Quantum coherence in the presence of unobservable quantities

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State representations summarize our knowledge about a system. When unobservable quantities are introduced the state representation is typically no longer unique. However, this nonuniqueness does not affect subsequent inferences based on any observable data. We demonstrate that the inference-free subspace may be extracted whenever the quantity's unobservability is guaranteed by a global conservation law. This result can generalize even without such a guarantee. In particular, we examine the coherent-state representation of a laser, where the absolute phase of the electromagnetic field is believed to be unobservable. We show that experimental coherent states may be separated from the inference-free subspaces induced by this unobservable phase. These physical states may then be approximated by coherent states in a relative-phase Hilbert space.

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I. INTRODUCTION

The representation of a state and its associated interpretation are fundamental issues in quantum mechanics. The state representation of a system summarizes our knowledge about that system; it summarizes the information about any observable we wish to measure. Conversely, experiments on a system allow us to determine the associated state representation.

Now it turns out that some observables are not accessible to experiment. Such quantities might be called *unobservables*, a term which we shall take to be a synonym for unmeasurable quantities. There are several mechanisms by which a quantity may be unobservable. One rigorous mechanism involves global conservation laws. In particular, the Wigner-Araki-Yanase (WAY) theorem says that any operator which does not commute with an operator corresponding to a global conservation law is not observable [4,5]. Therefore any system satisfying global conservation laws involves unobservable quantities.

In addition to those quantities whose unobservability is directly associated with global conservation laws, there are other quantities for which no rigorous argument currently exists guaranteeing their unobservability, but which are, nonetheless, generally accepted to be inaccessible to any experimental means. An example of such a quantity is the absolute phase of the electromagnetic field [1].

What are the consequences for the state representation of a system when unobservable quantities are involved? Since such a situation implies that there will be parts of the state's representation which cannot be examined this means that the state could equally well be represented in various functionally indistinguishable though distinct forms. Thus, an apparent consequence of dealing with unobservables is that the state representation is effectively nonunique.

In this paper we investigate the consequences of unobservable quantities on the state representation more fully. To do this, we consider the possibility that the nonuniqueness can be isolated in the state representation. More particularly, the state may be written in a form of a direct product of two states or components, where at least one of these components involves no nonuniqueness due to unobservability. This component involves a subspace which is spanned by only eigenstates of truly observable quantities. By contrast, the minimal nonunique component corresponds to that part of the state representation which cannot be determined by any experimental procedure; it corresponds to an inference-free subspace.

In Sec. II, we start by considering the case where the unobservability of quantities is guaranteed by global conservation laws and demonstrate that the inference-free subspace may be isolated and separated from the remainder of the state representation. A hint of how this may be achieved can be found in the WAY theorem itself, which also suggests that the unobservability of the noncommutative operators can be bypassed by taking into account their own relative quantities. We use the WAY theorem together with a consideration of state preparation to show that a relative-quantity Hilbert subspace can be constructed. In Sec. III, we consider a generalization of this argument, which may be applied to other cases involving unobservable quantities. In particular, we look into the case of the laser output field, where its absolute phase is generally accepted to be unobservable, though a direct proof for this based on the WAY theorem is currently lacking. Notwithstanding this, we demonstrate how the inference-free component of the state description may be isolated as an excellent approximation.

II. THE WIGNER-ARAKI-YANASE THEOREM AND AN INFERENCE-FREE SUBSPACE

The WAY theorem gives us a playground of systems where unobservability of certain quantities is guaranteed by global conservation laws. The WAY theorem states that any operator which does not commute with an operator of the global conservation is not observable [4,5]. Consider a system which consists of the observed subsystem and its measuring apparatus. As the total momentum $\hat{\Pi}$ of the system is conserved, a position operator $\hat{x}$ of the observed subsystem is unobservable. This is because the position operator does not commute with the total momentum and such measurement processes violate the conservation law.

