A Real Symmetric Tridiagonal Matrix With a Given Characteristic Polynomial

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ABSTRACT

Given a polynomial $u(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_\nu \in \mathbf{R}$, $\nu = 0, 1, \dots, n-1$, having only real zeros, we construct a real symmetric tridiagonal matrix whose characteristic polynomial is equal to $(-1)^n u(x)$. This is a complete solution to a problem raised and partly solved by M. Fiedler.

1. INTRODUCTION AND STATEMENT OF RESULTS

A standard result in matrix theory states that the characteristic polynomial of a real symmetric matrix has all its roots real. At the International Colloquium on Applications of Mathematics in Hamburg (July 1990) Professor M. Fiedler presented a paper on a converse problem he has been thinking about for many years (see also [1]):

Construct a real symmetric matrix for which a given normed polynomial with only real roots will be the characteristic polynomial.

Of course, the construction should require only a finite number of numerical operations. He presented the following nice and simple solution [1].

THEOREM A. *Let $u(x)$ be a monic polynomial of degree $n \geqslant 1$; let $b_1, \dots, b_n$ be distinct numbers such that $u(b_k) \neq 0$ for $k = 1, \dots, n$. Set*

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$v(x) = \prod_{k=1}^{n}(x - b_k)$, and define the n-by-n matrix $A = (a_{ik})$ by

$$a_{ik} = -\sigma d_i d_k \qquad \text{if} \quad i \neq k,$$
$$a_{kk} = b_k - \sigma d_k^2, \qquad i, k = 1, \ldots, n,$$

where σ is a fixed nonzero number and d_k is a root of

$$\sigma v'(b_k) d_k^2 - u(b_k) = 0.$$

Then $(-1)^n u(x)$ is the characteristic polynomial of the symmetric matrix A. If λ is an eigenvalue of A, then $(d_i/(\lambda - b_i))$ is a corresponding eigenvector.

If $u(x)$ is a polynomial with n real distinct roots, then choosing b_i as any real numbers interlacing the roots of u, one can choose σ as $+1$ or -1 in such a way that d_i are all real and A thus real symmetric.

Fiedler also showed that by a modification of the construction the matrix A takes the form

$$A = \begin{pmatrix} b_1 & & & c_1 \\ & \ddots & & \vdots \\ & & b_{n-1} & c_{n-1} \\ c_1 & \cdots & c_{n-1} & d \end{pmatrix},$$

where nonzero off-diagonal entries may appear only in the last row and column.

An interesting aspect of these results is that they allow a lot of flexibility concerning the choice of the numbers b_k which determine the elements of the matrix A. On the other hand, to guarantee that the elements of A are all real, the choice of appropriate b_k's may require some preliminary calculations like construction of a Sturm sequence and performing bisections. Furthermore, the case of a polynomial with real but multiple zeros seems not to be settled.

The purpose of this note is to present an alternative solution to the above problem of Fiedler which provides a real symmetric tridiagonal matrix for *every* polynomial with only real zeros (not necessarily distinct). It is based on a modification of the Euclidean algorithm. There may be still other interesting solutions, and the reader is encouraged to look for them.

Let us start with a

NOTATION. For a polynomial

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$$

of degree $k \geqslant 0$ we write $c(f) := a_k$ so that $f/c(f)$ is a monic polynomial having the same zeros as f.

MODIFIED EUCLIDEAN ALGORITHM. For

$$u(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad a_\nu \in \mathbf{R}, \quad \nu = 0, 1, \ldots, n-1, \quad (1)$$

define

$$f_1(x) := u(x), \qquad f_2(x) := \frac{1}{n}u'(x)$$

and proceed recurrently as follows: If $f_{\nu+1}(x) \not\equiv 1$, then dividing f_ν by $f_{\nu+1}$ with remainder $-r_\nu$, we obtain

$$f_\nu(x) = q_\nu(x)f_{\nu+1}(x) - r_\nu(x) \qquad (2)$$

and define

$$\text{(i)} \qquad c_\nu := c(r_\nu), \quad f_{\nu+2}(x) := \frac{r_\nu(x)}{c_\nu} \qquad \text{if} \quad r_\nu(x) \not\equiv 0,$$

$$\text{(ii)} \qquad c_\nu := 0, \qquad f_{\nu+2}(x) := \frac{f'_{\nu+1}(x)}{c(f'_{\nu+1})} \qquad \text{if} \quad r_\nu(x) \equiv 0.$$

If $f_{\nu+1}(x) \equiv 1$, we terminate the algorithm, defining $q_\nu(x) := f_\nu(x)$.

