

Reconstructing a pure state of a spin s through three Stern–Gerlach measurements

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Abstract. Consider a spin s prepared in a *pure* state. It is shown that, generically, the moduli of the $(2s + 1)$ spin components along three directions in space determine the state unambiguously. These probabilities are accessible experimentally by means of a standard Stern–Gerlach apparatus. It is therefore possible to reconstruct a generic pure state on the basis of $6s$ independent measured intensities.

1. Introduction

The reconstruction of a particle density operator is possible, in principle, through repeated measurements on an ensemble of identically prepared systems [1, 2]. Quantum states of vibrating molecules [3], of trapped ions [4], as well as the state of atoms in motion [5] have been reconstructed successfully in the laboratory. Similarly, quantum optical experiments have been performed to reconstruct the quantum state of a light wave [6].

For a spin of length s , this question arises for states in a Hilbert space of finite dimension. There is an explicit expression for the density matrix ρ in terms of the moduli of spin components along $(4s + 1)$ appropriate directions in space [7]†. A standard Stern–Gerlach apparatus with variable orientation in space provides the corresponding probabilities in an experiment. Alternatively, a Wigner function defined on the discrete phase space associated with a finite-dimensional Hilbert space allows one to reconstruct quantum states [9]. This method has been adapted in [10] in order to determine a quantized electromagnetic mode of a cavity. Every proposed method of state reconstruction is bound to reflect on the link between the relative frequencies—the outcomes of a *finite* number of measurements obtained in an actual experiment—and the calculated probabilities associated with an *infinite* ensemble (see [11], for example).

Suppose now that the spin state to be reconstructed is known to be prepared in a *pure* state which is determined by fewer parameters than a mixed state. How should we exploit this additional knowledge in the most efficient way? For a *particle*, the problem of reconstructing a pure state had been raised by Pauli [12] as early as 1933 but he did not provide an answer. One solution of the spin version of the problem [13] makes use of a *Feynman filter*. This is an advanced version of a Stern–Gerlach apparatus which is assumed to reveal the relative phases of the expansion coefficients of a pure spin state. Other approaches relate expectation values of spin multipoles to the parameters which define the quantum state [7, 14].

† As it stands, the reduction to $(2s + 1)$ directions proposed in [8] is erroneous.

As shown in this paper, the pure state of a spin s is fixed unambiguously if the $3(2s + 1)$ intensities of the spin components are measured along *three* axes[†]. Compared to the $(4s + 1)$ axes required for a mixed state [7], the experimental effort to perform state reconstruction is thus reduced considerably for large spins. Further, this result is satisfactory from a mathematical point of view since it generalizes an earlier result: the intensities along two *infinitesimally close* axes spanning a plane define a unique pure state when complemented by the expectation value of a spin component ‘out of plane’ [15]. Effectively, this means measuring $(2s + 1)$ probabilities along a third direction.

2. Result

The purpose of this section is twofold. First, the setting of the problem is introduced and secondly, the result is stated. The states of a spin of magnitude s live in a Hilbert space \mathbf{H}^s of dimension $(2s + 1)$, which carries an irreducible representation of the group $SU(2)$. The components of the spin operator $\vec{S} \equiv \hbar \vec{s}$ with standard commutation relations $[s_x, s_y] = i s_z, \dots$, generate rotations about the corresponding axes. The standard basis of the space \mathbf{H}^s is given by the eigenvectors of the z component of the spin, denoted by $|s, \mu_z\rangle$, $-s \leq \mu_z \leq s$. The transformation under the anti-unitary time reversal operator T fixes their phases, $T|s, \mu_z\rangle = (-1)^{s-\mu_z}|s, -\mu_z\rangle$. When expanded in the z basis,

