# Minimal qudit code for a qubit in the phase-damping channel

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Using the stabilizer formalism we construct the minimal code into a $D$-dimensional Hilbert space (qudit) to protect a qubit against phase damping. The effectiveness of this code is then studied by means of input-output fidelity.

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## I. INTRODUCTION

Quantum error correction (QEC) theory [1–4] concerns the possibility to protect against environmental noise when storing or transmitting quantum information. This possibility relies, likewise, in the classical error correction theory, in *redundancy*. This implies embedding a quantum information unit (a qubit) belonging to a given Hilbert space ($\mathcal{H}_2$) into a larger one ($\mathcal{H}_D$ with $D>2$). The latter is usually chosen as $n$ times the tensor product of the former, so that $D=2^n$ and $\mathcal{H}_D=\mathcal{H}_2^{\otimes n}$. We generally refer to this kind of encoding

$$\mathcal{C}:\ \mathcal{H}_2 \to \mathcal{H}_2^{\otimes n}, \tag{1}$$

as *block encoding* of the qubit. However, it is obvious that whatever $\mathcal{H}_D$ (with $D>2$), extra state space is available and could potentially be exploited without any restriction. As a consequence, the alternative possibility is to embed a qubit into a $D$-dimensional quantum system, i.e., a qudit, where $D$ can be made even infinite (in the limit where the qudit becomes a quantum oscillator, i.e., a bosonic mode [5,6]). We refer to this kind of encoding

$$\mathcal{C}:\ \mathcal{H}_2 \to \mathcal{H}_D \neq \mathcal{H}_2^{\otimes n}, \tag{2}$$

as to *qudit encoding* of the qubit. In a standard QEC framework, such encodings are coupled to suitable decoding stages, where a recovery operation (e.g., syndrome extraction and error correction) restores the original quantum information by removing the (correctable) errors induced by the noisy action of the environment.

Note that the encodings of Eqs. (1) and (2) are not equivalent. A first simple reason relies in the available dimensions for qudit encoding, which are not necessarily restricted to powers of 2. Moreover, at a more fundamental level, the errors affecting the two storing systems are different, that is, they form two different algebras. For a block of qubits errors are given by combinations of bit and phase flips, which are representable in terms of products of Pauli matrices. A single qudit instead is affected by amplitude and phase shifts (implying that, asymptotically, a single bosonic mode is affected by diffusion in position and momentum [6]), which are represented by the unitarily generalized Pauli matrices for a single qudit.

Pioneering advances in the QEC with higher-dimensional spin systems [7,8] and bosonic modes [5] were achieved during the 1990s. More recently, Ref. [6] introduced novel kinds of codes for qudits, known as *shift-resistant* (SR) quantum codes. In its simplest formulation, a SR code corresponds to embedding a logical qubit into a larger qudit, followed by a recovery stage which restores the quantum information from a bounded set of quantum errors (i.e., amplitude and phase shifts whose *weight* is less than some critical value). In particular, Ref. [6] showed that a qudit of dimension $D=18$ represents the smallest quantum system able to protect a logical qubit from a single quantum error, where the corresponding five-qubit block code [[5,1,3]] of Ref. [9] needs a Hilbert space of dimension $D=2^5>18$. Let us underline that both of these codes are stabilizer codes [3] and are *perfect*, roughly meaning that they need minimal quantum resources for their task [10].

The latter peculiarity is very important since the primary issue for having experimentally feasible QEC codes consists in simplifying their complexity. In fact, the importance of using minimal resource codes relies on our current difficulty in performing high fidelity operations on a small number of qubits [11]. However, apart from the optimality of the above perfect codes (which are designed to defeat general quantum errors), it is still an open problem to find the most efficient quantum codes which enable QEC within *specific error models*. In fact, if the dominant decoherence process in a physical system is of a specific nature and well known, one can look for a corresponding quantum error correction scheme whose quantum complexity is as small as possible. Such a problem has been raised, for the first time, in Ref. [12] for protecting logical qubits against dephasing. Later, Ref. [13] proposed an optimal code embedding a qubit in a block of bosonic modes able to protect against the effect of amplitude damping.

To date, nobody has analyzed the same problem for qudit encoding, i.e., nobody has considered the engineering of a minimal single-qudit code able to protect a logical qubit against a specific kind of decoherence. Only Ref. [14] pointed out that qudit encoding is not effective by itself when specific error models are taken into account. That is, without a suitable error correction (recovery) operation, the extra space cannot be exploited to protect against errors. In this paper, we consider the qudit encoding in a QEC framework (i.e., with a suitable recovery stage) and we design the minimal codes which are able to protect a logical qubit against a single class of errors, such as amplitude or phase shifts. Note that we are here considering minimal codes which are *quasiclassical*. In fact, even if they encode quantum information (one logical qubit), the environmental error model here is *classical*, in the sense that the correctable errors occur only in a preferred basis. The two complementary bases of a single qudit are perfectly symmetric, being connected by a discrete Fourier transformation. Therefore, by fixing the unperturbed basis (pointer basis) to be the computational one, we can always define as phase damping the damping that affects the complementary basis. We will show the robustness of a minimal qudit code in preserving the encoded quantum information against this kind of error.

The layout of the paper is the following. In Sec. II we present the code's construction and its performance against shift errors. Section III is devoted to the phase-damping channel. Section IV shows the performance of the code against phase damping in terms of input-output fidelity. Finally, Sec. V is the conclusion.

