Classical Broadcasting is Possible with Arbitrarily High Fidelity and Resolution

Thomas A. Walker and Samuel L. Braunstein
Computer Science, University of York, York YO10 5DD, United Kingdom
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We quantify the resolution with which any probability distribution may be distinguished from a displaced copy of itself in terms of a characteristic width. This width, which we call the ϵ resolution, is well defined for any normalizable probability distribution. We use this concept to study the broadcasting of classical probability distributions. Ideal classical broadcasting creates two (or more) output random variables each of which has the same distribution as the input random variable. We show that the universal broadcasting of probability distributions may be achieved with arbitrarily high fidelities for any finite ϵ resolution. By restricting probability distributions to any finite ϵ resolution we have therefore shown that the classical limit of quantum broadcasting is consistent with the actual classical case.

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With the advent of quantum computing, information processing as a science has had an increasing role in physics and the stark contrast between quantum and classical information processing has been brought to light. For this contrast to be fully appreciated, we must understand the limitations and capabilities of classical information processing and, in particular, see how they relate to any classical limit of quantum information processing capabilities.

One of the earliest limitations of quantum information processing was the no-cloning theorem. It tells us that it is impossible to make two *independent* copies of a quantum state by any unitary process [1]. Even for mixed state inputs, an ideal $1 \to 2$ cloning machine would require that the two clones be independent

$$\rho \to \sigma_{AB} = \rho_A \otimes \rho_B. \tag{1}$$

An important variation of cloning, called broadcasting, was constructed in 1996 by Barnum *et al.* [2]. The idea is that there are many applications where the two clones will never be used together so that one need not worry about the actual independence of the clones, only that each subsystem (or marginal distribution) by itself looks just like the original. In other words, an ideal broadcast of quantum states would yield $\rho \to \tilde{\sigma}_{AB}$ with $\mathrm{Tr}_j(\tilde{\sigma}_{AB}) = \rho$ for each $j = A, B$. In *finite* dimensions Barnum *et al.* have shown that it is impossible to broadcast arbitrary mixed states [2,3]. In their proof they demonstrate that the only states that can be broadcast are those that mutually commute. Naively, this would suggest that in the classical limit all probability distributions could be broadcast. However, what about the infinite-dimension limit (e.g., the case of continuous variables); the case for the classical limit there appears more subtle.

Indeed, recently Daffertshofer *et al.* studied the cloning of classical proability distributions [4]. They considered two kinds of ideal cloning machines: one which is the analogue of Eq. (1), where the copied distributions are statistically independent; and a second where only the marginal distribution of the copies need mimic the original state [4]. In particular, this second machine then actually describes the broadcasting of a classical probability distribution. Using this language then, Daffertshofer *et al.* could be said to claim that it is impossible to broadcast classical probability distributions with unit fidelity in a deterministic manner assuming (i) Liouville evolution for the broadcasting process and (ii) infinitely-narrow distributions (i.e., delta-function distributions) are excluded. Naively, this result makes sense since a resource corresponding to an unphysically narrow delta function is required to ideally broadcast a classical probability distribution. However, this result would then appear to be at odds with the infinite-dimensional limit of the work of Barnum *et al.* [2] in that it suggests that for quantum states with continuous degrees of freedom that it may not be possible to broadcast even states limited to a self-commuting set.

In this Letter we therefore raise the question of how well can one perform universal broadcasting of continuous classical probability distributions? However, rather than simply excluding delta-function distributions, we shall define a resolution measure so that we may work with distributions down to any fixed resolution. Here shall we show that for input distributions of any finite (nonzero) resolution that deterministic universal broadcasting may be achieved with fidelity arbitrarily close to unity via Liouville dynamics.

The criteria we consider for the broadcasting of classical distributions is the exact analogue of that for the broadcasting of quantum states. Initially, one has a probability distribution $p_{\rm in}(x)$ to be broadcast and another fixed distribution $g(y, z, \ldots)$ corresponding to some standard resource. At the end of the process the two states appear on the marginal distributions for random variables x and y with all other variables acting as "work space." Thus, the broadcasting machine should act as

$$p_{\rm in}(x)g(y, z, \ldots) \to p_{\rm out}(x, y, z, \ldots), \tag{2}$$

with the marginal outcome distributions given by

$$p^{(1)}_{\rm copy}(x) = \int dydz \ldots p_{\rm out}(x, y, z, \ldots),$$
$$p^{(2)}_{\rm copy}(y) = \int dxdz \ldots p_{\rm out}(x, y, z, \ldots), \tag{3}$$

where the integrals are over all variables except x and y respectively. Ideal deterministic broadcasting would be achieved if both of these marginals are identical to the input distribution $p_{\rm in}(x)$, which would correspond to unit fidelity for each state.

