Five-wave-packet linear optics quantum-error-correcting code

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In this article we outline a method for generating linear optics circuits that encode quantum-error-correcting codes. Using this method we produce a single-error-correcting code encoding one wave packet over five which can be implemented using linear optics and feed-forward correction. This code improves on the capacity of the best known code that can be implemented using linear optics and saturates the lower bound for the number of carriers needed for a single-error-correcting code. Our code can correct arbitrary single errors that occur *randomly* on each wave packet corresponding to a non-Gaussian error model, thus circumventing the so-called no-go theorem for Gaussian quantum-error correction.

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I. INTRODUCTION

Quantum information can be encoded on a variety of systems. It is common to focus on individual information carriers. Working with the underlying Hilbert-space formalism, the only feature of a specific physical implementation that is used is the Hilbert space dimension of the system (the number of distinct states of the system). Thus we may characterize the fundamental information carriers through their dimension.

The simplest (and most common) of quantum information carriers is the qubit—having dimension two. More generally, for finite dimensions, the information carrier would be the qudit—of dimension d. The simplest way to generalize to infinite dimensions is to use so-called continuous-variable systems which, for example, can be used to describe the states of a harmonic oscillator. Although continuous-variable systems can be implemented in a number of ways [1], we will henceforth restrict our attention to those formed by the quantum state of the transverse mode structure of electromagnetic radiation—corresponding to the states typically analyzed in so-called quantum optics [1,2].

While various types of quantum information can have implementations differing dramatically, one feature common to all is susceptibility to noise. As a consequence of this susceptibility techniques for protecting against noise are needed. One such technique is quantum-error correction [3,4] which was first formulated for qubits and later extended to qudits [5] and continuous variables [6,7]. Naively, it may appear that quantum-error correction on continuous variables is more difficult than for their discrete counterparts as there is a much wider class of errors that can occur. However, quantum-error-correction techniques can be applied to continuous variables in much the same way as for discrete variables, providing protection against arbitrary errors on, say, up to a single continuous variable—a single subsystem, sometimes called a single wave packet.

The first continuous-variable quantum-error-correcting codes relied on operations analogous to the commonly used exclusive-OR (XOR) operation for qubits [6,7]. While these generalized XOR operations can be performed in a number of continuous-variable schemes their optical implementation can be very challenging as it relies on nonlinear optics (or so-called active optical components). However, it has been shown subsequently that it is possible to perform continuous-variable quantum-error correction using only linear optics and squeezed auxiliary states [8]. This discovery pushed quantum-error correction for continuous variables into the realm of current experimental feasibility. Indeed, the nine-wave-packet continuous-variable linear optics code has recently been implemented experimentally [9].

It has been shown that Gaussian operations, such as the passive linear optics components used in our protocol, cannot be used to implement the correction of a *Gaussian* error model [10]. However, our error model corresponds to stochastic errors occurring on each wave packet, which can occur in situations such as atmospheric transmission [11]. Thus, the Gaussian no-go theorem does not apply to our code.

To date, the only known continuous-variable code that can be implemented via linear optics requires nine wave packets, while the smallest known continuous-variable code based on active components is a five wave-packet code [6] saturating the lower bound on the number of carriers for single-error correction [12]. Smaller codes offer many advantages, such as increased data rates or decreased chance of further errors occurring. In this article we investigate the construction of such a compact code based on linear optics. Previous approaches to building continuous-variable codes have focused typically on analogies to existing qubit codes. The original continuous-variable codes [6,7] were constructed by producing continuous-variable versions of qubit-error-correcting circuits. The known linear optical variant is based on the realization that it is possible to construct states similar to the highly entangled nine wave-packet continuous-variable codewords produced by active components [7,13] using only linear optical components [8]. This current article uses a different approach to achieve a linear optics construction for a five-wave-packet code: we define the conditions under which a linear optical circuit will yield a continuous-variable quantum-error-correcting code. We then search numerically for circuits satisfying this criterion. This approach is similar in spirit to that of constructing stabilizer codes [14] since the codes are constructed to satisfy error-correcting properties.

This article is structured as follows: continuous-variable error-correcting codes and their linear optics implementations are briefly reviewed. We then provide the criterion for a linear optics circuit to encode a continuous-variable quantum-error-correcting code and discuss how this was used to search for suitable circuits. Following this we discuss methods used to improve the codes produced with respect to efficient

implementation and present the "best" code found together with its encoding circuit.

A. Linear optics

Linear optics circuits consist of beam-splitters and phase delays in any combination. Since they act linearly on annihilation or creation operators it is convenient to work in the Heisenberg picture. For two input modes with annihilation operators $\hat{a}_1$ and $\hat{a}_2$ sent into an arbitrary phase-free beam-splitter, the output modes will be given by

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}. \tag{1}$$

Often, the parameter, θ, is specified by an energy or intensity transmission coefficient $\cos^2\theta$. To remove any ambiguity, $\cos\theta$ and $\sin\theta$ are taken to be positive throughout this article whenever beam-splitters are specified in this manner. Phase delays multiply the annihilation operator by a phase factor $e^{i\phi}$, giving

$$\hat{a}' = e^{i\phi}\hat{a}. \tag{2}$$

To preserve the canonical commutation relations among the annihilation and creation operators any linear optics operator must have the form

$$\hat{a}_i' = \sum_j U_{ij}\hat{a}_j, \tag{3}$$

where $U = (\!(U_{ij})\!)$ is an $n \times n$ unitary matrix. Reck *et al.* [15] have shown that any such transformation can be performed by a fixed configuration of $n(n-1)/2$ beam-splitters and $n(n+1)/2$ phase delays with appropriate parameters. The generic configuration they use is shown in Fig. 1.

II. ERROR MODEL

In order to discuss error correction, we first need to discuss the error model at which it is targeted. The error model has a significant impact on the performance of different types of correction. One of the most common models assumes that each primitive error simultaneously affects e distinct information carriers out of a block of n carriers and occurs stochastically with some probability.

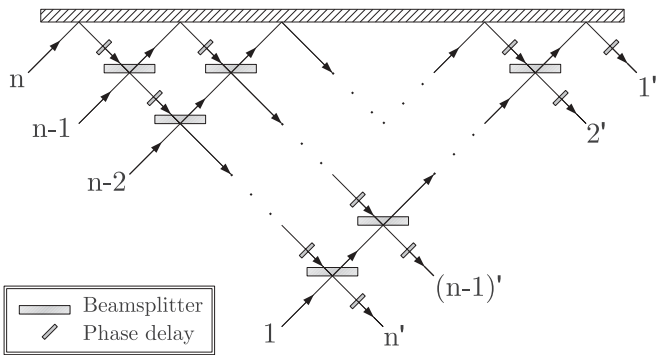

FIG. 1. A linear optics circuit that, depending on the choice of the n^2 parameters for the beam-splitters and phase delays, can perform any linear optics operation involving n modes [15]. The numbers 1 to n and $1'$ to n' label the input and output modes, respectively.