## II. CODE

### A. Qudits

Let us consider a qudit, i.e., a $D$-dimensional spin system. In its Hilbert space $\mathcal{H}_D$ we choose a computational basis $\{|j\rangle\}$ labeled by modular integers $j\in\mathbb{Z}_D:=\{0,\ldots,D-1\}$. An arbitrary unitary transformation $\mathcal{H}_D\to\mathcal{H}_D$ can be expanded in terms of $D^2$-*generalized* Pauli operators [15]

$$X^aZ^b,\quad a,b\in\mathbb{Z}_D, \tag{3}$$

which are defined by

$$X|j\rangle=|j\oplus 1\rangle,\quad Z|j\rangle=\omega^j|j\rangle, \tag{4}$$

where $j_1\oplus j_2:=j_1+j_2(\mathrm{mod}\ D)$ and

$$\omega:=\exp(i2\pi/D). \tag{5}$$

Such unitary operators satisfy the anticommutation relation

$$ZX=\omega XZ, \tag{6}$$

and their eigenstates are connected by

$$\widetilde{|i\rangle}=\sum_{j=0}^{D-1}H_{ij}|j\rangle, \tag{7}$$

where $X\widetilde{|i\rangle}=\omega^j\widetilde{|i\rangle}$, and $H$ is the $D\times D$ Fourier matrix with entries

$$H_{ij}:=\frac{\omega^{-ij}}{\sqrt{D}},\quad i,j\in\mathbb{Z}_D. \tag{8}$$

Accordingly, a general quantum error acting on the qudit can be decomposed in the error basis of Eq. (3). Its elements, i.e., the generalized Pauli operators, represent the basic quantum errors which a quantum correcting code must correct. According to Eq. (4) these are distinguished as amplitude shifts $X^a$ and complementary phase shifts $Z^b$. Multiplying by suitable phase factors $\omega^j$ the elements of Eq. (3), one defines the qudit Pauli group and, consequently, extends the stabilizer formalism [3,6]. These two kinds of errors, if considered separately, represent an Abelian group and their correction can thus be performed through quasiclassical codes.

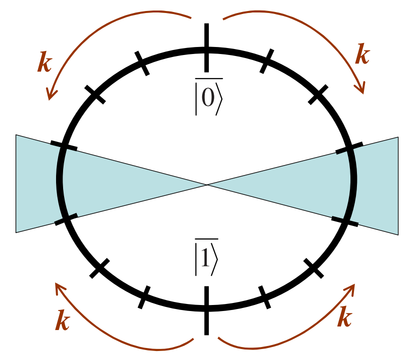


FIG. 1. (Color online) Pictorial view of errors' effect on code states. The shadowed area separates the correctable error spaces associated with the two code words.

### B. Single errors

It is natural to ask what is the smallest $D$-level system which protects an encoded qubit from a single amplitude shift $X^{\pm1}$. Let us first consider an example with $D=6$, so that $\omega=\exp(i\pi/3)$. A logical qubit can be encoded in the two code words stabilized by the generator $Z^2$, i.e.,

$$\overline{|0\rangle}:=|0\rangle,\quad \overline{|1\rangle}:=|3\rangle, \tag{9}$$

where

$$Z^2|0\rangle=|0\rangle,\quad Z^2|3\rangle=|3\rangle. \tag{10}$$

In such a case the measurement of the stabilizer preserves every coherent superposition $|\varphi\rangle=\alpha|0\rangle+\beta|3\rangle$, i.e.,

$$Z^2|\varphi\rangle=|\varphi\rangle, \tag{11}$$

while it detects single $X$ errors, i.e.,

$$Z^2X^{\pm1}|\varphi\rangle=\omega^{\pm2}X^{\pm1}|\varphi\rangle. \tag{12}$$

Alternately, one must consider the complementary generator $X^2$ for correcting single $Z^{\pm1}$ errors, i.e., one must encode the qubit into the code words

$$\overline{|+\rangle}:=\widetilde{|0\rangle},\quad \overline{|-\rangle}:=\widetilde{|3\rangle}, \tag{13}$$

where

$$X^2\widetilde{|0\rangle}=\widetilde{|0\rangle},\quad X^2\widetilde{|3\rangle}=\widetilde{|3\rangle}. \tag{14}$$

According to Eq. (7), one can express these code words in the computational basis as

$$\widetilde{|0\rangle}=\frac{1}{\sqrt{6}}\sum_{j=0}^{5}|j\rangle,\quad \widetilde{|3\rangle}=\frac{1}{\sqrt{6}}\sum_{j=0}^{5}(-1)^j|j\rangle. \tag{15}$$

### C. Multiple errors

From the previous example we argue that in order to correct $k$ shifts we need a qudit with

$$D=4k+2 \tag{16}$$

levels. This can be understood by means of the "clock" picture of Fig. 1, and is easily proven in the following. Consider, for example, the case of $k$ amplitude shifts $X^{\pm k}$, but the reasoning is perfectly symmetric for the complementary errors. In order to determine two possible code words, we must consider the eigenvalue equation

$$Z^G|j\rangle = \omega^{jG}|j\rangle, \tag{17}$$

where $j\in\mathbb{Z}_D$, $0\neq G\in\mathbb{Z}_D$, and $D$ must be determined. In Eq. (17), the state $|j\rangle$ is stabilized if and only if $\omega^{jG}=1$, and this happens in either the trivial case $j=0$ or in the case

$$jG = D. \tag{18}$$

Note that, since $j<D$, we must necessarily set $G\geq 2$ for the weight of the $Z$ generator. The resulting code will be able to correct all errors $X^0, X^{\pm 1}, X^{\pm 2}, \ldots, X^{\pm k}$ if and only if the stabilizer generator $Z^G$ will commute with $X^d$, where $d:=2k+1$ defines the distance of the code. Since

