

## Commensurate harmonic oscillators: Classical symmetries

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(Received 16 January 2002; accepted for publication 23 April 2002)

The symmetry properties of a classical  $N$ -dimensional harmonic oscillator with rational frequency ratios are studied from a global point of view. A *commensurate* oscillator possesses the same number of globally defined constants of motion as an *isotropic* oscillator. In both cases invariant phase-space functions form the *algebra*  $\mathfrak{su}(N)$  with respect to the Poisson bracket. In the isotropic case, the phase-space flows generated by the invariants can be integrated globally to a set of finite transformations isomorphic to the group  $SU(N)$ . For a commensurate oscillator, however, the *group*  $SU(N)$  of symmetry transformations is found to exist only on a *reduced* phase space, due to unavoidable singularities of the flow in the full phase space. It is therefore crucial to distinguish carefully between local and global definitions of symmetry transformations in phase space. This result solves the long-standing problem of which symmetry to associate with a commensurate harmonic oscillator. © 2002 American Institute of Physics. [DOI: 10.1063/1.1488672]

### I. INTRODUCTION

Harmonic oscillators are ubiquitous in physics. To lowest order, motion close to a stable equilibrium of a classical system is often described by a Hamiltonian of the form

$$H(q,p) = \sum_{n=1}^N \frac{\omega_n}{2} (p_n^2 + q_n^2), \quad \omega_n \in \mathbb{R}. \quad (1)$$

Here the (appropriately rescaled) canonical coordinates and momenta have Poisson brackets  $\{q_n, p_{n'}\} = \delta_{nn'}$ ,  $n, n' = 1, \dots, N$ . If the frequencies  $\omega_n$  are all equal,

$$\omega_n = \omega, \quad n = 1, \dots, N, \quad (2)$$

the Hamiltonian (1) describes an *isotropic*  $N$ -dimensional oscillator. This system is invariant under a set of transformations isomorphic to the group  $SU(N)$ : on the one hand, the quadratic form (1) in  $2N$  variables is obviously invariant under proper rotations  $SO(2N)$ —on the other hand, canonical transformations need to be symplectic, hence they are elements of  $Sp(N)$ . However, any transformation in  $\mathbb{R}^{2N}$  which is both (special) orthogonal and symplectic must be (special) unitary:<sup>1</sup>  $SU(N) = SO(2N) \cap Sp(N)$ . The group  $SU(N)$  is represented by  $(N^2 - 1)$  phase-space functions which, as constants of motion, generate symmetry transformations of the Hamiltonian. In fact, the isotropic oscillator is “maximally superintegrable” since it possesses the maximal number of  $(2N - 1)$  functionally independent constants of motion, exceeding by far the number of  $N$  globally defined invariants required for integrability.<sup>2</sup>

Suppose now that the frequency ratios  $\omega_n/\omega_{n'}$  are positive rational numbers,

$$\omega_n = \frac{\omega}{m_n}, \quad m_n \in \mathbb{N}_+, \quad \omega > 0. \quad (3)$$

This property defines a *commensurate* harmonic oscillator, or **m**-oscillator, with  $\mathbf{m} = (m_1, \dots, m_N)$ . As shown below, it also possesses  $(N^2 - 1)$  globally defined phase-space invariants, apart from the Hamiltonian. Their Poisson brackets form the Lie algebra  $\mathfrak{su}(N)$ , as for the isotropic oscillator. It is known that in both systems all orbits are closed. Nevertheless, some difference is to be expected, since all orbits of an isotropic oscillator have the same period, while commensurate frequencies allow for closed orbits with different periods. This is easily seen by exciting only individual degrees of freedom with frequencies  $\omega_n$ .