Before we proceed, we must introduce our measure of resolution for distributions. The standard deviation is not really suitable, since it excludes any distribution with infinite variance, which may be an artificial restriction. In fact, we shall see that such a restriction is unnecessary. Now there are numerous generalizations to the standard deviation, e.g., the full-width at half maximum, the 95% confidence interval etc., that are typically well defined provided only that the distribution itself is normalizable. Here we present a new generalization which rigorously captures the notion of the resolution of a probability distribution in a manner which is directly tied to the distinguishability between distributions. We call it this measure the ϵ resolution of a distribution.

Since we wish to mathematically describe the entire class of distributions with some finite (nonzero) resolution, we must begin by quantifying this concept. What can finite resolution mean? Presumably it must say something about how well one can distinguish some distribution $p(x)$ from a version of itself displaced by an amount δ, i.e., $D_\delta p(x) \equiv p(x - \delta)$.

Now recall that the trace norm between a pair of distributions p and q gives the optimal probability for being able to distinguish them as

$$d(p, q) \equiv \frac{1}{2} \int dx |p(x) - q(x)|, \tag{4}$$

(with the obvious generalization to higher dimensions) [5]. Thus, the optimal probability to be able to distinguish p from its displaced version $D_\delta p$ will be $d(p, D_\delta p)$. This allows us to define a natural resolution length-scale for any (normalizable) distribution as the largest ϵ such that

$$d(p, D_\delta p) \leq \frac{|\delta|}{\epsilon}, \quad \forall \ \delta. \tag{5}$$

Equivalently, we may write a formula for what we now call the ϵ *resolution* of a distribution as

$$\epsilon[p] \equiv \max_\delta \frac{|\delta|}{d(p, D_\delta p)}. \tag{6}$$

Why base our resolution measure on the trace norm instead of, say, the (Bhattacharyya) metric or its infinitessimal form, the Fisher information [6,7]? In principle, any reasonable regularization procedure might be expected to work equally well. One reason why we chose to base our measure on the trace norm was that its statistical interpretation applies even to single samplings, unlike the others for which achievable performance only occurs asymptotically for a large number of samplings [8].

In general the ϵ resolution can be difficult to calculate. However, we start by providing some results about it:

Lemma 1.—The ϵ resolution of a distribution is unchanged by any displacement of it, i.e., $\epsilon[D_\delta p] = \epsilon[p]$.

Lemma 2.—The ϵ resolution of a scaled distribution is itself scaled by the same amount; i.e., $\epsilon[S_s p] = \epsilon[p]/s$, where the squeeze operator acts as $S_s p(x) = sp(sx)$.

Our first nontrivial result about ϵ resolution is a simple formula for it for distributions which are symmetric and monotonically nonincreasing away from the origin:

Theorem 3.—For any monotonically nonincreasing symmetric distribution $p_{\rm mnsd}(x)$, its ϵ resolution is the reciprocal of its height. In other words

$$\epsilon[p_{\rm mnsd}] = \frac{1}{p_{\rm mnsd}(0)}. \tag{7}$$

Proof.—Since the ϵ resolution is independent of displacements, we may consider a more symmetric form of its definition in terms of

$$d(p, D_\delta p) = d(D_{\delta/2} p, D_{-\delta/2} p) = \frac{1}{2\pi} \int_0^\infty dx \int dk e^{ikx}(e^{-ik|\delta|/2} - e^{ik|\delta|/2})\chi(k), \tag{8}$$

where we have written the distribution as p in terms of its characteristic function via $p(x) = \int dk e^{ikx}\chi(k)/2\pi$.

Next, using the fact that [9]

$$\int_0^\infty dx e^{\pm ikx} = \pi\delta(k) \pm i\mathcal{P}\left(\frac{1}{k}\right), \tag{9}$$

where $\mathcal{P}$ stands for the principal-value function, we find that

$$d(p, D_\delta p) = \frac{i}{2\pi} \int dk \mathcal{P}\left(\frac{1}{k}\right)(e^{-ik|\delta|/2} - e^{ik|\delta|/2})\chi(k). \tag{10}$$