$$|\psi\rangle = \sum_{\mu_z=-s}^s \psi_{\mu_z} |s, \mu_z\rangle \quad (1)$$

a pure state is seen to be determined by $(2s + 1)$ complex coefficients $\psi_{\mu_z} \equiv \langle s, \mu_z | \psi \rangle$. If normalized, a ray $|\psi\rangle$ is thus specified by $4s$ real parameters. Two other bases of the space \mathbf{H}^s will be used to expand $|\psi\rangle$ as in equation (1): the sets $\{|s, \mu_x\rangle\}$ and $\{|s, \mu_y\rangle\}$ with $-s \leq \mu_x, \mu_y \leq s$, made up from the eigenvectors of the spin components $S_x = \vec{n}_x \cdot \vec{S}$ and $S_y = \vec{n}_y \cdot \vec{S}$, respectively, with unit vectors \vec{n}_x and \vec{n}_y pointing along the x and y axes. Rotations about appropriate axes by an angle $\pi/2$ map them to the z basis:

$$|s, \mu_z\rangle = e^{-i\pi s_y/2} |s, \mu_x\rangle = e^{i\pi s_x/2} |s, \mu_y\rangle. \quad (2)$$

A measurement of the intensities $\{|\langle s, \mu_z | \psi \rangle|^2\}$ does not fix a single state $|\psi\rangle$, since the phases of the coefficients ψ_{μ_z} remain undetermined. Measuring with respect to *two* axes provides $2(2s + 1)$ intensities which are usually compatible with a huge number of isolated states, in agreement with the result of [15]: the parameters fulfil nonlinear relations which may have multiple solutions. Enumerating the ensemble of possible ‘partner’ states is complicated, so a distinctive third measurement is included from the very beginning. It will be shown in the following that:

A generic spin state $|\psi\rangle \in \mathbf{H}^s$ is fixed unambiguously if $3(2s + 1)$ probabilities

$$p(\mu_k) = |\psi_{\mu_k}|^2 \quad k = x, y, z \quad (3)$$

are measured with a Stern–Gerlach apparatus along the *three* coordinate axes.

This result holds for *almost all* pure states (the ‘genericity’ will be defined below): there exist exceptional states of measure zero in Hilbert space \mathbf{H}^s such that the associated probabilities $p(\mu_k)$ are compatible simultaneously with a finite number of other states.

As it stands, statement (3) refers to three orthogonal axes, and, for simplicity, the proof will be carried out in this setting. The generalization to arbitrary axes not in a plane is possible as will be pointed out in the final section.

[†] Due to the normalization condition of the states only $2s$ out of the $(2s + 1)$ intensities with respect to a given axis are independent. Thus, only $6s$ independent numbers are obtained from measurements along three axes. These constraints should be kept in mind in the following.

It is useful to rephrase the statement at stake differently. According to (3) a state $|\tilde{\psi}\rangle$ gives rise to the *same* intensities as does $|\psi\rangle$ if its coefficients $\psi_{\mu_k} = \langle s, \mu_k | \tilde{\psi} \rangle$ differ from ψ_{μ_k} by phase factors only. The index k is understood to take the values x, y and z from now on. Using (1) one writes

$$\sum_{\mu_k=-s}^s \psi_{\mu_k} e^{i\chi_k(\mu_k)} |s, \mu_k\rangle = \exp[i\chi_k(s_k)] |\psi\rangle \quad (4)$$

with three polynomials $\chi_k(\mu)$ of order $2s$ in μ at most. The coefficients in (4) thus define three states $|\psi_k\rangle = W_k^s |\psi\rangle$, where $W_k^s = \exp[i\chi_k(s_k)]$ is a unitary operator diagonal in the k basis. Consequently, a state $|\tilde{\psi}\rangle$ compatible with (3) exists if and only if there are nontrivial unitary operators W_k^s such that

$$W_x^s |\psi\rangle = W_y^s |\psi\rangle = W_z^s |\psi\rangle \equiv |\tilde{\psi}\rangle. \quad (5)$$

It will turn out that—for almost all states $|\psi\rangle$ —this relation is satisfied only if the operators W_k^s are multiples of the identity, implying that $|\tilde{\psi}\rangle$ and $|\psi\rangle$ represent the *same* ray in Hilbert space.