In the following, the topological and group-theoretical impact of rational frequency ratios (different from one) will be made explicit. First, various papers dealing with commensurate oscillators are reviewed in Sec. II, which is independent of the later developments. The technical part starts with Sec. III, where, for simplicity, the class of two-dimensional **m**-oscillators will be studied in detail. The generalization to  $N \geq 3$ , given in Sec. IV, is *not* straightforward. Finally, the overall picture is summarized and conclusions are drawn. A study of *quantum mechanical* **m**-oscillators, including the classical limit to connect with the present results, will be presented elsewhere.<sup>3</sup>

## II. SYMMETRIES OF HARMONIC OSCILLATORS

The equations of motion of  $N$  harmonic oscillators can be solved analytically for arbitrary frequency ratios. In spite of this exceptional property many authors have wrestled with the *symmetries* of such systems, the question being how their symmetries depend on the (ir-) rationality of the frequency ratios. Most contributions are fostered by the difficulty to distinguish between local and global properties of phase space. Two-dimensional oscillators with rational or irrational frequency ratios are discussed almost exclusively. Surprising claims have been made in the attempt to generalize properties of the isotropic oscillator in  $N$  dimensions.

Jauch and Hill<sup>4</sup> address the problem of “accidental degeneracy” of quantum-mechanical energy eigenvalues. The obvious invariance of the three-dimensional harmonic oscillator (as well as the hydrogen atom) under the group of rotations in configuration space is not sufficient to explain the observed degeneracy of the energy levels. They conclude that additional constants of motion must exist which account for extra degeneracies in the quantum mechanical energy spectrum. In fact,  $(N^2 - 1)$  Hermitian operators can be specified which commute with the Hamiltonian of the isotropic harmonic oscillator in  $N$  dimensions. Their commutation relations turn out to be those of the algebra  $\mathfrak{su}(N)$ . Therefore, the oscillator is said to have the  $\mathfrak{su}(N)$  symmetry—which then leads to the correct degree of degeneracies of energy levels.

Pauli<sup>5</sup> and Klein<sup>6</sup> have pointed out that there is a connection between degeneracies of energy levels and the existence of further constants of motion in the associated *classical* system. Therefore, the result also should be manifest in the corresponding classical isotropic oscillator. Upon “dequantizing” the quantum invariants, one obtains indeed  $(N^2 - 1)$  constants of motion which constitute the  $\mathfrak{su}(N)$  algebra with respect to the Poisson bracket. Hence, the classical isotropic oscillator possesses indeed constants of motion other than the angular momentum. Its components generate obvious *geometrical* symmetry transformations while the additional constants are said to generate *dynamical* symmetry transformations. They cannot be visualized in configuration space because they mix coordinates and momenta.

However, to exhibit a set of conserved phase-space functions which form a particular algebra is *not* sufficient in order to prove invariance of the physical system in a global sense, i.e., in the entire phase space. Jauch and Hill assert that the “system of orbits” of a classical  $(m_1, m_2)$ -oscillator be invariant under a group of transformations isomorphic to the three-dimensional group of proper rotations  $\text{SO}(3)$ . However, this claim cannot be justified by local considerations only. In other words, global invariance under a particular group of transformations does not follow from specifying phase-space functions forming the corresponding algebra.

McIntosh reviews accidental degeneracy in classical and quantum mechanics in Ref. 7. He notes that the phase space of the isotropic harmonic oscillator in two dimensions foliates into hyperspheres, being surfaces of constant energy. A discussion of the canonical transformations generated by three constants of the motion quadratic in the coordinates and momenta follows. It becomes obvious that the group of symmetry transformations is the special unitary group in two dimensions,  $SU(2)$ —not the group of proper three-dimensional rotations,  $SO(3)$ , as Jauch and Hill suggested.

Dulock and McIntosh<sup>8</sup> devote a paper to the two-dimensional harmonic oscillator with arbitrary frequency ratio. Using classical variables which mimic quantum mechanical creation and annihilation operators, they write down three constants of motion with Poisson brackets isomorphic to the  $so(3)$  algebra relations. A Hopf mapping is performed in order to visualize “how the rotational symmetry of  $S^2$ , which is the three-dimensional rotation group, chances also to be the symmetry group of the harmonic oscillator.”<sup>8</sup> Formally, this method can be applied to oscillators with *arbitrary* frequency ratio. However, one of the transformations, which is one-to-one in the isotropic case, becomes a multiple-valued map. For rational frequency ratios there is a finite ambiguity, turning to infinite multiple-valuedness if the frequencies ratios are irrationally. In spite of this result, the authors claim that the set of symmetry transformations for *all* types of oscillators investigated is isomorphic to the group  $SU(2)$ —irrespective of the multiple-valuedness. Once more, the possibility to write down formal expressions which constitute particular algebraic relations is taken as a proof of the existence of an associated *group* of transformations.

