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Spatial squeezing of the vacuum and the Casimir effect ★

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Abstract

Free quantum electromagnetic radiation is enclosed in a one-dimensional cavity. The contribution of the kth mode of the field to the energy, contained in a region $\mathcal{R}$ of the cavity, is minimized. For the resulting *squeezed* state, the energy expectation in $\mathcal{R}$ is *below* its vacuum value. Pressing zero-point energy out of a spatial region can be used to temporarily *increase* the Casimir force.

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1. Introduction

If a harmonic oscillator is prepared in a coherent state, the product of the variances of position and momentum takes on the smallest possible value while their ratio is fixed. By using squeezed states [1] one is free to vary the ratio of the variances while keeping their product minimal; in other words, one can "distribute the uncertainty" over position and momentum at will. This possibility turns out to be of interest in quantum optics [2] and, especially, in the theory of optical communication [3]. To prepare a mode of the electromagnetic field in a squeezed state allows one to tune the signal-to-noise ratios of its quadrature components: one of the quadratures can be used to absorb the inevitable quantum-mechanical "noise" [4], while the other one serves to transmit an extremely well-defined signal. In the context of quantized electromagnetic fields, it seems natural to also investigate the *spatial* properties of the field (cf. Ref. [5]) if the harmonic oscillators associated with the normal modes of the system are prepared in squeezed states.

It is the purpose of this Letter to show that due to squeezing, quantum-mechanical fields can exhibit a property in configuration space which is unexpected from a classical point of view: it is possible to *decrease* the energy density locally below its value in the vacuum state by preparing the field in an appropriate state. This feature emerges naturally if one searches for those quantum-mechanical states of the field which, at a given time, minimize the energy contained in a prescribed spatial region, $\mathcal{R}$. Indeed, this requirement singles out a family of squeezed states of the field oscillators. As an application a modified version of the Casimir effect is proposed. The force acting on a reflecting plate at the boundary of the cavity is sensitve to the difference of the energy density across the plate. Since appropriately squeezed initial states decrease the energy density on one side of the plate below its vac-

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uum value, a temporary *increase* of the Casimir force is predicted.

Similar observations have been made in studies of systems (mostly cavities) with moving mirrors [6–9]. A typical feature of these time-dependent systems is the correlated excitation of modes into squeezed states. As a result, the vacuum state develops structure in the sense that the energy density is no longer spatially homogeneous. For an oscillating boundary, the emergence of "sub-Casimir" regions has been pointed out [10]. In the present work, the cavity has *fixed* boundaries, and the search is focussed on states of the field which minimize the expectation value of the energy in a prescribed region. Correlations between different modes will not be dealt with but individually squeezed modes will be seen to maximize the effect.

2. Quantized field

Consider a box of length $L = \Lambda + \lambda$. Its boundaries at $x_\ell = -\Lambda$ and $x_r = \lambda$ are assumed to consist of perfectly conducting mirrors. (This model is based on Ref. [11]; work performed under the auspices of N.G. van Kampen.) For simplicity, the box is assumed to be one-dimensional and a single polarization of the electric (and magnetic) field is taken into account only. Effectively, one is dealing now with a scalar theory, but the arguments given below are expected to hold in the general case, too. Units are chosen such that the dielectric constant ε of the vacuum, Planck's constant $\hbar$ and the velocity of light, c, are equal to one. Free electromagnetic radiation in the cavity is described quantum-mechanically in the Heisenberg picture by the Hamiltonian operator

$$\widehat{\mathcal{H}} = \int_{-\Lambda}^{\lambda} \mathrm{d}x\, \widehat{W}(x,t) = \tfrac{1}{2} \int_{-\Lambda}^{\lambda} \mathrm{d}x [\widehat{E}^2(x,t) + \widehat{H}^2(x,t)] = \sum_{k=1}^{\infty} \omega_k (a_k^\dagger a_k + \tfrac{1}{2}), \quad (1)$$

where ($u_k'(x) \equiv \mathrm{d}u_k(x)/\mathrm{d}x$)

$$\widehat{E}(x,t) = \sum_{k=1}^{\infty} u_k(x) \widehat{e}_k(t),$$

$$\widehat{H}(x,t) = \sum_{k=1}^{\infty} u_k'(x) \widehat{h}_k(t), \quad (2)$$

are the operators for the electric and the magnetic field (at position x and time t), respectively. The functions $u_k(x)$ follow from Maxwell's equations as solutions of