Before turning to the proof, the intensities $p(\mu_k)$ in (3) are represented in a more compact way. Define three functions $m_k(\alpha)$ of a complex variable $\alpha \in C$ by

$$m_k(\alpha) = \langle \psi | U_k^s(\alpha) | \psi \rangle \equiv \sum_{\mu_k=-s}^s e^{i\mu_k \alpha} p(\mu_k) \quad (6)$$

where the operator $U_k^s(\alpha) = \exp(i\alpha s_k)$ rotates a state $|\psi\rangle$ about the k axis if $\alpha \in R$. Equation (6) is inverted easily using the orthogonality of the functions $\exp[-i\mu_k \alpha]$ on the interval $0 \leq \alpha < 2\pi$.

3. Proof

The proof that the data (3) are sufficient to characterize a unique state is divided into five steps:

- (a) A 2^{2s} -dimensional ‘parent’ space \mathcal{H}^s is introduced which contains the Hilbert space \mathbf{H}^s of the spin s as a subspace.
- (b) To each state $|\psi\rangle \in \mathbf{H}^s$ an equivalence class of product states $\{|\Psi\rangle \in \mathcal{H}^s\}$ is associated.
- (c) A natural definition of *generic* states emerges for *product* states in \mathcal{H}^s and, *a fortiori*, in \mathbf{H}^s .
- (d) An appropriate set of expectation values of the parent states $|\Psi\rangle$ is shown to fix them uniquely.
- (e) Finally, it is shown that all states $|\tilde{\psi}\rangle$ satisfying (5) have parents in the *same* equivalence class as the original $|\psi\rangle$. Consequently, the (generic) state $|\psi\rangle$ is the only one giving rise to the intensities (3).

Let us now turn to the individual steps.

(a) The 2^{2s} -dimensional ‘parent’ space \mathcal{H}^s of \mathbf{H}^s is obtained from tensoring $2s$ copies of the Hilbert space C^2 of a spin $\frac{1}{2}$:

$$\mathcal{H}^s = \bigotimes_{r=1}^{2s} C_r^2. \quad (7)$$

A basis of C^2 is given by the eigenstates $|\sigma\rangle \equiv |s = \frac{1}{2}, \mu_3 = \frac{1}{2}\sigma\rangle$, $\sigma = \pm 1$, of the third component of the spin $\frac{1}{2}$: $\sigma_3 |\sigma\rangle = \sigma |\sigma\rangle$. This choice induces a basis of \mathcal{H}^s formed by all product states

$$|\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} |\sigma_r\rangle. \quad (8)$$

The parent space \mathcal{H}^s decomposes into a subspace $\mathcal{H}_{\text{sym}}^s$ and its complement,

$$\mathcal{H}^s = \mathcal{H}_{\text{sym}}^s \oplus (\mathcal{H}_{\text{sym}}^s)^\perp \quad (9)$$

where $\mathcal{H}_{\text{sym}}^s$ is spanned by the $(2s+1)$ states obtained from completely symmetrizing those in (8):

$$\begin{aligned} |s, \mu_3\rangle &= \mathcal{S}_{2s} |\{\sigma_r\}\rangle \\ &\equiv N_{\mu_3}^s \sum_{\{\sigma_r\}} \delta(\sigma_1 + \dots + \sigma_{2s} - 2\mu_3) |\{\sigma_r\}\rangle \end{aligned} \quad (10)$$

where $-s \leq \mu_3 \leq s$, using a symmetrizer of $2s$ objects, \mathcal{S}_{2s} , and the normalization factor $N_{\mu_3}^s = ((s - \mu_3)!(s + \mu_3)!/(2s)!)^{1/2}$. The space $\mathcal{H}_{\text{sym}}^s$ is important here because it carries a $(2s+1)$ -dimensional irreducible representation of the group of rotations, $SU(2)$, obtained upon reducing the product representation [16]

$$\mathcal{U} |\{\sigma_r\}\rangle = \bigotimes_{r=1}^{2s} \sum_{\sigma'_r = \pm 1} |\sigma'_r\rangle \langle \sigma'_r | u_r | \sigma_r \rangle \quad (11)$$

where u_r is the r th copy of a rotation $u \in SU(2)$ of the fundamental representation acting on \mathbb{C}^2 , and \mathcal{U} is an operator defined on \mathcal{H}^s . Since Hilbert spaces of the same dimension are isomorphic, $\mathcal{H}_{\text{sym}}^s$ and \mathbf{H}^s can and will be identified from now on.