One method of achieving robustness against this type of error is via error-correcting codes. The information is encoded over many carriers within blocks that allow correction when errors occur on a limited number of the underlying carriers. This technique was first applied to qubits by Shor [3] and independently shortly afterward by Steane [4] and then extended to qudits [5]. Similarly this type of error model, which is, for example, responsible for stochastic losses of optical signals traveling through the atmosphere [11], can be corrected for continuous-variables quantum systems. However, other error models common in continuous-variable scenarios may need other strategies, since codes do not allow for correction of Gaussian-type errors, where each mode has a small Gaussian error rather than a small probability p of a (possibly) large error occurring on that mode, using only linear optics [16]. In the following section we briefly outline the extension of such stochastic-error-model quantum-error-correcting codes to continuous variables.

A. Error correction over continuous variables

Since continuous variables can take on a range of values over the whole real line the range of errors that can occur is likewise infinitely larger than for discrete variables. Given this, one might expect that quantum-error correction on continuous variables may be more difficult or impossible. However, it turns out that such quantum-error correction can be implemented in much the same way as for discrete variables. This was first shown independently by Braunstein [6] and Lloyd and Slotine [7].

As with discrete quantum-error correction, it is enough to be able to correct for a much more limited class of errors [6]. As with qubits, it is sufficient to consider the most general error on a wave packet (an individual continuous-variable information carrier) to be written as a function of the position and momentum operators $\hat{E}(\hat{x},\hat{p})$ transforming an input state $|\psi\rangle$ via

$$|\psi\rangle \to \hat{E}(\hat{x},\hat{p})|\psi\rangle. \tag{4}$$

If E is described in terms of its Fourier transform $\tilde{E}$ on both variables, the state with error may be written as

$$\frac{1}{\pi}\int du\,dv\; e^{i(u\hat{x}+v\hat{p})}\tilde{E}(u,v)|\psi\rangle. \tag{5}$$

The operator $e^{i(u\hat{x}+v\hat{p})}$ corresponds to a shift in position by v and a kick in momentum by u. Thus, the state is a superposition of various phase-space displacements of the original state $|\psi\rangle$. If the values of u and v could be measured, then the superposition would collapse into one in which the state was displaced by known values, allowing correction by reversing the displacement.

Let us consider for simplicity a single-wave-packet *position* error that can be written as a function of the momentum operator $\hat{E}(\hat{p})$. The effect of such an error on a position eigenstate would be [6]

$$\begin{aligned}\hat{E}(\hat{p})\,|x\rangle &= \frac{1}{\pi}\int dy\,dp\, e^{2ip(y-x)}E(p)\,|y\rangle \\ &= \int dy\,\tilde{E}(y)|x-y\rangle,\end{aligned} \tag{6}$$

where, $\tilde{E}(y)$ is the Fourier transform of $E(p)$. So a position error will produce a superposition of the original state shifted by various y's. If this shift can be determined then the superposition of shifts would collapse into the original state shifted by a single known value. The following encoding allows just such a measurement for position errors (using two zero-position eigenstate auxiliary modes)

$$|x\rangle\,|0\rangle\,|0\rangle \rightarrow |x\rangle\,|x\rangle\,|x\rangle, \tag{7}$$

it is sufficient to see this in action on an encoded basis state $|x\rangle$. If a position error now occurs on the first encoded wave packet, say, then the overall state with the error will be

$$\int dy\, \tilde{E}(y)\,|x-y\rangle\,|x\rangle\,|x\rangle. \tag{8}$$

Correction is achieved by measuring the results of subtracting the wave packets' positions via homodyne detection [17] from three pairs (1,2), (2,3), (3,1) to form a syndrome vector. In the case of a correctable error, the difference between the pair not involving the corrupted wave packet will be zero identifying the location of the error. In the case of the error in Eq. (8) the syndrome vector $(-y,0,y)$ is found for some y with probability density $|\tilde{E}(y)|^2$ and the state has collapsed to one in which a single shift of size y has occurred on the first wave packet, $|x-y,x,x\rangle$. Thus we can recover the original encoded state by applying a shift of $-y$ to the first wave packet.

In a similar manner, this procedure could be applied to provide correction against momentum errors using the analogous code in the momentum basis. By applying the position correction encoding to a single wave packet, and the momentum correction encoding to each of the outputs of the position encoding it is possible to provide correction for a single arbitrary error the type in Eq. (5) [8].

B. Quantum-error correction using linear optics

The original continuous-variable codes [6,7] make use of nonlinear operations, such as the subtraction of wave packets' positions used in the syndrome measurement above. In optical schemes this operation is nonlinear as it does not preserve the total number of photons. Because nonlinear operations are typically much more difficult to implement it is worthwhile understanding to what extent they may be replaced by purely linear optics operations. Indeed, it turns out that it is possible to construct quantum-error-correcting codes, to correct against a stochastic-error model of the type described above that need only linear optics and a source of squeezed states [8].

The encoding of Eq. (7) is nonlinear, but by scaling the codeword it may replace it with the encoding

$$|x\rangle \rightarrow \left|\frac{1}{\sqrt{3}}x\right\rangle\left|\frac{1}{\sqrt{3}}x\right\rangle\left|\frac{1}{\sqrt{3}}x\right\rangle. \tag{9}$$

This encoding can be performed using linear optics. In particular, the pair of beam-splitters shown in Fig. 2 combine to form a so-called tritter arrangement that maps state $|x\rangle$ in mode 1 and two auxiliary zero-position eigenstates $|0\rangle\,|0\rangle$ in modes 2 and 3 into the encoding of Eq. (9).

Readout and correction peformed may be by running the encoding circuit (in this case the tritter) "backwards" and making position readouts on the auxiliary modes.

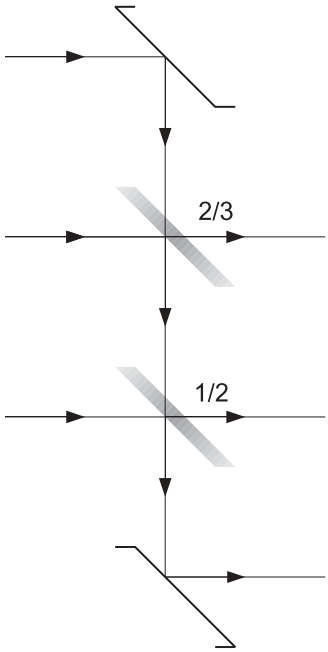

FIG. 2. An arrangement of beam-splitters representing the tritter circuit, encoding a state to protect against position errors. The numbers here (2/3 or 1/2) next to each phase-free beam splitter denote the energy transmission coefficient for that component.