$$Z^G X^d = \omega^{dG} X^d Z^G, \tag{19}$$

this will happen if and only if

$$dG = D, \tag{20}$$

i.e., if the weight of the generator corresponds to the ratio between the dimension of the qudit and the code's distance. By comparing Eqs. (18) and (20) we must conclude that

$$j = d = 2k+1. \tag{21}$$

Then, by making the "minimal" choice $G=2$ in Eq. (18), we get the minimal dimension $D$ of Eq. (16). In conclusion, $k$ amplitude shifts are corrected by encoding a qubit into the code words

$$\overline{|0\rangle} := |0\rangle, \quad \overline{|1\rangle} := |2k+1\rangle, \tag{22}$$

of a $(4k+2)$-dimensional qudit. These code words are stabilized by the operator $Z^2$ and are connected by the logical flip gate $\bar{X} := X^{2k+1}$. Analogously, $k$ phase shifts are corrected by means of the code words

$$\overline{|+\rangle} := \widetilde{|0\rangle} = \frac{1}{\sqrt{4k+2}} \sum_{j=0}^{4k+1} |j\rangle, \tag{23}$$

$$\overline{|-\rangle} := \widetilde{|2k+1\rangle} = \frac{1}{\sqrt{4k+2}} \sum_{j=0}^{4k+1} (-1)^j |j\rangle, \tag{24}$$

which are stabilized by $X^2$ and are connected by the logical phase gate $\bar{Z} := Z^{2k+1}$. Both these codes are perfect, since the correctable error spaces (of dimension $d$) associated to their code words just barely fit in the qudit space (of dimension $D=2d$). We may refer to these codes as to *minimal amplitude code* $\{|0\rangle, |2k+1\rangle\}$ and *minimal phase code* $\{\widetilde{|0\rangle}, \widetilde{|2k+1\rangle}\}$, respectively. It is clear that they are equivalent up to a (discrete) Fourier transformation.

Note that, correspondingly, the minimal qubit block code which is able to correct $k$-phase (or amplitude) error flips works via majority voting and, therefore, needs a block of $2k+1$ qubits. This is equivalent to considering a Hilbert space of dimension $D=2^{2k+1}$, which is exponential rather than polynomial in $k$. This means that the qubit code is exponentially more demanding than the corresponding shift-resistant qudit code at given weight $k$.

### D. Syndrome extraction and error recovery

#### *1. Amplitude errors*

Consider an arbitrary coherent superposition of orthogonal code words of the amplitude code $\{|0\rangle, |2k+1\rangle\}$, i.e.,

$$|\varphi(0)\rangle = \alpha|0\rangle + \beta|2k+1\rangle. \tag{25}$$

Suppose that an amplitude shift error $X^s$, with *syndrome* $-k\leq s\leq k$, occurs on this superposition. Then, the logical state of Eq. (25) becomes

$$|\varphi(s)\rangle := X^s|\varphi(0)\rangle = \alpha|0\oplus s\rangle + \beta|2k+1\oplus s\rangle. \tag{26}$$

According to Sec. II C, such an error is detected by measuring the complementary generator $Z^2$. In fact, this measurement gives

$$Z^2|\varphi(s)\rangle = \omega^{2s}|\varphi(s)\rangle, \tag{27}$$

i.e., the syndrome is unambiguously extracted via the eigenvalue $\omega^{2s}$ of $Z^2$ (nondegeneracy of the code), while the corrupted state is preserved in the process. In order to realize this kind of quantum nondemolition measurement, we must append an ancillary system to the signal, let the joint system evolve according to a suitable unitary interaction, and finally, measure the ancilla. Since we must distinguish $2k+1$ orthogonal errors ($k$ positive shifts, $k$ negative shifts, and the no-shift), we need an ancillary system having at least $2k+1$ orthogonal states, that we label by $|l\rangle_A$ with $l\in\mathbb{Z}$, $|l|\leq k$.

In detail this correction process goes as follows. Let us introduce the $2k+1$ projectors

$$P(s) := |0\oplus s\rangle\langle 0\oplus s| + |2k+1\oplus s\rangle\langle 2k+1\oplus s|, \tag{28}$$

and construct the following unitary operation [generalized controlled-NOT (CNOT) gate or operation]:

$$N := \sum_{s=-k}^{k} P(s) X_A^s. \tag{29}$$

It is then easy to check that $N$ realizes the syndrome extraction. In fact, its effect on the joint system signal plus ancilla corresponds to leaving the corrupted state unchanged while shifting the ancilla by a quantity equal to the syndrome, i.e.,

$$N(|\varphi(s')\rangle \otimes |0\rangle_A) = |\varphi(s')\rangle \otimes |s'\rangle_A. \tag{30}$$

At this point, the measurement of the ancilla provides the syndrome $s'$ and one restores the original signal state by applying the corresponding inverse operator $X^{-s'}$ to $|\varphi(s')\rangle$.