Now we give a way to construct a relative-quantity subspace. According to the WAY theorem, a relative quantity of the unobservable absolute operators can be observable. By constructing a subspace where the relative quantity can be well-defined, we isolate the inference-free component. Here we show an example which can be easily generalized. A relative position operator of the observed system to the apparatus $\hat{x}_1-\hat{x}_2$ $(=\hat{x}_r)$ commutes with the total momentum and hence is observable, where $\hat{x}_1$ and $\hat{x}_2$ are the absolute positions of the observed system and of the apparatus, respectively. We take eigenstates of an operator $\hat{x}_a=\hat{x}_1+\hat{x}_2$ to construct the entire Hilbert space together with eigenstates of $\hat{x}_r$. The Hilbert space for the entire system (the observed system and the apparatus) can be expanded by $\{|x_r\rangle\otimes|x_a\rangle\}$ as well as by $\{|x_1\rangle\otimes|x_2\rangle\}$. To construct a relative-position Hilbert space, we start with separable states given by

$$|\psi\rangle=|\psi_r\rangle\otimes|\psi_a\rangle, \tag{1}$$

where

$$|\psi_r\rangle=\int dx_r\psi_r(x_r)|x_r\rangle,$$

$$|\psi_a\rangle=\int dx_a\psi_a(x_a)|x_a\rangle. \tag{2}$$

Here the state is separable in terms of the two subspaces of $\{|x_r\rangle\}$ and $\{|x_a\rangle\}$.

As the operator $\hat{x}_a$ is not observable, the state $|x_a\rangle$ must be considered as a label of the equivalence class of states [3], hence the state $|x_a\rangle\langle x_a|$ implies a set of the states

$$\int dXP(X)e^{-iX\Phi}|x_a\rangle\langle x_a|e^{iX\Phi}, \tag{3}$$

with all possible prior distributions $P(X)$. Using this representation, the total state can be represented as

$$\rho_{ra}=\int dXP(X)e^{-iX\hat{\Pi}}|\psi\rangle\langle\psi|e^{iX\hat{\Pi}}. \tag{4}$$

The operator $\hat{x}_r$ commutes with the total momentum Π, then the state $|\psi_r\rangle$ is preserved under the action of the displacement operator $e^{-iX\Phi}$. This allows the density matrix to be

$$\rho=|\psi_r\rangle\langle\psi_r|\otimes\rho_a, \tag{5}$$

where

$$|\psi_r\rangle=\int dx_r\psi_r(x_r)|x_r\rangle$$

$$\rho_a=\int\int\int dXP(X)dx_adx'_a\psi_a(x_a)\psi_a^*(x'_a)\times e^{-iX\hat{\Pi}}|x_a\rangle\langle x'_a|e^{iX\hat{\Pi}}.$$

The relative-position state $|\psi_r\rangle$ is on the relative-position Hilbert space and the relative-quantity operators can be defined on this subspace. The state ρ_a constructs an inference-free component and the inference-free subspace is constructed to be completely free from a choice of the prior distribution.

Now, let us generalize the argument to entangled states. The general state can be represented by

$$|\phi\rangle=\int dx_rdx_a\psi(x_r,x_a)|x_r,x_a\rangle. \tag{6}$$

In the case where the state is entangled, the function $\psi(x_r,x_a)$ cannot be written as $\psi_r(x_r)\psi_a(x_a)$. This state in the same representation with a prior is

$$\rho=\int\cdots\int dx_adx_{a'}dx_rdx_{r'}\psi(x_r,x_a)\psi^*(x'_r,x'_a)\times|x_r\rangle\langle x'_r|\otimes\int dXP(X)|x_a+X\rangle\langle x'_a+X|. \tag{7}$$

By contrast to the separable case, it seems nontrivial to construct an inference-free subspace. Such entangled states can be obtained by assuming arbitrary separable states and some entangling operators. For instance, a product state of $|x_r\rangle$ and a superposition of the total momentum eigenstates is separable by definition and yet can generate entanglement with some entangling operator such as a SUM gate $[\exp(-i\hat{x}_r\otimes\hat{\Pi})]$. In fact, the SUM gate commutes with the total momentum and hence such an operation is allowed, so it seems that we can create an entangled state not violating the conservation low. However, the essential issue here is to consider the state representation process to obtain a consistent state representation under the global symmetries. Next, we will show that the global symmetries impose restrictions in the state representation process.

