^{\bar{p}_m(x)} dp\, W_m(x,p) W_{\rm TSS}(x,p;\bar{n}), \tag{5.6}$$

here $\bar{p}_m(x) = \sqrt{\frac{1}{2}\rho_m - x^2}$ and ρ_m denotes the largest zero of the mth Laguerre polynomials. The integration of the two contributing integrals is carried out in Appendix B. The result for real α and in the limit of strong squeezing is

$$P_m(\bar{n}) \simeq \frac{2}{\sqrt{2\pi(2\bar{n}+e^{2r})(m+\frac{1}{2}-\alpha^2e^{-2r})}} \times\left[\exp\left(-2\frac{m+\frac{1}{2}-\alpha^2e^{-2r}}{2\bar{n}+e^{2r}}\right) + \exp\left[-2(2\bar{n}+e^{-2r})(m+\tfrac{1}{2}-\alpha^2e^{-2r})\right] \times\cos(2\phi_m)\right], \quad (5.7)$$

where ϕ_m is defined in Eq. (B8) as

$$\phi_m = \int_{\sqrt{2}\alpha e^{-r}}^{\sqrt{2m+1}} \sqrt{2m+1-x^2}\,dx - \frac{\pi}{4} = (m+\tfrac{1}{2})\arctan\frac{\sqrt{m+\frac{1}{2}-\alpha^2e^{-2r}}}{\alpha e^{-r}} - \alpha e^{-r}\sqrt{m+\tfrac{1}{2}-\alpha^2e^{-2r}} - \frac{\pi}{4}. \quad (5.8)$$

We conclude with a discussion on the condition for the appearance of the oscillations (our discussion follows that of Ref. [16] closely). The second term in large square brackets in Eq. (5.7) will display oscillations as long as the decay length $1/(4\bar{n}+2e^{-2r})$ is large compared to the separation between the first maxima and the first minima. Taking $\alpha^{-1}e^r\sqrt{m+\frac{1}{2}-\alpha^2e^{-2r}} \ll 1$, we have [16] that the separation is $(9\pi\alpha e^{-r})^{2/3}/4$. That is, the condition to ensure that we see oscillations is

$$\bar{n} \lesssim \frac{1}{(9\pi\alpha e^{-r})^{2/3}} - \frac{e^{-2r}}{2}, \quad (5.9)$$

which is consistent with condition (2.9) when $\bar{n}=0$. For $\alpha = 7\sqrt{21}$ and $r = \ln 21/2$ this condition tells us that we are ensured of seeing oscillations if $\bar{n} \lesssim 0.006$. In fact, a look at Fig. 2 shows that this condition [like that of Eq. (2.9)] is rather conservative and that somewhat higher temperatures can be tolerated by the oscillations.

VI. CONCLUSIONS

We have studied one model for the addition of thermal noise to quantum states. For states with Gaussian Wigner functions we have argued that all such models will be equivalent up to a suitable mapping of their parameters, since for these states the thermalized Wigner function will still be a Gaussian. For our model of thermalization we have given several equivalent representations including a master equation representation, which could in principle be implemented in a laboratory. We found that this model leads to a very simple understanding of the loss of photocount oscillations in squeezed states of light under the addition of thermal noise. In particular, we have derived a diffusionlike equation that describes the "evolution" of the photocount statistics as the "temperature is raised" (in fact, this "diffusion" equation applies to the thermalization of any state). Finally, we have investigated the loss of these oscillations in terms of the interference-in-phase-space picture. Using this approach we have derived an analytic bound on the temperature which ensures that the oscillations will be visible.

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APPENDIX A

In this appendix we relate the TSS of Eq. (3.1) with that of $\hat{\rho}'_{\rm TSS}$ of Marian and Marian [3]:

$$\hat{\rho}'_{\rm TSS} = \hat{D}(\alpha')\hat{S}(-r')\hat{\rho}'_T\hat{S}^\dagger(-r')\hat{D}^\dagger(\alpha'), \quad (A1)$$

where $\hat{\rho}'_T$ is a thermal state with $\bar{n}'$ mean photon number. According to Ref. [3] the characteristic function of this state is

$$\chi_{\rm TSS} = \exp\left[-(\bar{n}'+\tfrac{1}{2})\cosh(2r')|\lambda|^2 + \tfrac{1}{2}(\bar{n}'+\tfrac{1}{2})\sinh(2r')(\lambda^2+\lambda^{*2}) + \alpha'^*\lambda - \alpha'\lambda^*\right]. \quad (A2)$$

Instead, from Eq. (3.1), using the identities

$$\hat{D}^\dagger(\alpha)\hat{D}(\beta)\hat{D}(\alpha) = \exp(\beta\alpha^*-\beta^*\alpha)\hat{D}(\beta)$$
$$\hat{S}^\dagger(r)\hat{D}(\alpha)\hat{S}(r) = \hat{D}(\alpha\cosh r+\alpha^*\sinh r), \quad (A3)$$

we may calculate the characteristic function Eq. (3.4) as

$$\chi_{\rm TSS} = \int\frac{d^2\beta}{\pi\bar{n}}\exp(-|\beta|^2/\bar{n}) \times\langle 0|\hat{D}^\dagger(\alpha)\hat{S}^\dagger(r)\hat{D}^\dagger(\beta)\hat{D}(\lambda)\hat{D}(\beta)\hat{S}(r)\hat{D}(\alpha)|0\rangle = \exp\left[-(\bar{n}+\tfrac{1}{2}\cosh 2r)|\lambda|^2 - \tfrac{1}{4}\sinh(2r)(\lambda^2+\lambda^{*2}) + \alpha_r^*\lambda - \alpha_r\lambda^*\right], \quad (A4)$$