Maiella and Vitale<sup>9</sup> react to the claim that “every classical system should possess a ‘dynamical’ symmetry larger than the ‘geometrical’ one.”<sup>9</sup> Using action-angle variables, they provide three constants of motion for the two-dimensional oscillator which form the  $su(2)$  algebra. However, for irrational frequency ratio the invariants are not single-valued—hence they consider the “ $su(2)$  symmetry” to be of “formal value” only. It is claimed to acquire physical relevance only for commensurate and, *a fortiori*, isotropic oscillators. At the same time, no argument is given which would forbid the existence of the group  $SU(2)$  for the irrational oscillator. The authors do not investigate whether, in the commensurate case, the invariants generate indeed finite single-valued phase-space transformations in  $SU(2)$ .

Maiella<sup>10</sup> extends this discussion to the  $N$ -dimensional oscillator and emphasizes that only single-valued constants of the motion generate actual symmetry transformations. Initially, the group of all contact transformations for a given dynamical system is considered. Any subgroup of transformations which is generated by single-valued constants of motion and leaves the Hamiltonian invariant is called an “invariance group.” The classical degree of degeneracy determines the number of its generators: each linear relation between the classical frequencies of the system with *rational* coefficients is accompanied by the appearance of a single-valued constant of motion. Subsequently, phase-space functions are given in action-angle variables which realize the algebra  $su(N)$  for an isotropic oscillator and the algebra  $su(n)$ ,  $2 \leq n < N$ , for smaller degeneracy. However, it is again not proven explicitly that the generators actually give rise to globally well-defined transformations.

In the late 1960s, successful application of group theoretical concepts in elementary particle physics renewed the interest in symmetries of classical Hamiltonian systems and stimulated more general approaches. The invariance of the three-dimensional Kepler problem under the group of four-dimensional rotations,  $SO(4)$ , was explicitly shown by Moser<sup>11</sup> in 1970 for the first time. Already in 1965 Bacry, Ruegg and Souriau<sup>12</sup> proved that there exists a set of global symmetry transformations for the Kepler problem being isomorphic to the group  $SO(4)$ . The transformations presented, however, do not act on variables in phase-space. The transformations of phase-space manifolds are parametrized by the components of angular momentum and of the Runge–Lenz vector. Representing only five independent constants of motion, the time  $t$  at which the particle passes the perihelion of the orbit is taken as sixth parameter.

Dulock and McIntosh<sup>13</sup> claim that the Kepler problem has not only the symmetry  $SO(4)$  but  $SU(3)$ . Two papers by Bacry, Ruegg and Souriau<sup>12</sup> and by Fradkin<sup>14</sup> generalize this statement: all classical central potential problems should possess the dynamic symmetries  $O(4)$  and  $SU(3)$ . This

surprising statement is subject to the same criticism as the following, even more general claim by Mukunda:<sup>15,16</sup> *all* classical Hamiltonian systems with  $N$  degrees of freedom have  $O(N)$  and  $SU(N)$  symmetries. If this statement were true, then there would exist just one and only one global phase-space structure for systems with  $N$  degrees of freedom—the well-established distinction between regular and chaotic systems would have no meaning at all.