For a single-wave-packet error, after running the tritter backward the output will be one of

$$\int dy\, \tilde{E}(y)\left|x+\sqrt{\frac{1}{3}}y\right\rangle\left|-\sqrt{\frac{2}{3}}y\right\rangle|0\rangle\,, \tag{10}$$

$$\int dy\, \tilde{E}(y)\left|x+\sqrt{\frac{1}{3}}y\right\rangle\left|\sqrt{\frac{1}{6}}y\right\rangle\left|-\sqrt{\frac{1}{2}}y\right\rangle, \tag{11}$$

$$\int dy\, \tilde{E}(y)\left|x+\sqrt{\frac{1}{3}}y\right\rangle\left|\sqrt{\frac{1}{6}}y\right\rangle\left|\sqrt{\frac{1}{2}}y\right\rangle, \tag{12}$$

corresponding to errors on the first, second, and third wave packets, respectively. Once the two auxiliary modes are measured in the position basis the error "collapses" to produce the syndrome pattern. In the above cases these are

$$\left(\sqrt{\frac{2}{3}}y,0\right), \tag{13}$$

$$\left(\frac{1}{\sqrt{6}}y,-\frac{1}{\sqrt{2}}y\right), \tag{14}$$

$$\left(\frac{1}{\sqrt{6}}y,\frac{1}{\sqrt{2}}y\right), \tag{15}$$

respectively, for some y. Since these patterns are maximally distinguishable for $y \neq 0$, the location and size of the error may be optimally identified from the syndrome. Once the collapsed error has been identified it may be corrected using a shift of $-y/\sqrt{3}$ in each case. Note that the $y=0$ case reduces to no error and so no correction is required.

The above three-wave-packet encoding can correct a single position error. It may be transformed into an analogous single

momentum error-correcting code by phase delays of $\pi/2$—which effectively interchanges x and p. By concatenating these codes a nine-wave-packet code analogous to the nine-qubit Shor code [3] can then be constructed.

So far, the operation of continuous-variable error-correction-codes have only been discussed under ideal circumstances—with perfect zero-position eigenstates as auxiliaries. Since these ideal auxiliaries are physically unrealizable they must be replaced with some realizable approximation in any actual implementation, typically position squeezed vacuum states [18]. This approximation introduces noise into the measured syndromes. This leads to two potential problems: The first is that the noise in a measured syndrome may move it far enough from the correct syndrome that it ends up being mistaken for the syndrome of a different error. If this occurs then we would "correct" for the wrong error which will typically result in a failure of the correction procedure. In the case of the position code above, this will be avoided if the width of the auxiliaries are smaller in magnitude than the collapsed error y, so the distributions of the auxiliary states must be sufficiently narrow compared to the size of the smallest error to be corrected.

The second potential problem, is that the accuracy of correction is limited by the precision with which the syndrome can be determined. This in turn depends on the width of the initial auxiliary states. In the ideal case of perfect syndrome measurement we would end up with the pure state

$$\int dx\, dx'\, \psi^*(x)\psi(x')\,|x'\rangle\,\langle x|\,. \tag{16}$$

However, when we have imprecise syndrome measurement we would end up with

$$\int dx\, dx'\, d\eta\, \psi^*(x)\psi(x')p(\eta)\,|x'-\eta\rangle\,\langle x-\eta|\,, \tag{17}$$

where $p(\eta)$ is the probability distribution over the linearly combined random variable $(s_1v_1+s_2v_2)/[\sqrt{3}(s_1^2+s_2^2)]$, where s_1 and s_2 are the constant factors multiplying y in Eq. (15) and v_1 and v_2 are distributed according to the position noise in the initial auxiliary states. Thus, in order to avoid failure of the correction procedure, $p(\eta)$ must be narrow compared to any important length scales in the unencoded wave function $\psi(x)$, i.e., the code cannot protect a state with features smaller than the initial widths of the auxiliary states.

Linear optics quantum-error-correcting codes allow for any quantum state to be protected against a given set of stochastic errors. However, in certain scenarios, where only a limited set of states need to be protected, then it may be possible to exceed the bounds imposed by the requirements of general error correction. For example, quantum erasure correcting codes for continuous variables allow for the protection of coherent states against probabilistic losses while needing only four wave packets [19] rather than the five required for full quantum-error correction [12].

The discovery of linear optics circuits for continuous-variable quantum-error correction significantly simplifies its implementation. Indeed, the above-mentioned nine-wave-packet linear optics code has already been implemented experimentally [9]. However, this code needs nine optical modes, while the best known continuous-variable code using active (nonlinear) operations requires only five modes [6]. It would be desirable, therefore, to have a linear optics continuous-variable code using a five-wave-packet encoding. We construct such a code below.

III. CONSTRUCTION OF A LINEAR OPTICS CONTINUOUS-VARIABLE QUANTUM-ERROR-CORRECTING CODE

In this section we explain our methodology for constructing linear optics continuous-variable quantum-error-correcting codes. We start by describing the form of the error-correction circuits—the key point being that they will operate in a manner similar to the original nine-wave-packet code [8]. We then derive the criterion that needs to be satisfied for a successful error-correction circuit for both nondegenerate and degenerate codes. Next we discuss how this criterion may be effectively evaluated. Finally, we describe how we go about the numerical search for examples of successful linear optics continuous-variable quantum-error-correcting codes.

A. Form of the linear optics quantum-error-correcting code

The codes we seek should operate in the same manner as the linear optics nine-wave-packet code [8]. An n-wave-packet code would encode m wave packets using $n-m$ auxiliary states, ideally zero position eigenstates. A linear optics circuit (an n-mode interferometer) described by an $n\times n$ unitary matrix U then maps the m initial "message" wave packets and $n-m$ auxiliary modes into an n-wave-packet code. In the simplest case a single wave packet ($m=1$) is encoded into an n-wave-packet code.

We shall suppose that an arbitrary error may occur on some limited number of wave packets (only one in the simplest nontrivial case) after which decoding, syndrome detection, and appropriate corrective feed-forward steps are performed. For production of the syndrome the encoding circuit is run backward (effectively running the "inverted" interferometer described by the matrix $U^\dagger$). In the absence of an error the auxiliary modes would be mapped back to $n-m$ zero position eigenstates. If an error occurred before decoding these $n-m$ modes it would be mapped to nonzero values (in the position basis). Therefore measuring the positions of the $n-m$ auxiliary modes yields a syndrome for a properly designed code. After syndrome recognition an arbitrary syndrome error should be reduced to a known position shift and momentum kick on identified wave packets. This phase-space displacement may be subsequently reversed via a feed-forward action based on the classical information obtained during syndrome measurement and recognition.

It remains to determine suitable linear optics circuits, described by their unitary matrix U, which can achieve this. The following section discusses the criterion for the unitary U that must be satisfied for a successful continuous-variable quantum-error-correcting circuit.

B. Criterion for a circuit to perform continuous-variable quantum-error correction

Here we construct in detail the criterion [20] that a linear optics circuit must satisfy in order to represent an error-correcting

circuit as outlined above. As discussed above, any linear optics circuit acting on n modes is equivalent to an $n \times n$ unitary matrix U.