As we have discussed above, a superposition can generate entanglement with some entangling operator, while by any of the allowed operations an eigenstate of the total momentum cannot be entangled with the relative-position subspace. This leads us to a question if any creation of superposition can be allowed under the global symmetries. It is not difficult to see that none of the operators which generate a superposition from an eigenstate is allowed under the conservation law. Any creation of superposition necessitates a third system to be involved in the state preparation process. This is inconsistent with the global symmetries. This concludes that considering the state preparation process, only the eigenstates of the total momentum are consistent with the global symmetry. These constraints on states allowed in the system could be considered as a superselection rule. The original work by Wigner [4] and the following works [6] have allowed the system to prepare an arbitrary state, in particular superpositions so that measurement of nonobservables in the sense of the WAY theorem can be arbitrarily precisely done. However, even if the system of the observed system and the apparatus recovers the conservation of the total momentum after the state preparation, the system cannot completely eliminate the third system. For example, a closed system with the momentum conservation is invariant under transformation by its absolute position, so different values of the total momentum give the same state to the system. Two different values of the *total momentum* $\hat{\Pi}$ become distinct when these are realized in the extended system. Thus, the extended system is necessary for the physical meaning of superpositions and the superposition states have to be captured in a relative-quantity subspace in the extended system. For a closed system with momentum conservation, as the eigenstate of the total momentum is the only state consistent, any state can be represented as Eq. (1) and hence the relative-position subspace always can be constructed.

III. THE CASE OF LASER OUTPUT FIELD

In this section, we generalize the argument to cases where unobservability is not guaranteed by global conservation laws. Our particular interest of such cases here is the laser output field. Despite a lack of rigorous proof, the absolute phase of an electromagnetic field has been considered to be nonobservable [1,2]. Due to the nonobservability of absolute phase ϕ, the state representation for the laser output field has an inference in terms of this quantity, and is written as

$$\rho=\int_0^{2\pi}\frac{d\phi}{2\pi}P(\phi)||\alpha|e^{-i\phi}\rangle\langle|\alpha|e^{-i\phi}|, \tag{8}$$

where $P(\phi)$ is an untestable prior distribution function, which is inference in the state representation. Now we take the state representation of the laser output field as an example of the general case to consider construction of the relative-quantity subspace. The previous argument suggests to generalize the state to two modes and take a relative phase of a two-mode coherent state. However, the argument about state preparation does not apply here, as unobservability of absolute phase is a weaker condition than global conservation law, then we have to find an alternative way to construct an inference-free subspace.

A two-mode coherent state is given as

$$|\alpha,\beta\rangle=||\alpha|e^{-i\phi_\alpha}\rangle\otimes||\beta|e^{-i\phi_\beta}\rangle \tag{9}$$

$$|\alpha,\beta\rangle=e^{-(|\alpha|^2+|\beta|^2/2)}\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{\alpha^{n_1}\beta^{n_2}}{\sqrt{n_1!n_2!}}|n_1,n_2\rangle. \tag{10}$$

The total photon number of the state is $N=n_1+n_2$ and the difference photon number is $M=(n_1-n_2)/2$ which is either integer (for even total photon numbers) or half integer (for odd total photon numbers). The state (10) can be alternatively expanded by the eigenstates characterized by these quantum numbers N and M as

$$|\alpha,\beta\rangle=e^{-\langle\hat{N}\rangle/2}\sum_{N=0}^{\infty}\sum_{M=-N/2}^{N/2}\frac{\alpha^{N/2+M}\beta^{N/2-M}}{\sqrt{\left(\frac{N}{2}+M\right)!\left(\frac{N}{2}-M\right)!}}|N,M\rangle, \tag{11}$$

where $\langle\hat{N}\rangle=|\alpha|^2+|\beta|^2$. Obviously this state is not separable in terms of the two subspaces $\{|N\rangle\}$ and $\{|M\rangle\}$. Unobservability of absolute phase is weaker as a restriction to the system, so the superposition rule for the conserved system does not apply here. Such a case, in general, does not allow us to simply isolate an inference-free subspace and requires an ingredient to approximately do so. In this case of coherent states, we take large total photon number limits to construct a relative-phase subspace.

Taking a set of parameters as

$$\frac{|\alpha|}{\langle\hat{N}\rangle^{1/2}}=-\sin\frac{\theta}{2},$$

$$\frac{|\beta|}{\langle\hat{N}\rangle^{1/2}}=\cos\frac{\theta}{2},$$

$$\phi_\alpha-\phi_\beta=\phi_r. \tag{12}$$

The two-mode coherent state can be written as the sum of spin coherent states, yielding

$$|\alpha,\beta\rangle=e^{-\langle\hat{N}\rangle/2}\sum_{N=0}^{\infty}\frac{(\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta})^N}{\sqrt{N!}}|N\rangle\otimes|\theta,\phi_r\rangle_N. \tag{13}$$

Here $|\theta,\phi_r\rangle_N$ is a spin-$N/2$ coherent state with the parametrization (12). Alternatively, the spin coherent state may be parametrized by ξ $(=-(|\alpha|/|\beta|)e^{-i\phi_r})$ as

$$|\theta,\phi_r\rangle_N=|\xi\rangle_N=\sum_{M=-N/2}^{N/2}\begin{pmatrix}N\\ \frac{N}{2}-M\end{pmatrix}^{1/2}\times(1+|\xi|^2)^{-N/2}\xi^{N/2+M}|M\rangle. \tag{14}$$

If the spin coherent state $|\xi\rangle_N$ is not dependent on the total photon number N, then the state for the total photon number can be realized as a coherent state of $|\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}\rangle$.