Mukunda argues on the basis of a theorem by Eisenhart.<sup>17</sup> Consider, in a Hamiltonian system with  $N$  degrees of freedom,  $n < N$  independent functions of canonically conjugate variables (subjected to weak conditions). They can always be supplemented by  $(2N - n)$  phase-space functions such that  $N$  pairs of canonically conjugate variables result which define a symplectic basis of phase space. Hence, starting with the Hamiltonian of the system under consideration one can find (i) a variable being canonically conjugate to the Hamiltonian and (ii)  $(N - 1)$  additional pairs of phase-space functions with Poisson brackets equal to one, all commuting with the first pair and therefore with the Hamiltonian. Consequently, this theorem is a blueprint to construct  $(2N - 1)$  independent constants of motion in any Hamiltonian system with  $N$  degrees of freedom. The particular form of the Hamiltonian does not even enter into the construction. Next, two different sets of phase-space functions are defined in terms of the  $(2N - 1)$  functions of this particular basis. Their Poisson brackets realize the relations characteristic of the algebras  $O(N)$  and  $SU(N)$ , respectively. In a footnote, the author restricts the applicability of the results: “We concern ourselves only with constructing realizations of Lie algebras, not of Lie groups. Even when we talk of invariance under the  $O(4)$  group, for example, we really intend invariance under the algebra.”<sup>15</sup> Consequently, “invariance under the algebra” is a *local* concept only, so that Mukunda’s construction has formal value only. Actually, the phase-space functions written down by Mukunda do not neatly map phase space onto itself: the functions become imaginary if the range of the canonical variables is not restricted artificially. The lesson to be learned is obvious: in order to establish the invariance of a system under a *group* of phase-space transformations it is not sufficient to realize specific Poisson-bracket relations with invariants.

A related position is put forward by Stehle and Han.<sup>18,19</sup> To identify a particular algebra by constants of motion does not guarantee the presence of a “higher symmetry”—a single-valued, or at most finitely many-valued, realization of the group must exist in phase-space. To show this, they show that a system is classically degenerate if the Hamilton–Jacobi equation of a particular system is separable in a continuous family of coordinate systems. This property is observable. Compare the Fourier series representation of one specific orbit described with respect to two different (continuously connected) coordinate systems. For consistency, the frequencies appearing in its Fourier decomposition must be rationally related, which corresponds to a classical degeneracy. It is important to note that the transformation from one coordinate system to the other be single-valued, otherwise the argument does not hold. Any phase-space function and, consequently, any constant of motion generates a transformation of phase-space onto itself; alternatively, it can be viewed as the generator for a transition to another coordinate system such that the Hamiltonian remains invariant. Only single-valued constants of motion generate global single-valued transformations—infinately many-valued “constants of motion” represent formal expressions only, not necessarily related to the existence of classical degeneracy. Therefore, they do *not* establish a higher symmetry group of the system.

To sum up, the construction of an algebra from constants of motion is only the first step in the proof of the existence of a potential higher symmetry group. It needs to be supplemented by a global investigation of the generated transformations.

### III. THE TWO-DIMENSIONAL COMMENSURATE OSCILLATOR

This section deals with the symmetry properties of a two-dimensional commensurate harmonic or  $(m_1, m_2)$ -oscillator described by the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{\omega}{2} \left( \frac{1}{m_1} (p_1^2 + q_1^2) + \frac{1}{m_2} (p_2^2 + q_2^2) \right), \quad m_1, m_2 \in \mathbb{N}_+, \quad (4)$$

where the integers  $m_1$  and  $m_2$  have no common divisor. Two pairs of canonical variables,  $q_n, p_n \in (-\infty, \infty), n=1,2$ , label points in phase space  $\Gamma \sim \mathbb{R}^4$ , the only nonvanishing Poisson brackets being given by

$$\{q_1, p_1\} = \{q_2, p_2\} = 1. \tag{5}$$

It will be useful to introduce two other sets of canonical variables. First, combine each pair into a complex variable

$$\alpha_n = \frac{1}{\sqrt{2}}(q_n + ip_n), \quad n=1,2, \tag{6}$$

with nonvanishing brackets

$$\{\bar{\alpha}_1, \alpha_1\} = \{\bar{\alpha}_2, \alpha_2\} = i, \tag{7}$$

where  $\bar{\alpha}$  denotes the complex conjugate of  $\alpha$ . Second, action-angle variables  $I_n \in [0, \infty)$  and  $\varphi_n \in [0, 2\pi), n=1,2$ , are determined through modulus and phase of  $\alpha_n = \sqrt{I_n} \exp[i\varphi_n]$ . Their nonzero brackets read

$$\{I_1, \varphi_1\} = \{I_2, \varphi_2\} = 1. \tag{8}$$

These coordinates provide alternative forms of the Hamiltonian,

$$H = \omega \left( \frac{\bar{\alpha}_1 \alpha_1}{m_1} + \frac{\bar{\alpha}_2 \alpha_2}{m_2} \right) = \omega \left( \frac{I_1}{m_1} + \frac{I_2}{m_2} \right). \tag{9}$$