We start with a total of n modes. Their annihilation operators $\hat{a}_i$, will be transformed by the encoding circuit U into annihilation operators $\hat{b}_i$ satisfying

$$\hat{b}_i = \sum_{j=1}^{n} U_{ij}\hat{a}_j. \tag{18}$$

Now, suppose some error occurs to the encoded state on wave packets with indices from some set ϵ. After such an error occurs the annihilation operators are denoted by $\hat{c}_i$ and related to the encoded modes $\hat{b}_i$ by

$$\hat{c}_i = \begin{cases} \hat{b}_i, & i \notin \epsilon \\ f_i(\hat{b}_{\epsilon_1}, \ldots, \hat{b}_{\epsilon_l}), & i \in \epsilon, \end{cases} \tag{19}$$

where the f_i are arbitrary (even nonlinear) functions of the relevant annihilation and creation operators. In order to examine the effect of the errors, it is useful to introduce operators $\hat{e}$ representing the change in the encoded operator as a consequence of an error. The $\hat{c}_i$ may therefore be rewritten as

$$\hat{c}_i = \hat{b}_i + \hat{e}_i, \tag{20}$$

with $\hat{e}_i = \hat{c}_i - \hat{b}_i$, so the $\hat{e}_i$ are zero for modes not affected by the error.

Next, the state is decoded by the circuit corresponding to $U^\dagger$, yielding annihilation operators $\hat{d}_i$ satisfying

$$\begin{aligned} \hat{d}_i &= \sum_{j=1}^{n} U_{ij}^\dagger \hat{c}_j = \sum_{j=1}^{n} U_{ij}^\dagger (\hat{b}_j + \hat{e}_j) \\ &= \hat{a}_i + \sum_{j=1}^{n} U_{ij}^\dagger \hat{e}_j. \end{aligned} \tag{21}$$

Decomposing the error into its effect on position and momentum operators, i.e., writing $\hat{e}_i = (\hat{x}_i + i\,\hat{p}_i)/\sqrt{2}$, allows the decoded modes $\hat{d}_i$ to be expressed as

$$\begin{aligned} \hat{d}_i &= \hat{a}_i + \sum_{j\in\epsilon} U_{ij}^\dagger \hat{e}_j \\ &= \hat{a}_i + \frac{1}{\sqrt{2}} \sum_{j\in\epsilon} (U_{ij}^\dagger \hat{x}_j + i U_{ij}^\dagger \hat{p}_j). \end{aligned} \tag{22}$$

For a code that encodes m wave packets into n, the syndrome is generated by measuring the position of wave packets $m+1$ through n (the last $n-m$ wave packets)

$$\begin{aligned} S_{i-m} &= \frac{\hat{d}_i + \hat{d}_i^\dagger}{\sqrt{2}} \\ &= \hat{x}_i^{\text{input}} + \sum_{j\in\epsilon} [\mathrm{Re}(U_{ij}^\dagger)\hat{x}_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}_j], \end{aligned} \tag{23}$$

for $m+1 \leqslant i \leqslant n$. Ideally, the auxiliary modes are initially zero position eigenstates, thus $\hat{x}_i^{\text{input}}$ may be replaced by its eigenvalue 0, leaving the components of the syndrome operator

$$\hat{S}_{i-m} = \sum_{j\in\epsilon} [\mathrm{Re}(U_{ij}^\dagger)\hat{x}_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}_j]. \tag{24}$$

We shall suppose the correctable errors correspond to arbitrary errors on up to l wave packets—a successful code encoding m wave packets into n might be labeled as an $[[n,m,2l+1]]$ continuous-variable quantum-error-correcting code. To achieve this we require that every syndrome that can be generated by such errors leads to a unique correction, i.e., that the set does not contain two errors that have the same syndrome but have a different collapsed error on the output state.

If any error can be expressed in terms of the $\hat{x}_i$ and $\hat{p}_i$ changes on the affected modes then it is sufficient to consider the errors for each mode by breaking them down into combinations of these variables. Suppose we have a set of errors that are not correctable for the code represented by the matrix U. In this case there must be two errors, say $\hat{e}$ and and $\hat{e}'$, with the same syndrome but requiring different corrections. If $\hat{e}$ consists of a shift of $\hat{x}_i + i\,\hat{p}_i$ on the ith mode with only the modes $i \in \epsilon$ being nonzero, similarly $\hat{e}'$ consists of shifts of $\hat{x}'_j + i\,\hat{p}'_j$ for $j \in \epsilon'$. The equal components of the syndrome may then be written as

$$\begin{aligned} \hat{S}_{i-m} &= \sum_{j\in\epsilon} [\mathrm{Re}(U_{ij}^\dagger)\hat{x}_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}_j] \\ &= \sum_{j\in\epsilon'} [\mathrm{Re}(U_{ij}^\dagger)\hat{x}'_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}'_j], \end{aligned} \tag{25}$$

with the output modes not all equal, i.e.,

$$\begin{aligned} \hat{d}_i &= a_i + \frac{1}{\sqrt{2}} \sum_{j\in\epsilon} (U_{ij}^\dagger \hat{x}_j + i U_{ij}^\dagger \hat{p}_j) \\ &\neq \\ \hat{d}'_i &= a_i + \frac{1}{\sqrt{2}} \sum_{j\in\epsilon'} (U_{ij}^\dagger \hat{x}'_j + i U_{ij}^\dagger \hat{p}'_j), \quad \text{for some } i \leqslant m. \end{aligned} \tag{26}$$

Thus, a set of errors is uncorrectable if there exist two errors $\hat{e}$ and $\hat{e}'$ in the set such that

$$\sum_{j\in\epsilon} [\mathrm{Re}(U_{ij}^\dagger)\hat{x}_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}_j] = \sum_{j\in\epsilon'} [\mathrm{Re}(U_{ij}^\dagger)\hat{x}'_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}'_j], \tag{27}$$

for $m+1 \leqslant i \leqslant n$, but

$$\sum_{j\in\epsilon} (U_{ij}^\dagger \hat{x}_j + i U_{ij}^\dagger \hat{p}_j) \neq \sum_{j\in\epsilon'} (U_{ij}^\dagger \hat{x}'_j + i U_{ij}^\dagger \hat{p}'_j), \tag{28}$$

for some $1 \leqslant i \leqslant m$.

The above conditions are equivalent to the statement that all errors up to some maximum weight l will be correctable if all of the solutions of

$$\sum_{j\in\epsilon} \mathrm{Re}(U_{ij}^\dagger)\hat{x}_j - \mathrm{Im}(U_{ij}^\dagger)\hat{p}_j = 0, \tag{29}$$

for auxiliary modes with $m+1 \leqslant i \leqslant n$, with weight less than or equal to $2l$ [the maximum combined weight from $\hat{e}$ and $\hat{e}'$ in Eq. (27)] are also solutions of

$$\sum_{j\in\epsilon'} U_{ij}^\dagger \hat{x}_j + i U_{ij}^\dagger \hat{p}_j = 0, \tag{30}$$

for all output modes $1 \leqslant i \leqslant m$. The converse is also true [i.e., if all errors up to weight l are correctable then solutions

to Eq. (29) are also solutions to Eq. (30)], since if this were not the case we could write the solution satisfying Eq. (29) but not Eq. (30) as the sum of two nonzero vectors with weight less than or equal to l. Such errors would have the same syndrome, but with opposite signs, even though the same is not true for their effect on the message. Thus, on negating one of the pair of errors, we would have two errors with the same syndrome but different effects on the output; this would make it impossible to implement an error-correction strategy based solely on the syndromes.