THEOREM 1. *The polynomial* (1) *has only real zeros if and only if the modified Euclidean algorithm yields* $n-1$ *nonnegative numbers* $c_1, \ldots, c_{n-1}$, *and in this case*

$$u(x) = (-1)^n \det(T - xI),$$

where I is the n-by-n identity matrix and T is the tridiagonal matrix

$$T = \begin{pmatrix} -q_1(0) & \sqrt{c_1} & & & & \\ \sqrt{c_1} & -q_2(0) & \sqrt{c_2} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & \sqrt{c_{n-2}} & -q_{n-1}(0) & \sqrt{c_{n-1}} \\ & & & & \sqrt{c_{n-1}} & -q_n(0) \end{pmatrix}. \tag{3}$$

THEOREM 2. *The polynomial* (1) *has n real distinct zeros if and only if the modified Euclidean algorithm yields $n - 1$ positive numbers $c_1, \ldots, c_{n-1}$.*

REMARK. Suppose u has only real zeros. Let ϕ be the greatest common divisor of u and u'/n, so that $\varphi_1 := u/\phi$ is a polynomial of degree k, say. The proofs will show that the modified Euclidean algorithm may as well be started with $f_1(x) := u(x)$ and $f_2(x) := \phi(x)\varphi_2(x)$, where φ_2 is any monic polynomial of degree $k - 1$ whose zeros separate those of φ_1. An analogous modification is possible in the steps of type (ii). This way we can introduce $n - 1$ parameters into the construction of T, as in the result of Fiedler.

2. LEMMAS

We need the following auxiliary results, which are more or less known from discussions of Sturm sequences (see [2, Chapter 3, Section 4.2] or [3, §38]). To avoid vagueness and to make the paper self-contained we present proofs.

LEMMA 1. *Let f and g be monic polynomials of degrees $k + 1$ and k, respectively, whose zeros are all real and distinct and separate each other. Then the division transformation*

$$f(x) = q(x)g(x) - r(x) \tag{4}$$

yields a polynomial r of degree $k - 1$ with $c(r) > 0$ whose zeros are all real and distinct and separate those of g. Furthermore, q is of the form $x - a$ where $a \in \mathbf{R}$.

Proof. The form of q needs no explanation. Clearly, r is of degree at most $k - 1$. If

$$\xi_1 > \xi_2 > \cdots > \xi_k \tag{5}$$

are the zeros of g, then by (4) and the separation property of the zeros of the monic polynomials f and g we have

$$(-1)^{\nu} = \operatorname{sgn} f(\xi_\nu) = -\operatorname{sgn} r(\xi_\nu), \qquad \nu = 1, 2, \cdots, k.$$

Hence r must be of the degree $k - 1$, and have precisely one simple zero in each of the intervals $]\xi_{\nu+1}, \xi_\nu[$, where $\nu = 1, 2, \cdots, k - 1$. Furthermore,

$$\operatorname{sgn} r(x) = \operatorname{sgn} r(\xi_1) = 1 \qquad \text{for} \quad x \geq \xi_1,$$

which shows that $c(r) > 0$. ■

As a converse statement we shall need

LEMMA 2. *Let g and h be monic polynomials of degrees k and $k - 1$, respectively, whose zeros are all real, and distinct and separate each other. For $c > 0$ and $a \in \mathbf{R}$ define*

$$f(x) := (x - a)g(x) - ch(x). \tag{6}$$

Then f is a monic polynomial of degree $k + 1$ with distinct real zeros which are separated by those of g.

Proof. Obviously f is a monic polynomial of degree $k + 1$, and $f(x)$ is positive for large x. Denoting and ordering the zeros of g as in (5), we must have $h(\xi_1) > 0$, since h is monic and has no zero to the right of ξ_1. By (6) and the separation property of the zeros of g and h,

$$(-1)^{\nu+1} = \operatorname{sgn} h(\xi_\nu) = -\operatorname{sgn} f(\xi_\nu), \qquad \nu = 1, 2, \ldots, k.$$

Hence f has an odd number of zeros in $]\xi_1, +\infty[$ and in each of the intervals $]\xi_{\nu+1}, \xi_\nu[$, where $\nu = 1, 2, \ldots, k - 1$. But this is possible only if f has precisely one simple zero in each of these intervals and an additional zero in $]-\infty, \xi_k[$. ■

3. PROOFS OF THE THEOREMS

We start with the

Proof of Theorem 2. If the polynomial (1) has n distinct zeros, they are separated by those of u'. Now repeated application of Lemma 1 shows that the modified Euclidean algorithm runs through $n-1$ steps of type (i), yielding $n-1$ positive numbers $c_1, \ldots, c_{n-1}$.

Conversely, if the modified Euclidean algorithm yields $n-1$ positive numbers $c_1, \ldots, c_{n-1}$, it can only run through steps of type (i), providing $n-1$ monic polynomials $q_\nu(x)$ of degree 1 such that

$$f_\nu(x) = q_\nu(x) f_{\nu+1}(x) - c_\nu f_{\nu+2}(x), \qquad \nu = 1, 2, \ldots, n-1, \tag{7}$$

where $f_n(x) =: q_n(x)$ is of degree 1 and $f_{n+1}(x) \equiv 1$.