(b) There is a one-to-one relation between states $|\psi\rangle \in \mathcal{H}_{\text{sym}}^s$ and equivalence classes of product states $|\Psi\rangle \in \mathcal{H}^s$:

$$|\Psi\rangle \equiv |\{\Psi^r\}\rangle = \bigotimes_{r=1}^{2s} \left(\sum_{\sigma_r} \Psi_{\sigma_r}^r |\sigma_r\rangle \right). \quad (12)$$

The equivalence relation \sim is defined as follows: the projection of a state $|\Psi\rangle$ in (12) onto a basis state $|s, \mu_3\rangle \in \mathcal{H}_{\text{sym}}^s$ must equal the corresponding expansion coefficient of $|\psi\rangle$ in the z basis, i.e.

$$\langle s, \mu_3 | \Psi \rangle = N_\psi \langle s, \mu_z | \psi \rangle \quad -s \leq \mu_3 = \mu_z \leq s \quad (13)$$

and the factor $N_\psi > 0$ may depend on the state $|\psi\rangle$ under consideration but *not* on the index μ_z . Thus, $|\Psi\rangle \sim |\Psi'\rangle$ means that for a fixed $|\psi\rangle$, equations (13) hold for both product states, $|\Psi\rangle \sim |\Psi'\rangle$. The association of spin states $|\psi\rangle$ with product or ‘parent’ states $|\Psi\rangle$ is essential for the following.

In order to determine the class of states satisfying equation (13) for a prescribed vector $|\psi\rangle$ (with definite phase), multiply by the factor $1/N_\mu^s$, by powers $(-z)^{\mu+s}$ and sum all terms. Then, the right-hand side defines an analytic function

$$f_R(z) = N_\psi \sum_{\mu=-s}^s \frac{(-z)^{\mu+s}}{N_\mu^s} \psi_\mu \propto \prod_{r=1}^{2s} (z_r - z) \quad (14)$$

specified by the location of its $2s$ zeros z_r in the complex plane. The left-hand side of (13) yields a second analytic function of z ,

$$\begin{aligned} f_L(z) &= \sum_{\mu=-s}^s \sum_{\{\sigma_r\}} (-z)^{\mu+s} \delta(\sigma_1 + \dots + \sigma_{2s} - 2\mu) \Psi_{\sigma_1}^1 \dots \Psi_{\sigma_{2s}}^{2s} \\ &\equiv \prod_{r=1}^{2s} (\Psi_-^r - z \Psi_+^r) \quad \Psi_\pm^r \equiv \Psi_{\pm 1}^r. \end{aligned} \quad (15)$$

The $(2s + 1)$ equations (13) are satisfied if $f_L(z)$ and $f_R(z)$ coincide. Being two polynomials of degree $2s$, this requires them to have identical zeros,

$$\frac{\Psi_-^r}{\Psi_+^r} = z_r \quad r = 1, \dots, 2s \quad (16)$$

in addition, $f_L(0) = f_R(0)$ must hold. Due to the normalization $\langle \Psi^r | \Psi^r \rangle = |\Psi_+^r|^2 + |\Psi_-^r|^2 = 1$, one can write

$$\begin{pmatrix} \Psi_+^r \\ \Psi_-^r \end{pmatrix} = \frac{e^{i\kappa_r}}{\sqrt{1 + |z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix} \quad \kappa_r \in [0, 2\pi). \quad (17)$$

Thus, there are $2s$ undetermined phase factors $e^{i\kappa_r}$ with a product equal to 1 ($|\psi\rangle$ denotes a *vector*, not a ray). However, the overall ambiguity is even larger: when comparing the zeros of the functions $f_L(z)$ and $f_R(z)$, there is no rule which would indicate what order to choose when writing down the product state $|\{\Psi^r\}\rangle$. In other words, the equivalence class of states defined by (13) consists of all states with coefficients (17) distributed in any order over the $2s$ spinors in (12). All these states are parents of the same $|\psi\rangle$ since they satisfy equation (13).