C. Evaluating the criterion for successful error correction

Having derived the general criterion we now consider how it can be applied in specific circumstances, starting with the nondegenerate case. For nondegenerate codes each error corresponds to a unique syndrome. Thus, there can be no nontrivial solutions to Eq. (29), since from such solutions we could extract two different errors with the same syndrome contradicting nondegeneracy. When there are no nontrivial solutions to Eq. (29) its trivial solutions will automatically satisfy Eq. (30). Therefore we can determine if we have a nondegenerate error-correcting code by checking that all correctable errors yield a unique syndrome. This check only requires the decoding circuit $U^\dagger$.

As discussed above, it is sufficient to consider errors that can be written as a superposition of phase-space displacements on each affected wave packet. The syndrome for a unit position shift on mode i corresponds to a vector consisting of the real part of the ith column of matrix $U^\dagger$ with the top m rows removed. The syndrome for a unit kick in momentum on the ith wave packet corresponds to the negative of the imaginary part of the same column. Thus, the syndrome for any single mode error will be a member of the subspace of $\mathbb{R}^{n-m}$ spanned by these two syndrome vectors. Similarly, syndromes for errors on less than k wave packets are members of the subspace spanned by the $2k$ primitive position and momentum syndromes of the corresponding modes. In the nondegenerate case, the syndromes for different errors need to be unique. Thus, the subspaces of possible syndromes for errors on different wave packets must only intersect at the origin, corresponding to the requirement that the set of syndromes must be linearly independent.

Writing the syndrome table for the code as a matrix provides a convenient method of calculating the syndrome for any error. Define the $2n \times (n-m)$ matrix

$$M \equiv \begin{pmatrix} \mathrm{Re}(U^\dagger_{(m+1),1}) & \cdots & \mathrm{Re}(U^\dagger_{n,1}) \\ -\mathrm{Im}(U^\dagger_{(m+1),1}) & \cdots & -\mathrm{Im}(U^\dagger_{n,1}) \\ \vdots & \ddots & \vdots \\ \mathrm{Re}(U^\dagger_{(m+1),n}) & \cdots & \mathrm{Re}(U^\dagger_{n,n}) \\ -\mathrm{Im}(U^\dagger_{(m+1),n}) & \cdots & -\mathrm{Im}(U^\dagger_{n,n}) \end{pmatrix}. \tag{31}$$

The syndrome for any combination of primitive position and momentum errors can be calculated by writing them as a vector of the following form

$$\vec{v}^\dagger = (x_1, p_1, x_2, p_2, \ldots, x_n, p_n), \tag{32}$$

with corresponding syndrome

$$\vec{v}^\dagger M. \tag{33}$$

If the code corrects all errors on up to l wave packets, then there can be no two vectors of weight at most l (here weight refers to the number of pairs of x_j, p_j with at least one of x or p nonzero) sharing no modes in common but that have the same syndrome. In other words, there is no solution to

$$\vec{v}^\dagger M = \vec{v}'^\dagger M, \tag{34}$$

with $\vec{v}$ and $\vec{v}'$ sharing no modes and having weight at most l unless both are zero. Equivalently, by subtracting one vector from the other, we may rewrite this criterion as saying that there is no solution $\vec{v}$ to

$$\vec{v}^\dagger M = 0, \tag{35}$$

with weight less than or equal to $2l$. It follows that any set of $2l$ pairs of x_i and p_i rows from the matrix in Eq. (31) must be linearly independent. For small l, this criterion may be checked quickly on a computer.

For degenerate codes, the possibility of nontrivial solutions to Eqs. (29) and (30) needs to be considered more carefully, since these joint solutions correspond to degeneracies—different errors with the same syndrome and collapsed error on the output and thus the same correction. Returning to the matrix form of the decoding circuit, $U^\dagger$, a unit position shift error on the jth wave packet corresponds to a complex displacement by $U^\dagger_{ij}$ on the ith output mode. Similarly, for a unit momentum kick error on the jth mode we will have a complex displacement of $iU^\dagger_{ij}$ on the ith mode. Now, we need to check that when the syndromes of primitive errors on different modes can be equal that the complex displacements at the outputs will also be equal. A convenient way to do this is to augment the matrix of syndromes in Eq. (31). As discussed above, two sets of rows that are linearly independent allows equal syndromes for different errors. If the effects of the primitive errors, $U^\dagger_{ij}$ for unit position shifts and $iU^\dagger_{ij}$ for unit momentum kicks, are appended to the matrix as follows

$$\left(\begin{array}{ccc|ccc} \mathrm{Re}(U^\dagger_{(m+1),1}) & \cdots & \mathrm{Re}(U^\dagger_{n,1}) & U^\dagger_{1,1} & \cdots & U^\dagger_{m,1} \\ -\mathrm{Im}(U^\dagger_{(m+1),1}) & \cdots & -\mathrm{Im}(U^\dagger_{n,1}) & iU^\dagger_{1,1} & \cdots & iU^\dagger_{m,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}(U^\dagger_{(m+1),n}) & \cdots & \mathrm{Re}(U^\dagger_{n,n}) & U^\dagger_{1,n} & \cdots & U^\dagger_{m,n} \\ -\mathrm{Im}(U^\dagger_{(m+1),n}) & \cdots & -\mathrm{Im}(U^\dagger_{n,n}) & iU^\dagger_{1,n} & \cdots & iU^\dagger_{m,n} \end{array}\right), \tag{36}$$

then the syndrome and effects of a combination of shift and kick errors can be calculated in the same manner as the syndrome in Eq. (33). Only now the first $n-m$ real values in the resulting vector correspond to the syndrome and the following m complex values to the effects on the output. We now require that if the syndromes for primitive errors on different wave packets can be made equal then their effects will also be equal. One method of checking this is by the use of Gaussian elimination. For every set of partitions of $2l$ modes out of n we produce a matrix similar to Eq. (36) containing only the rows corresponding to those modes. Gaussian elimination

is then applied to the left-hand side of the matrix. If, after reduction, there is a row of zeros on the left, the same row must also have zeros on the right, since rows of zeros on the left correspond to solutions to Eq. (29) while rows of zeros on the right correspond to solutions to Eq. (30).

D. Numerical search for codes

The above criterion can be used to search for linear optical circuits that will describe successful continuous-variable quantum-error-correcting codes. Recall that every linear optics circuit may be decomposed into the form shown in Fig. 1 [15]. This construction makes use of $n(n-1)/2$ beam-splitters and $n(n+1)/2$ phase delays, each of whose parameters can be taken to range over $[0,2\pi)$. To find suitable circuits we search this space of n^2 parameters for those that generate a matrix that is *not* a solution to Eq. (29). Since these parameters are continuous and the search space is large an exhaustive search is not possible, instead, an optimisation technique was used.