Taking the derivative of this expression then yields

$$\frac{\partial}{\partial\delta} d(p, D_\delta p) = p(\delta/2), \tag{11}$$

which may in turn then be trivially integrated as

$$d(p, D_\delta p) = \int_0^{|\delta|} dx p(x/2). \tag{12}$$

We now note, that if $p(x)$ is monotonically nonincreasing from its central symmetric value then Eq. (12) describes a convex function of δ. For such a convex form an upper bound is completely determined by its behavior near the origin as δ approaches zero. Thus, $d(p, D_\delta p) \leq |\delta| p(0)$ is a tight linear upper bound, which is saturated as $\delta \to 0$, from which our result follows. □

From this theorem it is now trivial to compute the ϵ resolution for a number of cases: For example, consider a Gaussian distribution

$$p_{\text{Gaussian}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \quad (13)$$

with mean μ and standard deviation σ. From our theorem we have $\epsilon[p_{\text{Gaussian}}] = \sqrt{2\pi}\sigma$. In order to gain familiarity with the ϵ resolution measure, we provide some further examples in the appendix.

Before considering how to construct a finite-resolution broadcasting machine, let us review the dynamics we shall allow. Liouville dynamics consists of a deterministic system governed by the equations of motion

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}, t), \qquad \vec{x}, \vec{v} \in \mathbb{R}^N, \quad (14)$$

where $\vec{x}$ is a point in an N-dimensional space and t is time. It is then straightforward to derive the equation of motion for an ensemble of particles undergoing this motion. The probability distribution $P(\vec{x}, t)$ describing such an ensemble must satisfy [10]

$$\frac{\partial P(\vec{x}, t)}{\partial t} = -\vec{\nabla} \cdot [\vec{v}(\vec{x}, t)P(\vec{x}, t)]. \quad (15)$$

As a simple example, consider damping dynamics in N dimensions

$$\frac{d\vec{x}}{dt} = -\alpha\vec{x}; \quad (16)$$

it is trivial to integrate the initial distribution $P(\vec{x}, 0)$ forward in time to yield

$$P(\vec{x}, t) = e^{N\alpha t} P(e^{\alpha t}\vec{x}, 0). \quad (17)$$

This corresponds to a uniform shrinking of the N-dimensional space $\mathbb{R}^N$ at a rate α.

Having built a tool to characterize distributions with any finite resolution, let us now consider an evolution which we shall show can yield near-ideal universal broadcasting of classical distributions.

As we shall not require any extra "work space" we only have two random variables x and y. Initially, these are the variables of the input distribution $p_{\text{in}}(x)$ and the standard resource $g(y)$ (the analogue to the blank sheet in a photocopy machine), respectively. At the end of the evolution, these are the random variables for the first and second system, respectively.

Consider the Liouville evolution described by

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = x, \qquad t \in [0, 1]. \quad (18)$$

Integrating forward from $t = 0$ to 1 yields

$$x(1) = x(0), \qquad y(1) = y(0) + x(0), \quad (19)$$

or, in terms of the evolution on the distributions, this corresponds to

$$p_{\text{in}}(x)g(y) \to p_{\text{out}}(x, y) = p_{\text{in}}(x)g(y - x). \quad (20)$$

The marginal distribution for the first system is

$$p_{\text{copy}}^{(1)}(x) = \int dy\, p_{\text{out}}(x, y) = p_{\text{in}}(x), \quad (21)$$

which exactly reproduces the input distribution with unit fidelity. For the second the marginal distribution is

$$\begin{aligned} p_{\text{copy}}^{(2)}(y) &= \int dx\, p_{\text{out}}(x, y) = \int dx\, p_{\text{in}}(x)g(y - x) \\ &= \int dx\, p_{\text{in}}(y - x)g(x), \end{aligned} \quad (22)$$

which is just a convolution between the input distribution and the initial distribution of the universal broadcasting machine's "blank sheet."

Recall that the (Bhattacharyya) fidelity between a pair of distributions p and q is given by [5]

$$F(p, q) = \int dx \sqrt{p(x)q(x)}. \quad (23)$$

However, since the following inequality holds between the fidelity and the trace norm [5]

$$F(p, q) \geq 1 - d(p, q), \quad (24)$$

we shall see that it will be sufficient to compare the trace norm between our second system $p_{\text{copy}}^{(2)}$ and the input distribution p_{in}—which is just the optimal probability to distinguish between them.