It is known that the last procedure, i.e., the error correction stage, can be also implemented in a unitary manner. In fact, we can define the correction operator

$$C := \sum_{s=-k}^{k} X^{-s}|s\rangle_A\langle s|, \tag{31}$$

and applying it to the final state of Eq. (30). In such a way we get

$$C(|\varphi(s')\rangle \otimes |s'\rangle_A) = |\varphi(0)\rangle \otimes |s'\rangle_A,$$

thus recovering the initial encoded state of Eq. (25). Note that the two unitary operators of Eqs. (29) and (31) can be compacted together in a unique recovery operator

$$R: = CN = \sum_{r,s=-k}^{k} X^{-r}P(s) \otimes |r\rangle_A\langle r|X_A^s. \tag{32}$$

The above derivation simply shows how the recovery procedure works properly when a logical state $|\varphi(0)\rangle$ is affected by amplitude error shifts which are correctable, i.e., which fall within the distance of the code. More generally, we may ask how the recovery works when such errors are not necessarily correctable. This is a question that must be answered if we want to test these codes in a quantum communication scenario where the decoherence of a channel can be very strong. To this purpose we must first derive the effect of recovery on a completely arbitrary state of the qudit.

Thus, let us consider an arbitrary state

$$\rho = \sum_{i,j=0}^{D-1} \rho_{ij}|i\rangle\langle j|, \quad \rho_{ij}: = \langle i|\rho|j\rangle, \tag{33}$$

of a qudit with dimension $D=4k+2$. The joint action of the recovery operator reads

$$R(\rho \otimes |0\rangle_A\langle 0|)R^\dagger = \sum_{s,s'=-k}^{k} X^{-s}P(s)\rho P^\dagger(s')X^{s'} \otimes |s\rangle_A\langle s'|. \tag{34}$$

If we now trace out the ancilla, we get the *recovery map* acting on the qudit state

$$\mathcal{E}_R(\rho) = \sum_{s=-k}^{k} X^{-s}P(s)\rho P^\dagger(s)X^s. \tag{35}$$

By virtue of Eqs. (28) and (33), it is equal to

$$\mathcal{E}_R(\rho) = \Phi(0,0)|0\rangle\langle 0| + \Phi(0,2k+1)|0\rangle\langle 2k+1| + \Phi(2k+1,0) \times|2k+1\rangle\langle 0| + \Phi(2k+1,2k+1)|2k+1\rangle\langle 2k+1|, \tag{36}$$

where

$$\Phi(x,y): = \sum_{s=-k}^{k} \rho_{x\oplus s,y\oplus s}. \tag{37}$$

#### 2. Phase errors

For correcting phase errors the procedure is perfectly analogous to the previous one. It is sufficient to exchange the role of $X$ and $Z$ and account for the rotated code words of the phase code $\{\widetilde{|0\rangle},\widetilde{|2k+1\rangle}\}$. Thus, the action of recovery on an arbitrary state $\rho$ of the system is now described by the map

$$\tilde{\mathcal{E}}_R(\rho) = \tilde{\Phi}(0,0)\widetilde{|0\rangle}\widetilde{\langle 0|} + \tilde{\Phi}(0,2k+1)\widetilde{|0\rangle}\widetilde{\langle 2k+1|} + \tilde{\Phi}(2k+1,0) \times\widetilde{|2k+1\rangle}\widetilde{\langle 0|} + \tilde{\Phi}(2k+1,2k+1)\widetilde{|2k+1\rangle}\widetilde{\langle 2k+1|}, \tag{38}$$

with

$$\tilde{\Phi}(x,y): = \sum_{s=-k}^{k} \tilde{\rho}_{x\oplus s,y\oplus s}, \tag{39}$$

and $\tilde{\rho}_{ij}: =\widetilde{\langle i|}\rho\widetilde{|j\rangle}$. In order to express these formulas in the computational ($Z$) basis, we apply Eq. (7) yielding

$$\tilde{\rho}_{ij} = \sum_{l,m=0}^{D-1} H_{il}^*\rho_{lm}H_{mj} = \frac{1}{D}\sum_{l,m=0}^{D-1} \rho_{lm}\omega^{il-mj}. \tag{40}$$

Therefore,

$$\tilde{\Phi}(x,y) = \frac{1}{D}\sum_{l,m=0}^{D-1} \rho_{lm}(-1)^{xl-ym}\Delta(l-m,D), \tag{41}$$

where

$$\Delta(l-m,D): = \sum_{s=-k}^{k} \omega^{s(l-m)} = \frac{\sin\frac{\pi(l-m)}{2}}{\sin\frac{\pi(l-m)}{D}}, \tag{42}$$

and $D=4k+2$ as usual. The formula of Eq. (38) is crucial for our purposes. In fact, it will enable us to test the correcting performance of our minimal phase code $\{\widetilde{|0\rangle},\widetilde{|2k+1\rangle}\}$ in a quantum communication scenario where the prevalent effect of decoherence is ascribable to phase damping.

## III. PHASE-DAMPING CHANNEL FOR QUDITS

The phase-damping (or phase-flip) channel for a qubit can be defined by the following Kraus decomposition [16]:

$$\mathcal{E}(\rho) = \sum_{i=0}^{1} E_i\rho E_i^\dagger, \tag{43}$$

with Kraus operators

$$E_0 = \sqrt{\frac{1+\eta}{2}}I, \quad E_1 = \sqrt{\frac{1-\eta}{2}}Z, \tag{44}$$

where $I$ is the two-dimensional identity operator and $Z$ is given by Eq. (4) with $D=2$. One can describe the phase-damping channel in an equivalent way, by adopting two different Kraus operators, related to those of Eq. (44) by a unitary transformation

$$\mathcal{E}(\rho) = E_0'\rho E_0'^\dagger + E_1'\rho E_1'^\dagger, \tag{45}$$

where now

$$E_0' = |0\rangle\langle 0| + \eta|1\rangle\langle 1|, \quad E_1' = \sqrt{1-\eta^2}|1\rangle\langle 1|. \tag{46}$$