1. Differential evolution

For optimization, differential evolution [21] was chosen for the first attempt due to the simplicity of the algorithm. Differential evolution is a technique that attempts to find the global minimum over continuous-variable multidimensional spaces, making it suitable for optimizing over the space of n^2 parameters used to define a linear optics circuit.

Differential evolution works with a population of N_{pop} vectors containing random parameters for candidate solutions, typically these populations are referred to as generations and a new generation is created from the previous one by an update procedure. Differential evolution produces a new generation by looping over the current population. For each member of the population $\vec{r}$ three other members of the current population $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are selected at random, from these three members a vector $\vec{v}$ is formed according to

$$\vec{v} = \vec{u}_1 + F(\vec{u}_2 - \vec{u}_3). \tag{37}$$

Where F is a real number that controls the amplification of the variation. From the initial member $\vec{r}$ and newly created vector $\vec{v}$ a candidate $\vec{c}$ is constructed based on a crossover rate R_c with $0 < R_c \leqslant 1$. For each element of $\vec{c}$ there is a $1 - R_c$ probability that it has value equal to the equivalent member of $\vec{r}$ and a R_c probability it will be equal to the equivalent member of $\vec{v}$. Once $\vec{c}$ has been constructed we calculate its fitness using some cost function and compare it to the cost of the original member $\vec{r}$. If the cost of $\vec{c}$ is less than that of $\vec{r}$ then in the next generation $\vec{r}$ will be replaced by $\vec{c}$.

New generations are created until a particular criterion is satisfied.

2. Original cost function

The initial aim of the search was to produce any circuit capable of performing quantum-error correction with no regard for performance. Thus, the cost function was chosen to reflect the criterion for nondegenerate codes given in Eq. (29). However, the optimization requires that the cost reflects how suitable a particular candidate is rather than simply providing a Boolean criterion. Thus the cost function was designed to try and take into account how "near" a set of parameters is to a single-error-correcting code. This was achieved by taking the cost to be the sum of the number of linearly independent solutions to Eq. (29) over all possible pairs of modes. Thus, codes with fewer uncorrectable single-wave-packet errors would have a lower cost.

3. Progress of optimization

For the first attempt the population N_{pop} was chosen to be 1000, an amplification factor F as $1/2$ and the crossover rate R_c as $1/2$. Initially the algorithm was set to finish after a thousand generations; however, most of the time a successful code was found in either the first or second generation. Thus, allowing the problem to be solved by random search. Note, however, that this cost function only reflects the criterion that the circuit corresponds to an error-correcting code in the ideal case of auxiliary modes that are perfect zero position eigenstates. In order for a code to be of practical value it would need to still be capable of performing error correction under less than ideal circumstances. The next section examines a number of improvements made to the cost function to allow it to produce codes that may be more effectively experimentally realizable.

IV. FURTHER OPTIMIZING THE CODES PRODUCED

This section looks at improving generated circuits in two areas. First, improving the code represented by the circuit under nonideal conditions. Second, optimizing the circuit with respect to difficulties in implementing it.

A. Optimizing the code

The optimization of the code focuses on two aspects that are typically ignored in discrete codes. The first involves the separation of syndromes, in an attempt to improve the probability of correctly identifying the error under nonideal conditions. The second attempts to minimize the amount of noise added to the output as a consequence of performing the correction under nonideal conditions.

1. Separation of syndromes

Under ideal circumstances, including all auxiliary modes prepared in perfect zero position eigenstates, the different syndromes in our linear optics continuous-variable quantum-error-correcting code would be exactly distinguishable. However, any imperfection, such as using only finitely squeezed states for the auxiliary modes, will lead to some uncertainty or noise in the syndrome readout (an issue typically ignored in discrete-variable quantum-error-correcting codes). Now, the requirement of Eq. (29) that any set of syndromes are linearly independent does not place any constraint on the angle between those syndromes. For example both of the sets $\{(1,0),(\cos\theta,\sin\theta)\}$ with θ small but greater than zero, and $\{(1,0),(0,1)\}$ are linearly independent but the angle between the vectors in the first set is very small, while in the second this angle is maximal.

Clearly, it would be worthwhile, in addition to the ideal linear independence enshrined in Eq. (29) between the distinct syndromes, to maximize the angles between all pairs of syndromes corresponding to different errors (in the case

of nondegenerate codes). This will minimize the chance of misidentification of the syndromes and hence minimize the chance of performing the wrong correction step.

For an n wave-packet code encoding m modes, an error on a single mode has a syndrome belonging to a two-dimensional subspace of $\mathbb{R}^{n-m}$ spanned by the syndromes corresponding to shifts in position and momentum. In order to minimize the chance of misidentification, the angle between any vector in this subspace and any vector in a subspace corresponding to an error on another mode should be as large as possible. In other words, defining the subspace corresponding to each error as S_i we seek to minimize

$$\cos\theta_{ij} = \max_{\substack{\vec{u}\in S_i,\,\vec{v}\in S_j \\ \|\vec{u}\|_2=\|\vec{v}\|_2=1}} \vec{v}^T\vec{u}, \tag{38}$$

over all pairs of modes i, j. This can be calculated using the matrix-2-norm [22], defined by

$$|A|_2 = \max_{|x|_2=1} |Ax|_2, \tag{39}$$

i.e., the square of the maximum eigenvalue of A. Defining P_i as the projector onto S_i then

$$\cos\theta_{ij} = |P_iP_j|_2 = |P_jP_i|_2. \tag{40}$$

This part of the cost function in the numerical search was therefore chosen to be the largest value of $\cos\theta_{ij}$ over all distinct errors i, j so that it reflects the worst case of syndrome overlap.

2. *Optimizing the correction procedure for nonideal auxiliary modes*

Once the modes on which the error occurred have been determined the correction needs to be performed, the effect the error has on the output is calculated so that it can be reversed. The effect on the output is calculated by finding a linear combination of x and p syndromes on the affected modes that equals the measured syndrome (syndrome recognition). Once this combination is found the effect of the error is equal to a linear combination of the effects of the x and p shifts on each mode with the same coefficients.

In the case where the auxiliary modes are only finitely squeezed, the measured syndrome will not correspond exactly to the error since there will be noise as a result of the auxiliary states not being ideal. The measured syndrome can then be written as

$$\vec{S}_{\text{ideal}} + \vec{\mathcal{N}}, \tag{41}$$

where $\vec{S}_{\text{ideal}}$ is the ideal syndrome corresponding to the error and $\vec{\mathcal{N}}$ is the vector of noise terms corresponding to the initial uncertainty in the auxiliary states. Now, since the correcting displacements are linear, the calculated correction will be the sum of the corrections corresponding to $\vec{S}_{\text{ideal}}$ and the extra noise $\vec{\mathcal{N}}$. This extra noise should be minimized so that the damage caused by performing the error-correction procedure is as small as possible.