Further, since we will only ask our universal broadcasting machine to attempt to suitably broadcast inputs with some limited finite resolution, we shall suppose that there is some fixed ϵ, such that for any input p_{in} we give to the machine

$$\epsilon[p_{\text{in}}] \geq \epsilon > 0, \quad (25)$$

is satisfied. Thus, we have

$$\begin{aligned} d(p_{\text{in}}, p_{\text{copy}}^{(2)}) &\equiv \frac{1}{2}\int dx |p_{\text{in}}(x) - p_{\text{copy}}^{(2)}(x)| \\ &= \frac{1}{2}\int dx \left| \int dy g(y)[p_{\text{in}}(x) - p_{\text{in}}(x - y)] \right| \\ &\leq \frac{1}{2}\int dx dy g(y)|p_{\text{in}}(x) - p_{\text{in}}(x - y)| \\ &= \int dy g(y) \frac{1}{2}\int dx |p_{\text{in}}(x) - p_{\text{in}}(x - y)| \\ &\leq \frac{1}{\epsilon[p_{\text{in}}]}\int dy |y| g(y) \leq \frac{1}{\epsilon}\int dy |y| g(y). \end{aligned} \quad (26)$$

So we have proved that

$$F(p_{\rm in}, p^{(2)}_{\rm copy}) \geq 1 - \frac{1}{\epsilon}\int dy|y|g(y). \qquad (27)$$

Thus, by choosing the initial distribution of the universal broadcasting machine's resource $g(y)$ to be sufficiently narrow, we may make this fidelity as high as we wish while operating on any inputs $p_{\rm in}$ whose resolution is bounded.

In particular, taking $g(y)$ as a Gaussian distribution of width σ, centered at the origin, we have

$$\int dy|y|g(y) = \sqrt{\frac{2}{\pi}}\sigma. \qquad (28)$$

So by choosing $\sigma = \sqrt{\pi/2}\nu\epsilon$ the fidelity for either state $p^{(j)}_{\rm copy}$ relative to in the input $p_{\rm in}$ is

$$F(p_{\rm in}, p^{(j)}_{\rm copy}) \geq 1 - \nu, \qquad j = 1, 2. \qquad (29)$$

Now taking $\alpha = \ln(1/\pi\nu)$ in Eq. (17) allows us to prepare the machine state g with a sufficiently narrow resolution so as to achieve Eq. (29) from a Gaussian prepared with resolution ϵ. Thus, since ν may be chosen to be as small as we wish, we may obtain broadcast copies with as high a fidelity as we wish, yet still excluding infinitely narrow resources for the machine's operation.

Daffertshofer *et al.* have shown that it is impossible to perfectly broadcast a classical state with continuous degrees of freedom [4]. However, by relaxing the expectations of our broadcasting machine we have shown that we can have essentially an ideal classical universal broadcasting machine for continuous degrees of freedom. Our classical machine can broadcast near unit fidelity copies of any input probability distribution, provided only that these distributions have some fixed bound to their resolution. Thus, provided one uses a suitable regularization scheme, such as, for example, the notion of ϵ resolution, there is no inconsistency with the infinite-dimensional limit of the broadcasting of quantum states [2].

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Appendix.—Here we provide the ϵ resolution for some interesting examples with,

Lévy distribution.—The symmetric Lévy distribution is given by

$$p_{\rm Lévy}(x) = \frac{1}{2\pi}\int dk e^{ikx} e^{-\gamma^\alpha |k|^\alpha}, \qquad (A1)$$

where the index of stability $\alpha \in (0, 2]$. When $\alpha = 1$ it produces a Cauchy distribution and when $\alpha = 2$ it yields a Gaussian distribution, but it is always normalizable. Our Theorem 3 allows us to compute the ϵ resolution as

$$\epsilon[p_{\rm Lévy}] = \frac{\pi\gamma}{\Gamma(1 + 1/\alpha)}, \qquad (A2)$$

which as might be expected is just the width parameter γ up to a prefactor. As noted, the ϵ resolution is well defined even when the standard deviation is not.

Sawtooth distribution.—Consider the asymmetric sawtooth distribution

$$p_{\rm Sawtooth}(x) = \begin{cases} \frac{2}{w}(1 - \frac{x}{w}), & 0 \leq x \leq w, \\ 0, & \text{otherwise.} \end{cases} \qquad (A3)$$

A short calculation yields $\epsilon[p_{\rm Sawtooth}] = w/4$, which is half the reciprocal height and hence twice as narrow a resolution as compared to that of a symmetric sawtooth distribution with the same base width and height. It appears that the asymmetric vertical edge yields a more highly resolvable feature for its height under the action of displacements. Further, this example shows that for nonsymmetric or nondecreasing distributions that the computation of the ϵ resolution may become nontrivial, since we can no longer rely on the above theorem.

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