$$u_k''(x) + \omega_k^2 u_k(x) = 0, \quad k = 1, 2, \ldots, \quad (3)$$

vanishing at the boundaries of the cavity: $u_k(-\Lambda) = u_k(\lambda) = 0$. The frequencies are given by $\omega_k = k\pi/L$, $k = 1, 2, \ldots$. The normalized modes $u_k(x)$ read

$$u_k(x) = \sqrt{\frac{2}{L}} \sin[\omega_k(\Lambda + x)], \quad (4)$$

and they constitute a complete and orthonormal set of functions on the interval $-\Lambda \leqslant x \leqslant \lambda$,

$$\int_{-\Lambda}^{\lambda} \mathrm{d}x\, u_k(x) u_\ell(x) = \delta_{k\ell}, \quad (5)$$

$$\sum_{k=1}^{\infty} u_k(x) u_k(y) = \delta(x - y). \quad (6)$$

The operator-valued amplitudes in Eq. (2) are related by $\widehat{e}_k(t) = -\mathrm{d}\widehat{h}_k(t)/\mathrm{d}t$, and one has explicitly

$$\widehat{h}_k(t) = \frac{1}{\sqrt{2\omega_k}} (a_k \mathrm{e}^{-\mathrm{i}\omega_k t} + a_k^\dagger \mathrm{e}^{\mathrm{i}\omega_k t}). \quad (7)$$

The operators $a_k^\dagger$ and a_k create and annihilate excitations of the mode with frequency ω_k, and they fulfill the familiar relations

$$[a_k, a_\ell^\dagger] = \delta_{k\ell}, \quad [a_k, a_\ell] = [a_k^\dagger, a_\ell^\dagger] = 0, \quad k, \ell = 1, 2, \ldots . \quad (8)$$

3. Minimization and squeezed states

Consider, for a moment, a *classical* vibrating string between $-\Lambda$ and λ. The state with lowest energy, i.e., the ground state of the string, is given by its straight configuration with each point being at rest. Focus now on a finite part of the system, the region to the right of the origin, $\mathcal{R} = \{x | 0 < x \leqslant \lambda\}$, for example. Clearly, the energy contained in region $\mathcal{R}$ takes on its minimal value, 0, if the state coincides with the ground state for $0 < x < \lambda$.

The ground state of the quantum-mechanical system with Hamiltonian $\hat{\mathcal{H}}$ is the vacuum $|0\rangle = |0_1, 0_2, \ldots, 0_k, \ldots\rangle$, in which no mode is excited: $a_k|0_k\rangle = 0$. Each oscillator contributes an amount $\omega_k/2$, its zero-point energy, to the total energy which therefore diverges. Defining the energy operator $\hat{\mathcal{H}}_{\mathcal{R}}$ as integral over the energy density in region $\mathcal{R}$,

$$\hat{\mathcal{H}}_{\mathcal{R}}(t) = \int_{\mathcal{R}} \mathrm{d}x\, \hat{W}(x,t), \qquad (9)$$

one finds as contribution of a mode with frequency ω_k

$$\langle 0_k|\hat{\mathcal{H}}_{\mathcal{R}}(t)|0_k\rangle = (V_k^- + V_k^+)\frac{\omega_k}{2} = \frac{\lambda}{L}\frac{\omega_k}{2}, \qquad (10)$$

clearly indicating the spatially homogeneous distribution of the energy in the vacuum. The positive dimensionless quantities

$$V_k^{\pm} = \frac{1}{2}\frac{\lambda}{L}(1 \pm \sigma_k), \qquad (11)$$

are given in terms of the λ-dependent *asymmetries* $\sigma_k = \sin(2\omega_k\lambda)/2\omega_k\lambda$.

To determine the state which yields the smallest possible expectation value of the energy in region $\mathcal{R}$, one requires that at a given moment ($t = 0$, say)

$$\langle\psi|\hat{\mathcal{H}}_{\mathcal{R}}|\psi\rangle = \min, \qquad (12)$$

with a yet undetermined state $|\psi\rangle = |\psi_1, \psi_2, \ldots, \psi_k, \ldots\rangle$. One can write

$$\langle\hat{\mathcal{H}}_{\mathcal{R}}\rangle = \frac{1}{2}\int_0^{\lambda} \mathrm{d}x\,[\langle\hat{E}(x)\rangle^2 + \langle\hat{H}(x)\rangle^2] + \sum_{k=1}^{\infty}[V_k^-\langle(\Delta\hat{e}_k)^2\rangle + \omega_k^2 V_k^+\langle(\Delta\hat{h}_k)^2\rangle], \qquad (13)$$