The two Kraus decompositions of the phase-damping channel for qubits, Eqs. (43) and (45), suggest two different generalizations to the general case of dimension $D$. The decomposition of Eq. (43) can be straightforwardly generalized as

$$\mathcal{E}(\rho) = \sum_{m=0}^{D-1} E_m\rho E_m^\dagger, \tag{47}$$

with

$$E_m = \sqrt{\binom{D-1}{m}\left(\frac{1-\eta}{2}\right)^m\left(\frac{1+\eta}{2}\right)^{D-1-m}} Z^m, \qquad (48)$$

which can be seen as a particular example of a Weyl channel [17], which is generally defined as

$$\rho \mapsto \mathcal{E}(\rho) = \sum_{m,n=0}^{D-1} \pi_{m,n}(Z^n X^m)\rho(X^m Z^n)^\dagger, \qquad (49)$$

with $0 \leq \pi_{m,n} \leq 1$ such that $\sum_{m,n=0}^{d-1}\pi_{m,n}=1$.

There is, however, a different way to define the phase-damping channel for a $D$-dimensional spin system, which is more closely related to the usual physical meaning of phase damping. In fact, phase damping usually means that decoherence affects the elements of a given basis leaving the elements of the complementary basis unchanged. As a consequence, in the basis of the unaffected states, decoherence destroys only the off-diagonal elements of the density matrix of the qudit state. (It is evident that here we are conventionally fixing the elements $|j\rangle$ of the computational $Z$ basis as the unchanged ones, but it is understood that a complementary damping channel can be symmetrically defined.)

The second definition of the phase-damping channel for a qudit is suggested by Ref. [18] that has shown that the Kraus decomposition of Eq. (45) is equivalent to another Kraus decomposition,

$$\mathcal{E}(\rho) = \sum_{i=0}^{\infty} E_i \rho E_i^\dagger, \qquad (50)$$

with an *infinite* number of $E_i$, which in the case of qubits, are given by

$$E_i = \delta_{i0}|0\rangle\langle 0| + \frac{\eta(-2\ln\eta)^{i/2}}{\sqrt{i!}}|1\rangle\langle 1|. \qquad (51)$$

The straightforward generalization of such a channel in $D$ dimensions has been already studied in Ref. [14] and has Kraus operators

$$E_i := \sum_{j=0}^{D-1} \frac{[j\sqrt{-2\ln\eta}]^i \eta^{j^2}}{\sqrt{i!}}|j\rangle\langle j|. \qquad (52)$$

In all these examples, the parameter $\eta \in [0,1]$ describes the strength of the damping [14]. Such a parameter can be assumed independent from $D$ because it usually depends only on the bath characteristics, e.g., in the Markov approximation it depends upon the spectral density of the bath only [19]. One can parametrize $\eta = e^{-\gamma}$, where $\gamma$ is proportional to the probability of a phase error, so that phase damping is larger for smaller $\eta$. In particular, $\eta \to 1^-$ and $\eta \to 0^+$ correspond to the weak and strong damping limit, respectively.

It is easy to check from Eqs. (50) and (52), that an arbitrary density operator $\rho = \sum_{i,j=0}^{D-1}\rho_{ij}|i\rangle\langle j|$ is mapped into the output density operator given by

$$\mathcal{E}(\rho) = \sum_{i,j=0}^{D-1} \rho_{ij}\eta^{(i-j)^2}|i\rangle\langle j|, \qquad (53)$$

i.e., we have, as expected, partial suppression of only the off-diagonal matrix elements of the state in the computational $Z$ basis. This latter equation shows why this second definition of the phase-damping channel for qudits reproduces the usual phase decoherence effect.

## IV. INPUT-OUTPUT FIDELITIES

Once we have defined the two kinds of phase-damping channels for a qudit, the Weyl channel of Eqs. (47) and (48) and the conventional phase-damping channel of Eq. (53), we can consider their action upon our minimal phase code $\{\widetilde{|0\rangle}, \widetilde{|2k+1\rangle}\}$. Here, we encode a logical qubit into a qudit (with dimension $D=4k+2$) by means of these code words, and we analyze the effects of phase damping with and without error recovery. Such effects are quantified in terms of fidelity of the output logical state with respect to the input. The results are then compared to the case where a bare qubit is sent through the channel, i.e., when neither encoding nor decoding is performed.

Let us encode an arbitrary pure state $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ of a qubit into a coherent superposition of phase code words, i.e.,

$$|\theta,\phi\rangle = \cos\frac{\theta}{2}\widetilde{|0\rangle} + e^{i\phi}\sin\frac{\theta}{2}\widetilde{|2k+1\rangle}. \qquad (54)$$

In the computational basis, the logical state of Eq. (54) reads

$$|\theta,\phi\rangle = \frac{1}{\sqrt{D}}\sum_{l=0}^{D-1}\left[\cos\frac{\theta}{2} + (-1)^l e^{i\phi}\sin\frac{\theta}{2}\right]|l\rangle, \qquad (55)$$

and the corresponding density operator is given by

$$\rho(\theta,\phi) = |\theta,\phi\rangle\langle\theta,\phi| = \frac{1}{D}\sum_{l,m=0}^{D-1}\Omega_{lm}|l\rangle\langle m|, \qquad (56)$$

where

$$\Omega_{lm} := \left[\cos\frac{\theta}{2} + (-1)^l e^{i\phi}\sin\frac{\theta}{2}\right]\left[\cos\frac{\theta}{2} + (-1)^m e^{-i\phi}\sin\frac{\theta}{2}\right]. \qquad (57)$$