**A. Constants of motion and Lie algebras**

Commensurate harmonic oscillators possess a large number of constants of motion. The Hamiltonian itself is an invariant as  $\{H, H\} = 0$ . Motion of the system with given energy  $E$  is thus restricted to a three-dimensional hyper-surface, an ellipsoid  $\mathcal{E}(E)$  in phase space  $\Gamma$ . Further, the actions  $I_1$  and  $I_2$ , having zero Poisson brackets with the Hamiltonian and among themselves, render the  $(m_1, m_2)$ -oscillator integrable. For fixed values of the actions, Arnold's theorem<sup>2</sup> states that the motion takes place on a two-dimensional torus  $\mathcal{T}(I_1, I_2)$ . In fact, the *entire* phase space is foliated by tori with radii  $\sqrt{I_1}$  and  $\sqrt{I_2}$ , respectively. According to (9) the Hamiltonian  $H$  is a linear function of these invariants.

A third, functionally independent (complex) constant of the motion is given by the expression

$$K = \alpha_2^{m_2} (\bar{\alpha}_1)^{m_1}. \tag{10}$$

As mentioned in Ref. 4, both its real and complex parts are invariant which implies that the phase  $\chi$  of the function  $K$ ,

$$\chi = m_2 \varphi_2 - m_1 \varphi_1 \in [0, 2\pi), \tag{11}$$

is a constant of the motion, too. Considered as a generator of transformations in phase space, it connects energetically degenerate pairs of tori. The existence of a third invariant is expected to reduce the dimensionality of the accessible manifold. Indeed, fixing the values of the three invariants  $I_1, I_2$ , and  $K$  (or, equivalently,  $\chi$ ) singles out a one-dimensional orbit on the torus  $\mathcal{T}$  if the two frequencies are rationally related. Generic orbits,  $\alpha_n(t) = \sqrt{I_n} \exp(-i\omega t/m_n + \varphi_n(0)), n=1,2$ , retrace themselves after a characteristic time  $t_m = 2\pi m_1 m_2 / \omega$ , with winding numbers  $m_2$  for  $\alpha_1$  and  $m_1$  for  $\alpha_2$ . However, if the frequency ratio of the motion on the tori were *not* rational, an orbit would cover the torus  $\mathcal{T}$  densely—the function  $K$  would represent a *formal* constant of the motion only, without any physical impact on the motion of the system. An important difference to the isotropic

oscillator is due to the fact that an  $\mathbf{m}$ -oscillator has different types of orbits with frequencies  $\omega/(2\pi m_1)$  and  $\omega/(2\pi m_2)$ , respectively. This allows one to distinguish experimentally the two cases.

The phase space of an  $\mathbf{m}$ -oscillator has a particular *discrete* symmetry. Combine the variables  $\alpha_n$  into a column: now the Hamiltonian is obviously invariant under  $m_1 m_2$  finite rotations,  $\alpha \rightarrow R_1^{r_1} R_2^{r_2} \alpha$ ,  $r_n = 0, \dots, m_n - 1$ , or, explicitly,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{-i2\pi r_1/m_1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i2\pi r_2/m_2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{12}$$

These transformations map the phase-space  $\Gamma$  to itself. They form a cyclic group  $C_{m_1 m_2} = C_{m_1} \times C_{m_2}$ , the direct product of two cyclic groups with  $m_1$  and  $m_2$  elements, respectively. In Ref. 20,  $C_{m_1 m_2}$  has been called *ambiguity group*.