We define the cost of correction for an error on a particular mode to be the sum of the increase in variances of x and p on the output state. Restricting attention to single mode errors, for a syndrome $\vec{S}$ caused by an error on an identified mode, the required correction is

$$C\left(\frac{\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}.\vec{S}_x} + \frac{i\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}.\vec{S}_p}\right)\cdot\vec{S}, \tag{42}$$

where C is the *complex* displacement of the output if the error was a unit shift in x, $\vec{S}_{x/p}$ is syndrome for a unit shift in x/p and $\vec{S}_p^{\perp}$ is the component of $\vec{S}_x$ orthogonal to $\vec{S}_p$ and vice versa. The correction for position will be the real component of Eq. (42)

$$\left(\frac{\mathrm{Re}(C)\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}\cdot\vec{S}_x} - \frac{\mathrm{Im}(C)\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}\cdot\vec{S}_p}\right)\cdot\vec{S} \tag{43}$$

and the kick to correct momentum will be the imaginary component

$$\left(\frac{\mathrm{Im}(C)\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}\cdot\vec{S}_x} + \frac{\mathrm{Re}(C)\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}\cdot\vec{S}_p}\right)\cdot\vec{S}. \tag{44}$$

Now, assuming the inputs to all auxiliary modes are distributed with the same variance v (so they each correspond to finitely squeezed vacuum states with the same squeezing parameter), each component of $\vec{\mathcal{N}}$ is randomly distributed about 0 with variance v, then the increase in variance for x and p, respectively, will be

$$\left(\frac{\mathrm{Re}(C)\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}\cdot\vec{S}_x} - \frac{\mathrm{Im}(C)\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}\cdot\vec{S}_p}\right)\cdot\left(\frac{\mathrm{Re}(C)\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}\cdot\vec{S}_x} - \frac{\mathrm{Im}(C)\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}\cdot\vec{S}_p}\right)v, \tag{45}$$

$$\left(\frac{\mathrm{Im}(C)\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}\cdot\vec{S}_x} + \frac{\mathrm{Re}(C)\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}\cdot\vec{S}_p}\right)\cdot\left(\frac{\mathrm{Im}(C)\vec{S}_p^{\perp}}{\vec{S}_p^{\perp}\cdot\vec{S}_x} + \frac{\mathrm{Re}(C)\vec{S}_x^{\perp}}{\vec{S}_x^{\perp}\cdot\vec{S}_p}\right)v. \tag{46}$$

Adding these together to get the cost for correcting the error, Eq. (42), yields a net variance

$$v|C|^2\frac{|\vec{S}_x|^2+|\vec{S}_p|^2}{|\vec{S}_x|^2|\vec{S}_p|^2-(\vec{S}_x\cdot\vec{S}_p)^2}. \tag{47}$$

Since C and the components of the vector $\vec{S}_x - i\vec{S}_p$ come from a column of the unitary matrix representing the circuit, and denoting the angle between $\vec{S}_x$ and $\vec{S}_p$ as θ, this cost may be written as

$$v\,\frac{|C|^2(1-|C|^2)}{|\vec{S}_x|^2|\vec{S}_p|^2(1-\cos^2\theta)}. \tag{48}$$

Thus, the cost for an individual error is minimized when the syndrome vectors are orthogonal, reflecting the fact that only the orthogonal components of the syndromes can be used to calculate the correction, and $|\vec{S}_x| = |\vec{S}_p|$.

This part of the cost for an encoding matrix U was taken to be the sum of individual costs over all correctable errors. The minimum occurs when $|C_i|^2 = 1/n$ leading to a target cost of $4/(n-1)$, which for the five-wave-packet-code case would be one. For circuits encoding m wave packets, the cost for an individual error was modified to be the sum of the increase in variance on each of the outputs, in this case the minimum becomes $4m/(n-m)$.

B. Optimizing the circuit for ease of implementation

A number of criteria were chosen to try and make generated circuits easier to implement. We focused on two aspects: number of components and component simplicity.

1. Number of components

One important consideration is the number of components in the circuit. The circuit used for the creation of arbitrary unitary operators, Fig. 1, contains $n(n-1)/2$ beam-splitters and $n(n+1)/2$ phase delays. However, if the beam-splitter parameters are multiples of $\pi/2$ then they represent either the identity or a swap, with a possible phase flip. In these cases the beam-splitter can be removed from the circuit reducing the number of number of beam-splitters and phase delays by one. Having fewer components not only reduces the cost of implementing the circuit but also reduces potential sources of noise.

2. Simpler settings for components

While it is possible to produce beam-splitters and phase delays with any arbitrary setting for θ, in practice certain values are easier to achieve than others. In particular the standard 50:50 beam-splitter is cheaper typically than one with variable settings, although a variable splitter can always be constructed using two 50:50 splitters with a suitable phase delay along one path. The cost function was therefore modified to reflect these concerns. Since the beam-splitter settings also relate to the number of components these two costs were combined. Each beam-splitter parameter was assigned a cost of 0 if it corresponded to no beam-splitter, a cost of 1 if it corresponded to a 50:50 beam-splitter and all other settings a cost of 2. Since the search space for each parameter is continuous, there would be almost no possibility of these exact values being obtained using the standard differential search algorithm. To allow circuits to be generated with parameters corresponding to lower costs a threshold value ϵ was introduced. When calculating the cost of a parameter, if its value differs from one with a lower cost by less than ϵ then it is replaced with the lower cost value for the purposes of evaluating the circuit, and also in the final output. However, the unaltered value was used for generating the next generation.

Simple parameters for phase delays are preferred since it is beneficial for phase locking. For this reason the parameters were limited to multiples of $\pi/2$. This was achieved in the same manner as for beam splitters but with the threshold set to $\pi/4$ in order to always fix it to a multiple of $\pi/2$.

C. Overall cost function

The final cost function was chosen to be a weighted average of all the above considerations. Since the primary aim of the optimization is to produce a circuit that performs error correction, the cost function was chosen so that it always preferred circuits representing codes. This was achieved by attaching a much larger weight to the error-correction criterion so that it always outweighed other factors. We chose a weight of 10^4. The final cost function was chosen to be

$$10^4\, C_{\text{crit}} + 2\, C_{\text{sep}} + C_{\text{corr}} + \tfrac{1}{5}\, C_{\text{beam}}, \tag{49}$$

where C_{crit} represents the cost of the criterion for error correction as used in the original optimization, C_{sep} the cosine of the minimum angle between any two syndrome spaces, C_{corr} the cost of correction and C_{beam} the cost of the beam-splitters. For the beam-splitter cost the threshold ϵ was set to $\pi/20$.