where all expectation values are taken with respect to $|\psi\rangle$, and $\Delta\hat{g}_k$ is defined as $\Delta\hat{g}_k = \hat{g}_k - \langle\hat{g}_k\rangle$. The expectation value $\langle\hat{\mathcal{H}}_{\mathcal{R}}\rangle$ takes on its minimum value if each of the positive terms in Eq. (13) is as small as possible without violating the inequalities

$$\langle(\Delta\hat{e}_k)^2\rangle\langle(\Delta\hat{h}_k)^2\rangle \geqslant \tfrac{1}{4}, \quad k = 1, 2, \ldots, \qquad (14)$$

which follow from Eq. (8). The integral in (13) does not contribute if

$$\langle\hat{E}(x)\rangle = \langle\hat{H}(x)\rangle = 0, \quad x \in \mathcal{R}, \qquad (15)$$

representing a condition on the set $\{\langle\hat{e}_k\rangle, \langle\hat{h}_k\rangle\}$. The smallest contribution of the kth mode in (13) is given by

$$\langle(\Delta\hat{e}_k)^2\rangle = \frac{\omega_k}{2}\mu_k, \quad \langle(\Delta\hat{h}_k)^2\rangle = \frac{1}{2\omega_k}\frac{1}{\mu_k}, \qquad (16)$$

as follows from minimizing the kth term in the sum under the constraint (14); the positive real number μ_k is given by $\mu_k = (V_k^+/V_k^-)^{1/2}$.

The product of the variances given in Eqs. (16) fulfills the equality

$$\langle(\Delta\hat{e}_k)^2\rangle\langle(\Delta\hat{h}_k)^2\rangle = \tfrac{1}{4}, \qquad (17)$$

hence, for $\mu_k \neq 1$, the kth mode is in a *squeezed* state. In general, a squeezed state is obtained from an eigenstate of the annihilation operator,

$$a_k|\alpha_k\rangle = \alpha_k|a_k\rangle \qquad (18)$$

(i.e., a coherent state $|\alpha_k\rangle$), by applying to it the unitary squeezing (or dilation) operator [1], $\hat{S}_k(r_k) = \exp[r_k(a_k^2 - a_k^{\dagger 2})/2]$,

$$|r_k, \alpha_k\rangle = \hat{S}_k(r_k)|\alpha_k\rangle. \qquad (19)$$

In contrast to coherent states, which under harmonic time evolution remain states of minimum uncertainty, squeezed states only periodically recover the property of minimality. To squeeze a coherent state $|\alpha_k\rangle$ requires energy,

$$\Delta\mathcal{H}_k = \langle r_k, \alpha_k|\hat{\mathcal{H}}|r_k, \alpha_k\rangle - \langle\alpha_k|\hat{\mathcal{H}}|\alpha_k\rangle = \omega_k \sinh^2 r_k. \qquad (20)$$

The relation between the squeezing parameter r_k and μ_k in (16) is given by $r_k = -\frac{1}{2}\ln\mu_k$.

4. Spatial variation of the energy density

For the sake of simplicity, it will be assumed from now on that the expectation values $\langle\hat{E}(x)\rangle$ and $\langle\hat{H}(x)\rangle$ vanish everywhere, not only in the region $\mathcal{R}$. In this case one has $\alpha_k = 0$, and each mode is a squeezed vacuum state which is conveniently denoted by $|\sigma_k\rangle$ since its degree of squeezing is completely determined by the asymmetry σ_k.

The expectation value of $\widehat{\mathcal{H}}_{\mathcal{R}}$ in the state $|\sigma\rangle = |\sigma_1, \sigma_2, \ldots, \sigma_k, \ldots\rangle$ is found to be

$$\langle\sigma|\widehat{\mathcal{H}}_{\mathcal{R}}|\sigma\rangle = \sum_{k=1}^{\infty} \omega_k \sqrt{V_k^+ V_k^-} = \frac{\lambda}{L} \sum_{k=1}^{\infty} \frac{\omega_k}{2} \sqrt{1-\sigma_k^2}. \tag{21}$$

Since $V_k^+ + V_k^- > 2\sqrt{V_k^+ V_k^-}$ (if $V_k^+ \neq V_k^-$), one finds by comparison with (10) that the contribution of each mode is less than for the vacuum, $|0\rangle$. The energy difference is given by

$$\Delta\mathcal{H}_{\mathcal{R}}^k(\lambda) = \frac{\lambda}{L}\frac{\omega_k}{2}\left(\sqrt{1-\sigma_k^2} - 1\right) < 0. \tag{22}$$

This inequality represents the main result: just as one can distribute the uncertainty of a harmonic oscillator between position and momentum, it becomes possible to decrease the energy density in one spatial region compared to the vacuum value while enhancing it elsewhere. This result is in striking contrast with the impossibility of decreasing locally the energy of a classical vibrating string below its ground-state energy.