The Poisson bracket of two invariants results in a third invariant. Therefore, the collection of all invariants is a Lie algebra. Typically, it will contain an infinite number of elements, all of which depend functionally on a smaller number of invariants. By an appropriate choice of the invariants, however, algebras with a finite number of elements can be found. The simplest example is given by the three invariants  $I_1, I_2, K$  giving rise to the following brackets:

$$\{I_1, K\} = -im_1 K, \quad \{I_2, K\} = im_2 K, \quad \{I_1, I_2\} = 0. \tag{13}$$

The algebra contains three independent elements—it is not possible to find an algebra with fewer elements since the  $\mathbf{m}$ -oscillator has three invariants. It also contains two elements with vanishing Poisson bracket which, in a system with two degrees of freedom, is the maximum number of “commuting” functionally independent invariants.

There is an alternative set of four invariants  $J = (J_0, \vec{J})$ ,

$$J_0 = \frac{I_1}{2m_1} + \frac{I_2}{2m_2} = \frac{1}{2\omega} H, \tag{14}$$

$$J_1 = \sqrt{\frac{I_1 I_2}{m_1 m_2}} \cos \chi, \tag{15}$$

$$J_2 = \sqrt{\frac{I_1 I_2}{m_1 m_2}} \sin \chi, \tag{16}$$

$$J_3 = \frac{I_1}{2m_1} - \frac{I_2}{2m_2}. \tag{17}$$

Only three of these invariants are functionally independent because

$$J_0^2 - \vec{J}^2 = 0. \tag{18}$$

This constraint is conveniently rephrased by saying that the “four vector”  $J$  is “null” or “light like.” The functions  $J$  are particularly interesting since they form the basis of a Lie algebra isomorphic to  $u(2)$ ,

$$\{J_0, J_j\} = 0, \quad \{J_j, J_k\} = \sum_{l=1}^3 \epsilon_{jkl} J_l, \quad j, k = 1, 2, 3, \tag{19}$$



which has  $su(2)$  as a subalgebra, generated by the components of  $\vec{J}$ . Equations (19) have been at the origin of many attempts to associate a group  $SU(2)$  of symmetry transformations with the two-dimensional  $\mathbf{m}$ -oscillator.

**B. Reduced phase space and space of invariants**

Consider the complex variables

$$\beta_n = \frac{|\alpha_n|}{\sqrt{m_n}} \left( \frac{\alpha_n}{|\alpha_n|} \right)^{m_n} = \sqrt{\frac{I_n}{m_n}} \exp[im_n\varphi_n], \quad n = 1, 2, \tag{20}$$

which satisfy

$$\{\bar{\beta}_n, \beta_{n'}\} = i\delta_{nn'}. \tag{21}$$

In spite of these relations, the variables  $\beta_n$  do not define pairs of canonical coordinates of  $\Gamma$  since the map  $\alpha \rightarrow \beta$  is not a one-to-one transformation. The variables  $\beta_n$  are, however, canonical coordinates in the *reduced* phase space  $\Gamma_{\mathbf{m}}$ . The reduced space is obtained from identifying those  $m_1 m_2$  points of  $\Gamma$  which satisfy  $\beta(R_1^{r_1} R_2^{r_2} \alpha) = \beta(\alpha)$ ,  $R_n^{r_n} \in \mathcal{C}_{m_n}$ . The definition of the variables (20) is motivated by the invariance of the constants of motion in (14)–(17) under the ambiguity group  $\mathcal{C}_{m_1 m_2}$ .

The invariants (14)–(17) take a simple form when expressed in terms of the reduced variables,

$$J_0 = \frac{1}{2}(\bar{\beta}_1\beta_1 + \bar{\beta}_2\beta_2), \tag{22}$$

$$J_1 = \frac{1}{2}(\bar{\beta}_1\beta_2 + \bar{\beta}_2\beta_1), \tag{23}$$

$$J_2 = \frac{1}{2i}(\bar{\beta}_1\beta_2 - \bar{\beta}_2\beta_1), \tag{24}$$

$$J_3 = \frac{1}{2}(\bar{\beta}_1\beta_1 - \bar{\beta}_2\beta_2). \tag{25}$$

Using the two-component ‘‘Weyl spinor’’  $\beta = (\beta_1, \beta_2)$ , these invariants can be written

$$J_\nu = \frac{1}{2}\bar{\beta} \cdot \sigma_\nu \beta, \quad \nu = 0, \dots, 3, \tag{26}$$