The effects of the phase-damping Weyl channel and of the conventional phase-damping channel on this logical state can be described with a unified formalism. In fact, one can write the output state of the channel for the two cases as

$$\mathcal{E}[\rho(\theta,\phi)] = \frac{1}{D}\sum_{l,m=0}^{D-1}\Omega_{lm} f_r(\eta, l-m)|l\rangle\langle m|, \qquad (58)$$

where $r=1$ refers to the conventional phase-damping channel, $r=2$ to the Weyl channel, and

$$f_1(\eta, l-m) = \eta^{(l-m)^2} \qquad (59)$$

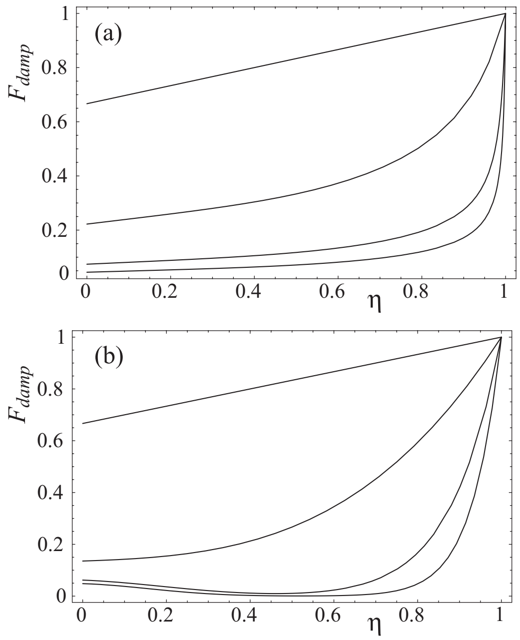


FIG. 2. Averaged input-output fidelities $F_{damp}$ in the case of a damped qudit. (a) refers to the conventional phase-damping channel, while (b) refers to the Weyl channel. Fidelities are plotted versus the damping parameter $\eta$ and for different dimensions $D=30,18,6,2$ from bottom to top.

$$f_2(\eta,l-m)=\left[\left(\frac{1-\eta}{2}\right)\omega^{(l-m)}+\left(\frac{1+\eta}{2}\right)\right]^{D-1}. \tag{60}$$

All the results can be expressed in terms of these two functions associated to each channel. In order to estimate the decoherence effects we compute the fidelity between the input and output states

$$\begin{aligned}F_{damp}(\theta,\phi)&:=\langle\theta,\phi|\mathcal{E}[\rho(\theta,\phi)]|\theta,\phi\rangle\\&=\frac{1}{D^2}\sum_{l,m=0}^{D-1}|\Omega_{lm}|^2 f_r(\eta,l-m),\end{aligned} \tag{61}$$

and, then, average this quantity over all the possible input states

$$\begin{aligned}F_{damp}&:=\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi F_{damp}(\theta,\phi)\\&=\frac{1}{3D^2}\sum_{l,m=0}^{D-1}[3+(-1)^{l-m}]f_r(\eta,l-m).\end{aligned} \tag{62}$$

The behavior of the averaged fidelity is shown for the two cases in Fig. 2, where (a) refers to the $r=1$ conventional phase-damping channel and (b) to the $r=2$ Weyl channel. In both cases we note that the decoherence effect of the channel increases with the dimension $D$.

Let us now apply the recovery map of Eq. (38) to the corrupted state of Eq. (58). The output (recovered) state is then given by

$$\rho_{rec}(\theta,\phi):=\tilde{\mathcal{E}}_R\{\mathcal{E}[\rho(\theta,\phi)]\}. \tag{63}$$

By inserting the matrix elements $\rho_{lm}=D^{-1}\Omega_{lm}f_r(\eta,l-m)$ of the corrupted state $\mathcal{E}[\rho(\theta,\phi)]$ into Eq. (41) we get the corresponding $\tilde{\Phi}$ coefficients

$$\tilde{\Phi}(x,y)=\frac{1}{D^2}\sum_{l,m=0}^{D-1}\Omega_{lm}f_r(\eta,l-m)(-1)^{xl-ym}\Delta(l-m,D), \tag{64}$$

which, in turn, must be substituted into Eq. (38) in order to give the final explicit expression of the recovered state $\rho_{rec}(\theta,\phi)$. The input-output fidelity then becomes

$$\begin{aligned}F_{rec}(\theta,\phi)&:=\langle\theta,\phi|\rho_{rec}(\theta,\phi)|\theta,\phi\rangle\\&=\frac{1}{D^2}\sum_{l,m=0}^{D-1}\Omega_{lm}f_r(\eta,l-m)\Delta(l-m,D)\\&\quad\times\left[\cos^2\frac{\theta}{2}+(-1)^{-m}\frac{\sin\theta}{2}e^{i\phi}+(-1)^{l}\frac{\sin\theta}{2}e^{-i\phi}\right.\\&\quad\left.+(-1)^{l-m}\sin^2\frac{\theta}{2}\right],\end{aligned} \tag{65}$$

and its average over all possible input states takes the form

$$\begin{aligned}F_{rec}&:=\frac{1}{4\pi}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi F_{rec}(\theta,\phi)\\&=\frac{1}{3D^2}\sum_{l,m=0}^{D-1}[3+(-1)^{l-m}]f_r(\eta,l-m)\Delta(l-m,D),\end{aligned} \tag{66}$$

Note that the recovery fidelity of Eq. (66) has the same form as the damped fidelity of Eq. (62) except for the presence of the kernel-like term $\Delta(l-m,D)$ in Eq. (42). This term formally takes the recovery operation into account and is therefore responsible for the very different behaviors of $F_{rec}$ and $F_{damp}$. Such a term is equal to 1 only in the trivial case of a qubit ($D=2\Leftrightarrow k=0$) for which we have $F_{rec}=F_{damp}$, as is intuitively expected. In fact, in this case, the logical qubit is simply encoded into another qubit and therefore remains unencoded.