A *given* product state $|\Psi\rangle$ with components

$$\langle \{\sigma_r\} | \Psi \rangle = \Psi_{\{\sigma_r\}} = \prod_{r=1}^{2s} \Psi_{\sigma_r}^r \quad (18)$$

has a unique ‘daughter’ $|\psi\rangle$ to be read off directly. Upon parametrizing each factor $|\Psi^r\rangle$ by a complex number z_r ,

$$\begin{pmatrix} \Psi_+^r \\ \Psi_-^r \end{pmatrix} = \frac{1}{\sqrt{1 + |z_r|^2}} \begin{pmatrix} 1 \\ z_r \end{pmatrix} \quad (19)$$

one sees that the ensemble $\{z_r\} \equiv (z_1, \dots, z_{2s})$ (*no* order implied) defines the daughter $|\psi\rangle$ completely, while a maximum of $(2s)!$ different parent states $|\Psi\rangle$ is associated with a given set $\{z_r\}$.

(c) Suppose that three ensembles of $2s$ real numbers each, $\{x_r\}$, $\{y_r\}$ and $\{|z_r|\} \equiv (|z_1|, \dots, |z_{2s}|)$ with $z_r = x_r + iy_r$ are given in disorder. If one is able to construct the disordered ensemble of $2s$ complex numbers $\{z_r = x_r + iy_r\}$ upon using the $2s$ conditions $|z_r|^2 = x_r^2 + y_r^2$, the equivalence class with representant $|\Psi\rangle$ is called *generic*. In other words, it must be possible to combine unambiguously real and imaginary parts into complex numbers z_r . In this spirit, a daughter $|\psi\rangle \in \mathcal{H}_{\text{sym}}^s$ will be called *generic* if it has generic parents $|\Psi\rangle$. The procedure does not work if equalities such as $x_r = \pm y_{r'}$, $r \neq r'$ exist; hence *exceptional* states have measure zero.

(d) It is shown now that the expectation values of rotations $\mathcal{U}_k(\alpha)$ about the axes x , y and z , fix generic product states $|\Psi\rangle = |\{\Psi^r\}\rangle$ up to a permutation of the factors $|\Psi^r\rangle$ and an overall phase factor. A generic $|\Psi\rangle \in \mathcal{H}^s$ leads to three expectation values

$$M_k(\alpha) = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle \equiv \prod_{r=1}^{2s} \langle \Psi^r | u_k^r(\alpha) | \Psi^r \rangle \quad (20)$$

where $u_k(\alpha) = \mathbf{1} \cos(\alpha/2) + i\sigma_k \sin(\alpha/2)$ represents a rotation about axis k in C^2 . Using the parametrization of equation (19), the functions $M_k(\alpha)$ defined in (20) read explicitly

$$M_x(\alpha) = \prod_{r=1}^{2s} \left(\cos \frac{\alpha}{2} + i \frac{2x_r}{1 + |z_r|^2} \sin \frac{\alpha}{2} \right) \tag{21a}$$

$$M_y(\alpha) = \prod_{r=1}^{2s} \left(\cos \frac{\alpha}{2} + \frac{2y_r}{1 + |z_r|^2} \sin \frac{\alpha}{2} \right) \tag{21b}$$

$$M_z(\alpha) = \prod_{r=1}^{2s} \left(\cos \frac{\alpha}{2} + i \frac{1 - |z_r|^2}{1 + |z_r|^2} \sin \frac{\alpha}{2} \right) \tag{21c}$$

where again $z_r = x_r + iy_r$. Denote by $|\tilde{\Psi}\rangle \equiv |\{\tilde{\Psi}^r\}\rangle$ another product state with expectations $\tilde{M}_k(\alpha)$

$$\tilde{M}_k(\alpha) = \langle \tilde{\Psi} | \mathcal{U}_k(\alpha) | \tilde{\Psi} \rangle \equiv \prod_{r=1}^{2s} \langle \tilde{\Psi}^r | u_k^r(\alpha) | \tilde{\Psi}^r \rangle. \tag{22}$$