Here, we consider a limit of large total photon number, $\langle\hat{N}\rangle^{1/2}\to\infty$. The contribution of components for small N to the sum is negligible and the main contribution is the terms of the order $N\simeq\langle\hat{N}\rangle^{1/2}$. In the large limit of N, the spin coherent state can be contracted to a Weyl-Heisenberg (WH) coherent state. When $|\alpha|\simeq|\beta|$, the state can be typically contracted to a WH coherent state,

$$|\theta,\phi_r\rangle\to|-\sqrt{2}|\alpha|e^{-i\phi_r}\rangle. \tag{15}$$

At the limit, this coherent state is approximately separable with the subspace of the total photon number, and can be extracted from the sum in Eq. (13) as

$$|\alpha,\beta\rangle\simeq|-\sqrt{2}|\alpha|e^{-i\phi_r}\rangle\otimes e^{-\langle\hat{N}\rangle/2}\sum_{N=0}^{\infty}\frac{(\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta})^N}{\sqrt{N!}}|N\rangle$$
$$=|-\sqrt{2}|\alpha|e^{-i\phi_r}\rangle\otimes|\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}\rangle. \tag{16}$$

The laser output state ρ in the equivalent class of this state with a prior $P(\phi_\beta)$ may be given as

$$\rho = \int d\phi_\beta P(\phi_\beta)(|-\sqrt{2}|\alpha|e^{-i\phi_r}\rangle\langle -\sqrt{2}|\alpha|e^{-i\phi_r}| \otimes |\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}\rangle\langle\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}|). \tag{17}$$

The space of the relative phase ϕ_r is independent from the integral of the absolute phase ϕ_β, yielding

$$\rho = |-\sqrt{2}|\alpha|e^{-i\phi_r}\rangle\langle -\sqrt{2}|\alpha|e^{-i\phi_r}| \otimes \rho_{\mathrm{P}}, \tag{18}$$

where

$$\rho_{\mathrm{P}} = \int d\phi_\beta P(\phi_\beta)|\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}\rangle\langle\langle\hat{N}\rangle^{1/2}e^{-i\phi_\beta}|. \tag{19}$$

Hence the relative-phase subspace can be approximately constructed. However, this approximation is not so useful as the limit brings $|\alpha|$ also to infinity with $\sqrt{2}\alpha \simeq \langle\hat{N}\rangle^{1/2}$. By contrast, when $|\alpha| \ll |\beta|$ is satisfied, the group contraction may be taken in the order of $\langle\hat{N}\rangle$. In this case the spin state $|\xi\rangle_N$ is contracted by a parameter $\epsilon = 1/|\beta|$ as

$$\xi = -\epsilon|\alpha|e^{-i\phi_r} \quad (\epsilon \to \infty). \tag{20}$$

In this contraction, the spin size given by $|\beta|^2$ goes to infinity with $\epsilon \to 0$ and the state is contracted to a WH coherent state $|-|\alpha|e^{-i\phi_r}\rangle$.

The coherent state from laser can be approximately represented as

$$|\alpha,\beta\rangle\langle\alpha,\beta| \simeq |-|\alpha|e^{-i\phi_r}\rangle\langle -|\alpha|e^{-i\phi_r}| \otimes \rho_{\mathrm{P}}, \tag{21}$$

under the condition

$$\langle\hat{N}\rangle \simeq |\beta|^2 \gg |\alpha|^2. \tag{22}$$

Hence the state on the subspace of the relative phase is a coherent state, which is, in fact, what we call a coherent state in experiments.

To conclude, we have shown the explicit construction of an approximate relative-phase Hilbert space. The two-mode coherent state can be represented as a pure coherent state in the relative-phase subspace under the condition (22). This state presentation of relative phase does not involve prior distribution, and hence circumvents the entire discussion about unknowable absolute phase.

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