Fixing $q_\nu(x)$ and c_ν ($\nu = 1, 2, \ldots, n-1$) and starting with $f_{n+1}(x)$ and $f_n(x)$, we may successively calculate $f_{n-1}(x), f_{n-2}(x), \ldots$ from (7). Applying Lemma 2 in each step, we find that $f_1(x) = u(x)$ has n distinct zeros. ■

Proof of Theorem 1. Suppose u has only real zeros, and let ϕ be the greatest common divisor of u and u'/n. Clearly, ϕ is a monic polynomial, which we may assume to be of positive degree $n-k$, say, since otherwise we are in the situation of Theorem 2. Consequently,

$$u(x) = f_1(x) = \phi(x)\varphi_1(x) \quad \text{and} \quad \frac{1}{n}u'(x) = f_2(x) = \phi(x)\varphi_2(x),$$

where φ_1 and φ_2 are monic polynomials of degrees k and $k-1$, respectively, whose zeros are real and distinct and separate each other. Furthermore, the modified Euclidean algorithm shows that for $\nu = 1, 2, \ldots, k-1$ the polynomial ϕ factors out in the recurrence relation (7) so that

$$f_\nu(x) =: \phi(x)\varphi_\nu(x), \quad \nu = 1, 2, \ldots, k, \qquad f_{k+1}(x) = \phi(x), \qquad \varphi_{k+1}(x) :\equiv 1$$

and

$$\varphi_\nu(x) = q_\nu(x)\varphi_{\nu+1}(x) - c_\nu \varphi_{\nu+2}(x), \qquad \nu = 1, 2, \ldots, k-1. \tag{8}$$

Since (8) is equivalent to the division of φ_ν by $\varphi_{\nu+1}$ with remainder $-c_\nu \varphi_{\nu+2}$, we may apply Lemma 1 successively for $\nu = 1, \ldots, k-1$ and find that the numbers c_ν are all positive and the degree of $f_{\nu+1}$ decreases by 1 if ν increases by 1. The calculation of f_{k+2} leads to a step of type (ii) which yields

$$c_k = 0, \qquad f_{k+2}(x) = \frac{1}{n-k}\phi'(x).$$

Continuing with f_{k+1} and f_{k+2}, we are in the same situation as at the beginning and may therefore repeat the above argumentation. This way we obtain precisely $n-1$ numbers c_ν, which are all nonnegative.

If the modified Euclidean algorithm yields $n-1$ nonnegative numbers c_ν, the relations (7) must hold with monic polynomials q_ν of degree 1. Since $q_n := f_n$, the matrix T in (3) exists and has real elements. Now define $T_1 := T$, and let T_k for $k > 1$ be the submatrix obtained from T by canceling the first $k-1$ rows and columns. For $\nu = 1, \dots, n$ denote by g_ν the characteristic polynomial of T_ν. Obviously,

$$\begin{aligned} g_n(x) &= -q_n(0) - x = -q_n(x) = -f_n(x), \\ g_{n-1}(x) &= q_{n-1}(x)q_n(x)c_{n-1} \\ &= q_{n-1}(x)f_n(x) - c_{n-1}f_{n+1}(x) = f_{n-1}(x). \end{aligned} \tag{9}$$

Expanding $\det(T_\nu - xI)$, where I is the identity matrix of the same size as T_ν, with respect to the first row, we find

$$\begin{aligned} g_\nu(x) &= [-q_\nu(0) - x]g_{\nu+1}(x) - c_\nu g_{\nu+2}(x) \\ &= -q_\nu(x)g_{\nu+1}(x) - c_\nu g_{\nu+2}(x). \end{aligned} \tag{10}$$

Comparing (9) and (10) with (7), we conclude that $f_\nu(x) = (-1)^{n-\nu+1}g_\nu(x)$ and hence

$$u(x) = f_1(x) = (-1)^n \det(T - xI).$$

Since T is real and symmetric, the zeros of u are all real. ■

4. EXAMPLES

For the polynomial $u(x) = x^5 - 5x^3 + 4x$ whose zeros are 0, ± 1, and ± 2 the matrix T becomes

$$T = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{\frac{7}{5}} & 0 & 0 \\ 0 & \sqrt{\frac{7}{5}} & 0 & \sqrt{\frac{36}{35}} & 0 \\ 0 & 0 & \sqrt{\frac{36}{35}} & 0 & \sqrt{\frac{4}{7}} \\ 0 & 0 & 0 & \sqrt{\frac{4}{7}} & 0 \end{pmatrix}.$$

For the polynomial $u(x) = x^5 - x^3$ the modified Euclidean algorithm runs through two steps of type (ii) and we obtain

$$T = \begin{pmatrix} 0 & \sqrt{\frac{2}{5}} & 0 & 0 & 0 \\ \sqrt{\frac{2}{5}} & 0 & \sqrt{\frac{3}{5}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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