where  $\sigma_0 = \mathbf{1}_2$ , and the Pauli matrices  $\sigma_k$ ,  $k = 1, 2, 3$ , generate the algebra  $su(2)$ . Consequently, the invariants, which span the space of invariants,  $Y$ , turn into *sesquilinear* expressions on the reduced phase space  $\Gamma_{\mathbf{m}}$ . Their structure is similar to those of the isotropic or (1,1)-oscillator: in this case, the reduced phase space and the original one coincide,  $\Gamma_{(1,1)} = \Gamma$ . In some sense, the non-bijective map  $\alpha \rightarrow \beta$  ‘‘linearizes’’ the invariants at the expense of accounting for a fraction of phase space only. It will be shown later that the concept of the reduced space  $\Gamma_{\mathbf{m}}$  is natural in the present context as it provides appropriate setting to derive global statements about symmetry transformations.

**C. Topological aspects**

Turn now briefly to the topology of the spaces involved. Consider the nontrivial transformations introduced so far: first, the original phase space has been mapped to the reduced phase space,

$$\psi: \Gamma \rightarrow \Gamma_{\mathbf{m}}: \alpha \mapsto \beta(\alpha); \tag{27}$$

second, introducing the invariants  $J$  maps the reduced variables to the space of invariants,  $Y$ ,

$$\phi: \Gamma_{\mathbf{m}} \rightarrow Y: \beta \mapsto J(\beta), \tag{28}$$

which is an upper cone in  $\mathbb{R}^4$  since  $J_0 = |\vec{J}|$ .

The reduced phase space  $\Gamma_{\mathbf{m}}$  has the structure of a well-known fiber bundle. To see this, consider an orbit  $\alpha(t)$  in phase space  $\Gamma$ . Its image in the reduced space  $\Gamma_{\mathbf{m}}$  is given by  $\beta(t) = e^{-i\omega t} \beta(0)$ . The maps  $\beta \rightarrow e^{i\gamma} \beta$  form a group  $U(1)$  which leaves invariant the map  $\phi$ ,  $J(e^{i\gamma} \beta) = J(\beta)$ , since the phase drops out from the sesquilinear expressions given in Eqs. (26). Therefore,  $\Gamma_{\mathbf{m}}$  is indeed a *fiber bundle*  $(Y, \phi, \mathcal{O})$ : the invariants  $Y$  form the base, each orbit  $\mathcal{O}_{\beta_0} = \{e^{i\gamma} \beta_0 \mid \gamma \in [0, 2\pi)\}$  is a fiber, and the map  $\phi$  is the projection. The global structure of the bundle follows from the fact that the restriction of  $\Gamma_{\mathbf{m}}$  to the submanifold  $\Gamma_{\mathbf{m}}(E)$  with points  $\vec{\beta} \cdot \beta = E/\omega$  is isomorphic to the sphere  $S^3$ —as is obvious from the quadratic form (4). Thus, the restriction of the map  $\phi$  to  $\Gamma_{\mathbf{m}}(E)$  defines the *Hopf fibration* of  $S^3$ . To each orbit  $\mathcal{O}_{\beta}$  in  $S^3$  corresponds a point  $\vec{J}(\beta)$  of the sphere of radius  $J_0 = E/(2\omega)$  and a circle in the tangent space at this point.

It is interesting to look at the space  $Y$  of invariants and the transformations among them from a general perspective. To do so, consider the complex instead of the real Lie algebra  $\mathfrak{su}(2)$  which also leaves invariant the Hamiltonian  $H \sim J_0$  in (22) invariant. This is the Lie algebra  $\mathfrak{sl}(2, C)$  associated with the group  $SL(2, C)$ , the universal covering of the Lorentz group. The Lorentz group induced by  $SL(2, C)$  in  $Y$  is the transitivity group of the upper (half-) cone.