Our qudit phase code can be compared with an $n$-qubit repetition code. Actually, for a given $D=4k+2$, the integer $n$ should be chosen as $\mathrm{odd}\lceil\log_2 D\rceil$, that is, as the odd integer closest to $\log_2 D$ from above. For an $n$-qubit repetition code, starting from Eq. (44), we straightforwardly get

$$\begin{aligned}F_{rec}=&\sum_{k=0}^{(n-1)/2}\binom{n}{k}\left(\frac{1+\eta}{2}\right)^{n-k}\left(\frac{1-\eta}{2}\right)^{k}+\frac{1}{3}\sum_{k=(n+1)/2}^{n}\binom{n}{k}\\&\times\left(\frac{1+\eta}{2}\right)^{n-k}\left(\frac{1-\eta}{2}\right)^{k}.\end{aligned} \tag{67}$$

Notice that this result holds for both channels, and moreover, for $n=2$ it corresponds to one qubit code, i.e., to unencoded qubit.

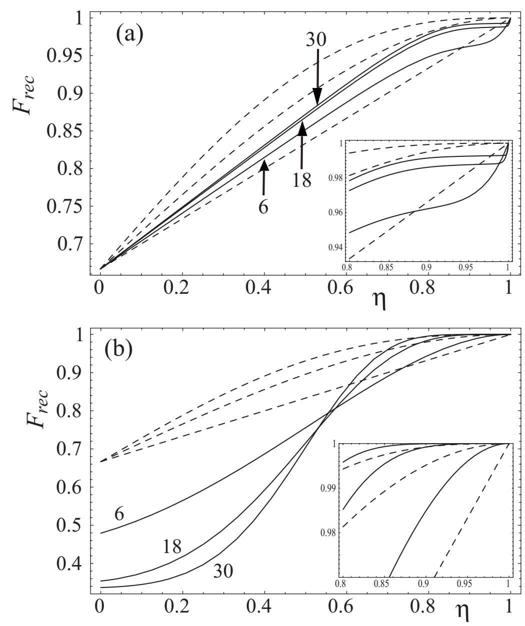


FIG. 3. Averaged input-output fidelities $F_{rec}$ in the case of a damped qudit in the presence of the recovery stage. (a) refers to the conventional phase-damping channel, while (b) refers to the Weyl channel. Fidelities are plotted versus the damping parameter $\eta$ and for different dimensions $D=30,18,6$. Dashed lines refer, from top to bottom, to repetition code of dimension 32,8,2 (using 5,3,1 quibts, respectively). In (a) the qudit code is always worse than the repetition code, but it performs better than the unencoded case except for $\eta \to 1$ (see the inset). In (b) the qudit code is effective at small phase damping $\eta \to 1$, where it can outperform the repetition code (see the inset).

The correcting power of our qudit phase code is shown in Fig. 3, where the recovery fidelity of Eq. (66) has been plotted as a function of the channel decoherence parameter $\eta$ for several dimensions $D=4k+2$, for the two examples of phase-damped channels, the conventional phase-damping channel in Fig. 3(a) and the Weyl channel in Fig. 3(b).

In Fig. 3(a) we see that in the case of conventional phase damping the qudit code performs better and better for increasing dimension and always outperforms the unencoded case $D=2$ [lower dashed line in Fig. 3(a)]. More precisely, in both limits $\eta \to 1^-$ (weak damping) and $\eta \to 0^+$ (strong damping) the correction scheme does not depend on $D$. However, the improvement with respect to the unencoded transmission of the qudit is remarkable in the intermediate regime [see Fig. 3(a)]. We have also made the comparison with the block encoding. Actually, the dotted lines from top to bottom in Fig. 3(a) refer to the repetition code of dimension 32,8,2 (using 5,3,1 qubits), respectively. This code always outperforms the qudit code proposed here for any $D$, showing that even though useful, our qudit codes are not optimal in the case of the conventional phase-damping channel. This is, however, not surprising because the qudit code is very different from repetition codes and it is not designed to cancel errors at first order in the error probability as do the latter codes.

The situation for the Weyl channel shown in Fig. 3(b) is more involved. In this case, the correcting power of the qudit code increases for increasing $D$ *only at small phase damping* $\eta \to 1$, while worsening for increasing dimension in the strong damping limit $\eta \to 0$. This means that for a Weyl channel the qudit code outperforms the unencoded case $D=2$ only at large $\eta$ ($\eta > 0.7$) and therefore it is useful only in the weak damping limit. In this limit however, contrary to what happens for the conventional phase-damping channel, the qudit code becomes particularly useful because it can outperform even the repetition code (dotted lines from top to bottom in Fig. 3(b) refer to the repetition code of dimension 32,8,2, respectively). In particular, while the qudit code of $D=6$ does not outperform the three-qubit repetition code, the qudit code of $D=18$ outperforms the three-qubit repetition code, and quite remarkably the qudit code of $D=30$ outperforms the five-qubit repetition code.

Finally, the qudit code turns into a noneffective code while decreasing $\eta$ (worsening for increasing dimension), because it is tailored to correct errors of weight up to $k$ while in such a limit errors of higher weight become more and more probable.