D. Progress of optimization

The differential evolution parameters were the same as those used in the original search for a code capable of performing error correction. As before, finding circuits capable of providing error correction was easy and was achieved typically within one or two generations when the initial population was 1000. However, optimizing for the additional considerations was much slower. Within a few thousand generations the values for the various costs were close to optimal, but it took millions of generations before they ceased to improve.

V. FIVE-WAVE-PACKET CODE USING LINEAR OPTICS

The encoding circuit for the five-wave-packet code with the lowest cost is presented in Fig. 3. This code has a cost of correction of 1, which is its lower bound and thus minimizes the increase in noise when correction is performed. The minimum angle between two syndrome subspaces is approximately $\pi/5$ so all syndromes are reasonably well separated.

In terms of components, the code uses seven beam-splitters of which five are non-50:50, this compares favorably to the nine-wave-packet code [8] which uses 12 50:50 beam-splitters. If the arbitrary beam-splitters were substituted for two 50:50 beam-splitters then our five-wave-packet code would also use 12. The phase delays of the code are all simple multiples of $\pi/2$ as specified by the cost function, although the other costs only marginally improved if this restriction were removed.

Finally, the unitary matrix representing the encoding circuit is

$$U \approx \begin{pmatrix} 0.45i & -0.51 & -0.60i & -0.37 & 0.20i \\ -0.45i & 0.51 & -0.60i & -0.37 & -0.20i \\ -0.45 & 0.20i & -0.37 & -0.60i & 0.51 \\ -0.45 & 0.20i & 0.37 & 0.60i & 0.51 \\ -0.45i & -0.63 & 0 & 0 & -0.63i \end{pmatrix} \tag{50}$$

to two significant figures. A circuit implementing it is given in Fig. 3.

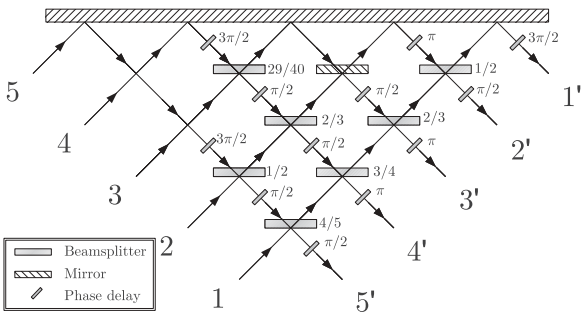

FIG. 3. The encoding circuit for the constructed five-wave-packet code. The numbers next to the beam-splitters are the energy transmission coefficients, and the numbers next to the phase delays denote the phase delay in radians.

TABLE I. Table of syndromes and effects on the output for primitive unit displacements of position and momentum for the five-wave-packet code generated by the circuit in Fig. 3.

	Unit position shift		Unit momentum kick	
Mode	Syndrome	Effect	Syndrome	Effect
1	$(-0.51,0,-0.37,0)$	$-0.45i$	$(0,-0.6,0,0.2)$	0.45
2	$(0.51,0,-0.37,0)$	$0.45i$	$(0,-0.6,0,-0.2)$	-0.45
3	$(0,-0.37,0,0.51)$	-0.45	$(0.2,0,-0.6,0)$	$-0.45i$
4	$(0,0.37,0,0.51)$	-0.45	$(0.2,0,0.6,0)$	$-0.45i$
5	$(-0.63,0,0,0)$	$0.45i$	$(0,0,0,-0.63i)$	-0.45

A. Demonstrating the correction criterion

In this section we demonstrate how the five-wave-packet code in Fig. 3 can be used to perform correction. As discussed previously, for nondegenerate codes we can perform correction if primitive displacement errors on different wave packets correspond to different syndromes. To determine this we use the decoding matrix which is the Hermitian conjugate of the encoding matrix. From this we find the syndromes for unit displacements which for position errors are the real components of the columns of the submatrix of $U^\dagger$ with the top row removed and the negative of the imaginary components for momentum kicks. The syndromes for the primitive unit displacements in position and momentum can be found in Table I.

On measurement, the error will collapse into a phase-space displacement associated with the measured syndrome. The syndrome will therefore be a linear combination of the unit position and momentum syndromes which span a two-dimensional subspace of $\mathbb{R}^4$. In order for error identification to succeed any two of these subspaces can only intersect at the origin, since otherwise we could find errors on both modes with the same syndrome and thus would not be able to determine the mode on which the error occurred. Examination of Table I shows that an intersection of any two such subspaces is impossible for our constructed five wave-packet code.

B. Using the code to perform correction

In order to perform the correction, we first identify the mode on which the error occurred. This can be done by determining which subspace of syndromes the measured syndrome comes from. One method of doing this is to calculate the size of the projection of the measured syndrome onto each of the syndrome subspaces. The mode on which the error occurred is then determined to be the one corresponding to the subspace with the largest projection. Since the syndromes in Table I for position and momentum errors on the same mode are already orthogonal the projection can be calculated trivially. For ideal case of zero noise in syndrome measurement, the size of the maximal projection will be the same as the size of the syndrome if the error is correctable since it is a linear combination of the position and momentum syndromes. However, if there is noise in the syndrome measurement such as that caused by the use of finitely squeezed states as auxiliaries then the projection will typically be smaller than the syndrome. Thus, the error is chosen to be the maximum, although in some cases it may be desirable to place a bound on the minimum relative size of the projection to identify uncorrectable errors.

Having determined the mode on which the error occurred, we can now calculate the displacement that must be performed to recover the original state. Since the syndromes for the unit position and momentum errors on the same mode are orthogonal, for syndrome $\vec{S}$, the effect the error has on the wave packet can be determined to be

$$\left(E_x \frac{\vec{S}_x}{|\vec{S}_x|^2} + E_p \frac{\vec{S}_p}{|\vec{S}_p|^2}\right) \cdot \vec{S}. \tag{51}$$

This simple form relies on the fact that our code has orthogonal syndromes. For each mode, the vector corresponding to the term inside the brackets can be calculated from information provided in Table I. The correcting phase-space displacement can then be applied using feed-forward to complete the error-correction protocol.

VI. DISCUSSION

One potential advantage of continuous-variable quantum systems is that they allow certain quantum information protocols to be implemented more simply and deterministically than is possible for discrete variables. For example, entanglement may be deterministically created by combining any pair of noncoherent pure states at a beam-splitter (a purely passive linear optical device). Similarly disentangling quantum continuous variables can be achieved by running the entanglement strategy in reverse.

In this article we have developed a criterion to identify when a linear optics circuit can be used to generate a continuous-variable quantum-error-correcting code. Using optimization techniques with a cost function based on this criterion we have found a five-wave-packet code which could be implemented using linear optics and a supply of position squeezed vacuum states. Thus, it can be implemented in the same manner as the nine-wave-packet code of Refs. [8,9]. This code could potentially be useful for protecting continuous variables against certain types of catastrophic errors.

However, error correction only works when the errors it is applied to correspond to the model for which it is designed. In many optical applications likely errors are not large changes in state but rather small "diffusive" errors. There are other schemes that allow qubits to be encoded over continuous variables [23] in such a manner that these small errors can be corrected.

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