Upon describing the state $|\tilde{\Psi}\rangle$ by the sequence $\{\tilde{z}_r\}$, the three functions $\tilde{M}_k(\alpha)$ are given by equations (21) after replacing each z_r by \tilde{z}_r . It is shown now that the conditions

$$\langle \tilde{\Psi} | \mathcal{U}_k(\alpha) | \tilde{\Psi} \rangle = \langle \Psi | \mathcal{U}_k(\alpha) | \Psi \rangle \quad k = x, y, z \tag{23}$$

necessitate $|\tilde{\Psi}\rangle \sim |\Psi\rangle$. Being analytic in the complex α plane, the functions $M_k(\alpha)$ and $\tilde{M}_k(\alpha)$ are equal if they have same zeros. The equality $\tilde{M}_z(\alpha) = M_z(\alpha)$ requires $|\tilde{z}_r| = |z_r|$. The condition $\tilde{M}_x(\alpha) = M_x(\alpha)$ in turn implies $\tilde{x}_r = x_r$; finally, $\tilde{y}_r = y_r$ follows from $\tilde{M}_y(\alpha) = M_y(\alpha)$. However, this procedure determines the ensembles $\{x_r\}$, $\{y_r\}$ and $\{|z_r|\}$ *without* any order of its members. Nevertheless, one can reconstruct the ensemble $\{z_r\}$ (*no* order implied) according to (c) if $|\Psi\rangle$ is *generic* providing thus a *unique* equivalence class. For exceptional states, the $2s$ complex numbers cannot be reconstructed unambiguously since they might allow for parents contained in different equivalence classes.

(e) Results (a)–(d) imply that the probabilities $p(\mu_k)$ for three directions x, y and z as given in equation (3) determine a generic state $|\psi\rangle$ unambiguously. According to equation (5), a state $|\tilde{\psi}\rangle$ gives rise to the same probabilities as does $|\psi\rangle$ if one has

$$|\psi_x\rangle = |\psi_y\rangle = |\psi_z\rangle = |\tilde{\psi}\rangle. \tag{24}$$

For parent states $|\Psi_k\rangle$ of $|\psi_k\rangle$ this relation says that

$$|\Psi_x\rangle \sim |\Psi_y\rangle \sim |\Psi_z\rangle \sim |\tilde{\Psi}\rangle. \tag{25}$$

This implies that the mean values $\langle \Psi_k | \mathcal{U}_k(\alpha) | \Psi_k \rangle$ of the operator $\mathcal{U}_k(\alpha) = \otimes_r \exp[i\alpha\sigma_x/2]$ are equal for $k = x, y, z$: as products they are invariant under a permutation of their factors. This also holds for expectation values of the operators $\mathcal{U}_y(\alpha)$ and $\mathcal{U}_z(\alpha)$. Write the parent states $|\Psi_k\rangle$ in the form $\mathcal{W}_k|\Psi\rangle$ with operators $\mathcal{W}_k(\{\alpha_{k,r}\}) = \otimes_r \exp[i\alpha_{k,r}\sigma_k/2]$ defined on the parent space \mathcal{H}^s such that they have W_k^s as component acting in $\mathcal{H}_{\text{sym}}^s$. Contrary to the rotations $\mathcal{U}_k(\alpha)$ which depend linearly on the generators s_k , the operators W_k^s are *nonlinear* functions $\chi(s_k)$ of them, cf equation (4). Therefore, the operators $\mathcal{W}_k(\{\alpha_{k,r}\})$ depend on a set of $2s$ *different* angles $\{\alpha_{k,r}\}$. Using (25) one concludes that

$$\begin{aligned} \langle \tilde{\Psi} | \mathcal{U}_k | \tilde{\Psi} \rangle &= \langle \Psi_k | \mathcal{U}_k | \Psi_k \rangle \\ &= \langle \Psi | \mathcal{W}_k^\dagger \mathcal{U}_k \mathcal{W}_k | \Psi \rangle = \langle \Psi | \mathcal{U}_k | \Psi \rangle. \end{aligned} \tag{26}$$