where

$$\alpha_r \equiv \alpha\cosh r - \alpha^*\sinh r. \quad (A5)$$

Equating Eqs. (A2) and (A4) in powers of λ, we find the relationship between them as being

$$\sinh 2r = -2(\bar{n}'+\tfrac{1}{2})\sinh 2r',$$
$$\bar{n}+\tfrac{1}{2}\cosh 2r = (\bar{n}'+\tfrac{1}{2})\cosh 2r', \quad (A6)$$
$$\alpha' = \alpha_r = \alpha\cosh r - \alpha^*\sinh r.$$

We can invert these relationships to yield

$$\tanh 2r' = \frac{-\sinh 2r}{2(\bar{n} + \frac{1}{2}\cosh 2r)},$$
$$(\bar{n}' + \tfrac{1}{2})^2 = \frac{1}{4}(2\bar{n} + e^{2r})(2\bar{n} + e^{-2r}) \; ; \tag{A7}$$

these inverted relations make it clear that

$$\bar{n} = 0 \iff \bar{n}' = 0 \, . \tag{A8}$$

As an example, the set of parameters $\bar{n} = 0$, $\alpha = 7\sqrt{21}$, and $r = \ln(21)/2$ corresponds to $\bar{n}' = 0$, $\alpha' = 7$, and $r' = -\ln(21)/2$.

APPENDIX B

In this appendix we carry out the integrals in Eqs. (5.5) and (5.6). In the limit of strong squeezing and low temperature ($\bar{n} \lesssim e^{-2r}$) the narrow Gaussian in x behaves like a δ function

$$\exp\left(-\frac{(x - \sqrt{2}\alpha e^{-r})^2}{2\bar{n} + e^{-2r}}\right) \simeq \sqrt{\pi(2\bar{n} + e^{-2r})}\,\delta(x - \sqrt{2}\alpha e^{-r}), \tag{B1}$$

so integration over the x variable will leave Eq. (5.5) in the form

$$P_m^{(1)}(\bar{n}) = \frac{2(-1)^m}{\sqrt{\pi(2\bar{n} + e^{2r})}} \int_{\bar{p}_m(x)}^{\infty} dp\, e^{-x^2-p^2} \times L_m(2x^2 + 2p^2) \times \exp\left(-\frac{p^2}{2\bar{n} + e^{2r}}\right)\Bigg|_{x=\sqrt{2}\alpha e^{-r}} . \tag{B2}$$

Using the result [see Eq. (D3) of Ref. [17] for $m \to \infty$]

$$\int_{\bar{p}_m(x)}^{\infty} dp\, e^{-x^2-p^2} L_m(2x^2 + 2p^2) \simeq \frac{(-1)^m}{2\sqrt{2(m + \frac{1}{2} - \frac{1}{2}x^2)}} \tag{B3}$$

and evaluating the slowly varying exponential function $\exp[-p^2/(2\bar{n} + e^{2r})]$ at the Bohr-Sommerfeld trajectory $p_m^2 = 2(m + \frac{1}{2} - \alpha^2 e^{-2r})$ we obtain

$$P_m^{(1)}(\bar{n}) \simeq \frac{1}{\sqrt{2\pi(2\bar{n} + e^{2r})}} \frac{\exp\left(-2\frac{m + \frac{1}{2} - \alpha^2 e^{-2r}}{2\bar{n} + e^{2r}}\right)}{\sqrt{m + \frac{1}{2} - \alpha^2 e^{-2r}}} \tag{B4}$$

Next, we turn to the calculation of $P_m^{(2)}(\bar{n})$, i.e., Eq. (5.6). We perform the integral in p by taking $\exp[-p^2/(2\bar{n} + e^{2r})] \simeq 1$ and using [Eq. (D6) of Ref. [17]]

$$\int_{-\bar{p}_m(x)}^{\bar{p}_m(x)} dp\, e^{-x^2-p^2} L_m(2x^2 + 2p^2) \simeq \frac{(-1)^m \cos[2S_m(x) - \pi/2]}{\sqrt{2(m + \frac{1}{2} - \frac{1}{2}x^2)}}, \tag{B5}$$

where the phase

$$S_m(x) = \int_x^{\sqrt{2m+1}} du\, p_m(u) \, . \tag{B6}$$

Finally, neglecting the slow variation in $p_m(x)$ compared to $\cos[2S_m(x) - \pi/2]$ we have

$$P_m^{(2)}(\bar{n}) \simeq \frac{2}{\sqrt{2\pi(2\bar{n} + e^{2r})}} \times \frac{\exp[-2(2\bar{n} + e^{-2r})(m + \frac{1}{2} - \alpha^2 e^{-2r})]}{\sqrt{m + \frac{1}{2} - \alpha^2 e^{-2r}}} \times \cos(2\phi_m) \, , \tag{B7}$$

where

$$\phi_m = \int_{\sqrt{2}\alpha e^{-r}}^{\sqrt{2m+1}} du\, p_m(u) - \frac{\pi}{4} \, . \tag{B8}$$

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