If an arbitrarily large but finite number (cf. below) of the asymmetries σ_k is different from zero, one can conclude immediately from (22) that

$$\langle\sigma|\widehat{\mathcal{H}}_{\mathcal{R}}|\sigma\rangle < \langle 0|\widehat{\mathcal{H}}_{\mathcal{R}}|0\rangle. \tag{23}$$

A better understanding of this result is obtained if one studies the spatial variation of the energy density associated with the kth mode, for example. It turns out to be distributed inhomogeneously in the cavity (cf. (13)),

$$\langle\sigma_k|\widehat{W}(x)|\sigma_k\rangle = \frac{\omega_k}{2L}\left(\mu_k \sin^2 \omega_k(\Lambda + x) + \frac{1}{\mu_k}\cos^2 \omega_k(\Lambda + x)\right), \tag{24}$$

as shown in Fig. 1. There is a sequence of regions within which the energy density either exceeds the homogeneous energy distribution of the vacuum of mode k, or remains below it. The asymmetry σ_k has a value such that in the region $\mathcal{R}$ the sum of the (hatched)

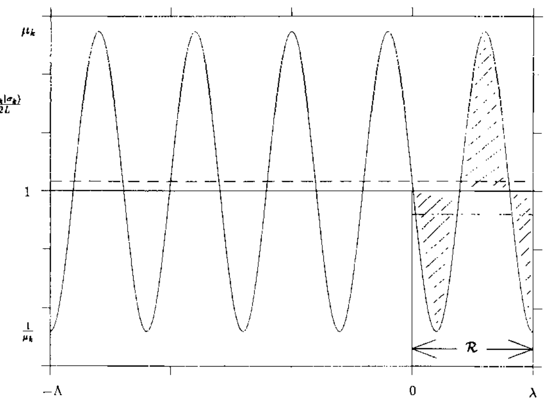

Fig. 1. Energy density of a squeezed mode $|\sigma_k\rangle$ with squeezing parameter $\mu_k > 1$ (note especially the low values at the boundaries). The energy density of the vacuum state $|0_k\rangle$ is equal to $\omega_k/2L$ (full line). The energy contained in mode k is given by $\omega_k(\mu_k + \mu_k^{-1})/2$ (dashed line). Summing up the hatched areas in the region $\mathcal{R}$ with appropriate signs yields a value less than $\lambda\omega_k/2L$ (dash-dotted line).

areas below the full line is larger than of those above it. The pronounced dip of the energy density at the end of the cavity, x_r, suggests that the Casimir force on the plate can be increased if the kth mode is in the state $|\sigma_k\rangle$.

5. Modified Casimir effect

According to the Casimir effect (see Ref. [12] for a review) two parallel, perfectly conducting plates in the vacuum at a distance L attract each other with a force proportional to $L^{-(d+1)}$, d being the dimension of the space. From a microscopic point of view, this force is understood as a residual macroscopic signature of the van der Waals force acting between the particles in the mirrors. A simple way to derive it is to consider the difference of the zero-point energy densities in the regions enclosed by the plates and outside, respectively; similarly, one can attribute the Casimir force to the radiation pressure exerted on the plates due to the vacuum field [13].

In three spatial dimensions, the force acting on a volume V (with surface A) is proportional to the rate of change of the momentum contained in this volume. The expectation value of this force, $\boldsymbol{F}$, when the field is in state $|\psi\rangle$ can be expressed as a surface integral of appropriate components of the stress–energy tensor $\hat{\sigma}$,

$$F_j = \int_A \mathrm{d}A \, \langle\psi|\hat{\sigma}_{jk}|\psi\rangle n_k, \quad j,k = x,y,z, \tag{25}$$

where $\boldsymbol{n}$ is a unit vector normal to the surface A, pointing outwards. The components of $\hat{\sigma}$ are given by

$$\hat{\sigma}_{jk} = \frac{1}{8\pi}[\hat{E}_j\hat{E}_k + \hat{E}_k\hat{E}_j + \hat{H}_j\hat{H}_k + \hat{H}_k\hat{H}_j - \delta_{jk}(\hat{\boldsymbol{E}}^2 + \hat{\boldsymbol{H}}^2)]. \tag{26}$$

Choosing the volume to be a thin slice containing (a unit area of) one of the plates, one finds from (25) (after appropriate regularization) the Casimir force on the plate in the vacuum state of the field [13].