The third equality follows because \mathcal{W}_k and \mathcal{U}_k do commute, both being functions of s_k only. Equation (26) comes down to saying that the functions $M_k(\alpha)$ and $\tilde{M}_k(\alpha)$ coincide for all k

and α . One concludes thus with (d) that the state $|\tilde{\Psi}\rangle$, a parent of $|\tilde{\psi}\rangle$, is necessarily a member of the *same* equivalence class as the parent $|\Psi\rangle$ of $|\psi\rangle$. In other words, the application of the operators \mathcal{W}_k on a parent $|\Psi\rangle$ does not map it into another equivalence class. In the generic case, there is thus no state different from $|\psi\rangle$ with the same data (3) which was to be shown.

4. Discussion

The characterization of a pure state $|\psi\rangle$ by the probabilities (3) is almost always possible. If one deals with an exceptional state as defined in (c), different scenarios are possible. First, the set of data (3) might nevertheless be sufficient to fix the underlying state unambiguously. This is due to the fact that the original conditions on the moduli are stronger than those formulated in the parent space. Suppose that the numbers $\{z_r\}$ are associated with a parent state $|\Psi\rangle$ and $\{z'_r\}$ with another one, $|\Psi'\rangle$, where both sets of complex numbers are obtained from the ensembles $\{x_r\}$ and $\{y_r\}$ through $|z_r|^2 = x_r^2 + y_r^2$. This does not necessarily imply the existence of an independent $|\psi'\rangle \neq |\psi\rangle$ since it is the basic conditions $|\psi'_{\mu_k}| = |\psi_{\mu_k}|$ which must be satisfied. Secondly, a numerical coincidence of parameter values might indeed destroy the one-to-one relation between daughter states and (equivalence classes of) parent states. This case fits the result of [15] where intensities are measured along three infinitesimally close axes: in this situation, one can write down explicitly non-generic states for which the reconstruction scheme does not work.

Since the ambiguities are due to the *numerical* values of expansion coefficients, it is expected that they do not persist if one were to measure intensities with respect to another set of orthogonal axes close to the original ones. Although plausible, there seems to be no straightforward way to prove this statement. The difficulty here is due to the nonlinearity of the underlying equations leading to nearly unpredictable behaviour of solutions even if the parameter values (orientation of the axes) are modified only slightly. If, however, the state at hand does possess a particular ‘symmetry’, the ambiguity is not automatically removed by a slight reorientation of the axes. As explicit calculations for low values of spin s show, the ambiguity persists, for instance, if $\psi'_{\mu_k} = \psi_{\mu_k}^*$, corresponding to $\langle \psi | s_y^{n+1} | \psi \rangle = 0$ with $1 \leq n \leq [2]$. Similar ambiguities exist with respect to the x and z axes.

It is worthwhile to point out that the reasoning of the previous section remains valid if one measures the intensities along directions characterized by non-orthogonal unit vectors \mathbf{n}_ζ , \mathbf{n}_η , and \mathbf{n}_ξ : the derivation does not make use of the fact that the three bases $|\mu_k\rangle$, $k = x, y, z$, of Hilbert space are associated with *orthogonal* axes. It is essential, however, that the vectors be linearly independent, that is, they have to span a *volume* in space: $\mathbf{n}_\zeta \cdot \mathbf{n}_\eta \times \mathbf{n}_\xi \neq 0$. For each choice of directions in space—be they orthogonal or not—there will be a set of exceptional states such that the data (3) do not single out one individual state. If two triples $(\mathbf{n}_\zeta, \mathbf{n}_\eta, \mathbf{n}_\xi)$ and $(\mathbf{n}_{\zeta'}, \mathbf{n}_{\eta'}, \mathbf{n}_{\xi'})$ can be mapped onto each other by a rigid rotation, the respective ensembles of exceptional states will also be mapped onto each other.

To sum up, the pure state of a spin s is generically fixed by the $6s$ independent moduli of the spin components with respect to three spatial directions not in a plane. Compared with a constructive method for which the number of intensities is quadratic in s , the number of experimentally determined parameters is considerably smaller for the non-constructive method presented here, growing only linearly in s .

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