### A. State-dependent fidelity

Besides the average fidelity, it is also interesting to analyze the state-dependent fidelity $F_{rec}(\theta,\phi)$ of Eq. (65). Both phase-damping channels act on the phase of the states and therefore in both cases the eigenstates of $Z$ are unaffected by decoherence, as can be easily checked. It is interesting to see what happens to the encoded states in the presence of the recovery procedure of Sec. II D. By construction, the two quantum code words $\overline{|+\rangle} := \widetilde{|0\rangle}$ and $\overline{|-\rangle} := \widetilde{|2k+1\rangle}$ of our phase code are stabilized by $X^2$ and connected by the logical phase gate $\bar{Z} := Z^{2k+1}$. As a consequence, they are significantly affected by both phase-damping channels even in the presence of error correction. For the conventional phase-damping channel the two fidelities satisfy the simple relation

$$F_{rec}(0,0) = F_{rec}(\pi,0) \to \frac{1}{2} \quad \text{for} \quad \eta \to 0^+, \tag{68}$$

i.e., they are completely dephased under the effect of strong phase damping. In the Weyl channel case the effect is even stronger and for strong phase damping $\eta \to 0$ one has $F_{rec}(0,0) = F_{rec}(\pi,0) \simeq 0$ at large enough $D$. However, one can still find an encoded basis which is unaffected by phase damping in the presence of error correction. Such a basis is formed by the *rotated* code words given by the two simple superpositions of the initial code words,

$$|\zeta_0\rangle := \frac{\overline{|+\rangle} + \overline{|-\rangle}}{\sqrt{2}} = (2k+1)^{-1/2} \sum_{n=0}^{2k} |2n\rangle, \tag{69}$$

$$|\zeta_1\rangle := \frac{\overline{|+\rangle} - \overline{|-\rangle}}{\sqrt{2}} = (2k+1)^{-1/2} \sum_{n=0}^{2k} |2n+1\rangle. \tag{70}$$

One can easily check that these are the eigenstates of the logical phase gate $\bar{Z} := Z^{2k+1}$ and are connected by the single shift operator $X$, i.e.,

$$\bar{Z}|\zeta_0\rangle = |\zeta_0\rangle, \quad \bar{Z}|\zeta_1\rangle = -|\zeta_1\rangle, \tag{71}$$

and

$$X|\zeta_0\rangle = |\zeta_1\rangle, \quad X|\zeta_1\rangle = |\zeta_0\rangle. \tag{72}$$

In other words, these *rotated* code words $\{|\zeta_0\rangle, |\zeta_1\rangle\}$ behave like the $Z$ eigenstates $|0\rangle$ and $|1\rangle$ of a qubit. In fact, the corresponding fidelities satisfy, for both phase-damping channels,

$$F_{rec}(\pm\pi/2,0) = 1, \quad \forall \quad k,\eta, \tag{73}$$

which is exactly the behavior of the $Z$-basis eigenstates $|0\rangle, |1\rangle$ under the action of phase damping (i.e., they remain unchanged). However, let us remark that these correspondences with the single-qubit eigenstates hold if and only if the qudit code words are subject to error correction.

One can explicitly show that the two code words $\{|\zeta_0\rangle, |\zeta_1\rangle\}$ are perfectly restored by error recovery for any value of $\eta$ by making use of Eqs. (38) and (41), from which one can verify that

$$\begin{aligned}\tilde{\Phi}(0,0) &= (-1)^u \tilde{\Phi}(0,2k+1) \\ &= (-1)^u \tilde{\Phi}(2k+1,0) \\ &= \tilde{\Phi}(2k+1,2k+1) = 1/2,\end{aligned} \tag{74}$$

which implies

$$\tilde{\mathcal{E}}_R[\mathcal{E}(|\zeta_u\rangle\langle\zeta_u|)] = |\zeta_u\rangle\langle\zeta_u|. \tag{75}$$

As a final remark, note how the error correcting properties of the orthogonal choices $\overline{|+\rangle}, \overline{|-\rangle}$ and $|\zeta_0\rangle, |\zeta_1\rangle$ are perfectly the same in the regime of weak damping (where they correct up to $k$ phase shifts with exactly the same ability), while their performances dramatically split when the phase damping becomes heavier and it is no-more reducible to the standard QEC regime.

## V. CONCLUSION

In conclusion, we have addressed the problem of how to profitably exploit the extra space available by embedding a quantum system into a "larger" one (qudit encoding). Such an approach can be useful from the point of view of the experimental feasibility of quantum error correction schemes, since the dimension $D$ of the encoding Hilbert space remains reasonably low. In particular, we have considered the minimal $D$ which enables the construction of qudit codes able to restore a logical qubit in specific decoherence models. These minimal codes are then proven to be efficient in protecting quantum information against the detrimental effects of phase damping. This study could shed further light into the role that Hilbert space dimensions play in quantum error correction. The opposite problem of quantum data compression could be considered in the same light. That is, data compression from $\mathcal{H}_2^{\otimes n}$ to $\mathcal{H}_D$ with $2^n > D$, as it has been considered in Refs. [20] and [21].

Possible experimental implementations of these codes and the corresponding recovery operations require the ability to efficiently implement the generalized Pauli operators $X$ and $Z$ in an effective $D$-dimensional system. An interesting opportunity is provided by ring-shaped optical lattices, which have been proposed as a possible quantum simulator of periodic one-dimensional quantum systems [22]. If we place a single atom in a ring-shaped lattice with $D$ sites, the ground states in each site are the basis states. As a consequence, the amplitude shift $X$ is realized by tunneling, while the phase shift $Z$ could be realized by applying controlled local Stark shifts to the atom.

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