For a one-dimensional model with one polarization of the field only, the stress energy tensor reduces to $\hat{\sigma}_{jk} = -\delta_{jk}\hat{W}/4\pi$, and the expression for the force simplifies to

$$F_x(x) = \langle\psi|\hat{\sigma}_{xx}|\psi\rangle = -\frac{1}{4\pi}\langle\psi|\hat{W}(x)|\psi\rangle. \tag{27}$$

In the state $|\sigma_k\rangle$ the energy density near the plate (that is, if x approaches λ from the left) is found from (24) to have the value

$$\langle\sigma_k|\hat{W}(\lambda)|\sigma_k\rangle = \frac{1}{\mu_k}\frac{\omega_k}{2L} = \frac{1}{\mu_k}\frac{k\pi}{2L^2}. \tag{28}$$

If the squeezing parameter μ_k is larger than one, Eq. (24) implies that the contribution of this mode to the force on the plate is reduced by a factor $1/\mu_k$. For $\mu_k \to \infty$, this contribution can be made vanishingly small, at the expense of a high energy, as seen from (20). Since the force resulting from Eq. (28) is proportional to L^{-2} (for $d = 1$) just as is the Casimir force, it is possible to strongly reduce it by moderately squeezing many modes.

6. Initial states

As long as an arbitrary large but *finite* number of modes are squeezed the resulting state is a possible initial state of the field. If, however, one wishes to squeeze infinitely many squeezed modes in order to enhance the effect, the situation becomes subtle. Suppose for a moment that *all* asymmetries σ_k were nonzero. In this case, one has for all k

$$\tilde{\omega}_k \equiv \omega_k\sqrt{1-\sigma_k^2} < \omega_k, \quad k = 1,2,\dots\,. \tag{29}$$

Thus, each term in the sum (21) has been replaced by a smaller one, tempting one to conclude that the sum should be correspondingly smaller. However, the net effect is just the opposite since the summation extends to infinity: the density of modes *increases* by "rescaling" each frequency according to (29) and, effectively, this leads to a *higher* energy density contained in $\mathcal{R}$. In other words, more modes contribute up to the cutoff frequency [15] which is introduced in order to regularize the divergent zero-point energy. However, choosing all asymmetries $\sigma_k \neq 0$ is *not* a sensible state of the electromagnetic field since the energy difference between the states $|0\rangle$ and $|\sigma\rangle$ is logarithmically divergent as follows from (20),

$$\sum_{k=1}^{\infty}\Delta\mathcal{H}_k \sim \sum_{k=1}^{\infty}\omega_k\sigma_k^2 \sim \sum_{k=1}^{\infty}\frac{\sin^2 2\omega_k\lambda}{k}. \tag{30}$$

This sum diverges for almost all values of λ; convergence is possible only if λ is commensurate with $L/2$. Consequently, the state $|\sigma\rangle$ is, in general, *not* an element of the Hilbert space built upon the vacuum state $|0\rangle$. These difficulties are not present if either an arbitrarily large but finite number of the asymmetries σ_k are different from zero, or if the (infinitely many) nonzero σ_k are distributed in such a way that

$$\langle\sigma|\hat{\mathcal{H}}^2|\sigma\rangle - \langle 0|\hat{\mathcal{H}}^2|0\rangle < \infty. \tag{31}$$

This condition guarantees both, the existence of the time evolution of the state $|\sigma\rangle$ under $\hat{\mathcal{H}}$, and that $|0\rangle$ and $|\sigma\rangle$ are elements of the same Hilbert space.

7. Measurement

The questions arise of how to initially prepare the electromagnetic field in an appropriately squeezed state, and how to perform a measurement. Various (more or less efficient) techniques are known in order to squeeze states [2,3]. For the present purpose it seems promising to generate squeezed modes inside the cavity by an instant change of the length of the cavity, from L' to L, say. This causes a "frequency jump" which is known to imply squeezing [16].

If the squeezed modes have been generated, the modification of the Casimir force could be measured as follows. After the time $1/4\omega_k$ the maxima and minima shown in Fig. 1 have exchanged their positions.

Thus, time intervals during which the force on the plate is smaller than the standard Casimir force are followed by other intervals during which it is larger: this necessitates time-resolved measurements since, compared to the Casimir force, the average force on the plate is increased due to the radiation pressure of the photons enclosed in the cavity. If one were able to continuously monitor the force on the plate an oscillatory behaviour about the vacuum value measured in Ref. [14] would emerge. Perhaps the most realistic approach is to perform a resonance measurement in the spirit of Ref. [17] exploiting the periodic variation of the force (cf. also Ref. [18]).

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