The elements of  $SL(2, C)$  can be written as  $u(\tau, \gamma) = \exp[g(\tau, \gamma)]$ , where  $\tau$  and  $\gamma$  are two real parameters, and each  $g$  is a traceless complex matrix,

$$g(\tau, \gamma) = \frac{1}{2} (\gamma \vec{\nu} \cdot \vec{\sigma} + i \tau \vec{n} \cdot \vec{\sigma}). \tag{29}$$

The matrices  $u(\tau, 0)$  belong to the group  $SU(2)$ . Thus, they generate rotations and infinitesimal transformations which can be written in terms of a Poisson bracket:

$$\frac{d\vec{J}(\tau)}{d\tau} = \vec{n} \wedge \vec{J}(\tau) \equiv \{\vec{J}, \vec{n} \cdot \vec{J}\}. \tag{30}$$

The subsets  $u(0, \gamma)$  represent Lorentz boosts mapping a point  $\beta$  according to

$$u(0, \gamma) \beta = (\cosh(\gamma/2) + \sinh(\gamma/2) \vec{\nu} \cdot \vec{\sigma}) \beta \equiv \beta(\gamma). \tag{31}$$

On the invariants, the transformation

$$J_{\nu} = \vec{\beta} \cdot \sigma_{\nu} \beta \mapsto J_{\nu}(\beta) = \vec{\beta}(\gamma) \cdot \sigma_{\nu} \beta(\gamma) \tag{32}$$

is induced. Hence, the sphere  $S^3$  of radius  $J_0 = H/(2\omega)$  is mapped to a sphere of radius  $J_0(\gamma)$  with

$$\frac{dJ_0}{d\gamma} = \vec{\nu} \cdot \vec{J}, \quad \frac{d\vec{J}}{d\gamma} = J_0 \vec{\nu}. \tag{33}$$

This is an infinitesimal Lorentz transformation which maps the upper cone  $Y \in \mathbb{R}^4$  to itself as is obvious from  $d(J_0(\gamma)^2 - \vec{J}(\gamma)^2)/d\gamma = 0$  and  $J_0$  remaining positive. Contrary to (30), it is not possible to express the right-hand sides of (33) by means of Poisson brackets. This can be understood from a quantum mechanical point of view. A classical theory can only manage Boltzmann statistics whereas in quantum (field) theory, due to the *anticommutativity* of Weyl spinors, it would be possible to find a commutator to express the derivatives  $dJ_{\nu}/d\gamma$ .



**IV. GLOBAL INVARIANT VECTOR FIELDS**

Each phase-space function generates a flow in phase space  $\Gamma$ , as well as in the reduced phase space  $\Gamma_m$ , and in the space of invariants  $Y$ . The invariants generate flows which commute with the Hamiltonian vector field. To be more specific, consider any element  $J_\nu, \nu=0\dots3$ , of the Lie algebra  $u(2)$ . When acting on an observable  $f$  through the Poisson bracket,

$$V_\nu = \{f, J_\nu\}, \quad \nu = 0, \dots, 3, \tag{34}$$

it defines a vector field  $V_\nu$  in  $\Gamma$ . Its integral lines satisfy the differential equation

$$\frac{df}{d\tau} = V_\nu. \tag{35}$$

The solution of this differential equation is a map  $f(\tau)$  which will be written in the form

$$f(\tau) = \text{Exp}[\tau J_\nu](f) = \sum_{k=0}^{\infty} \{f, J_\nu\}_k \frac{\tau^k}{k!}, \tag{36}$$

where

$$\{g, h\}_{k+1} = \{\{g, h\}_k, h\}, \quad k = 1, 2, \dots, \quad \{g, h\}_0 = g, \tag{37}$$

with smooth phase-space functions  $g$  and  $h$ . In a simplified notation, the solutions (36) are written as

$$S_\nu[\tau] \equiv \text{Exp}[\tau J_\nu], \quad \nu = 0, \dots, 3, \quad S_{\vec{n}}[\tau] \equiv \text{Exp}[\tau \vec{n} \cdot \vec{J}], \quad |\vec{n}| = 1, \tag{38}$$

each unit vector  $\vec{n}$  being associated with a point of the unit sphere  $S^2$ .

The crucial question now is to investigate whether the flow (34) and hence the maps (38) are defined everywhere in the space under consideration. Only in this case, the algebra formed by the closed set of Poisson brackets among the invariants integrates to a group of symmetry transformations. More specifically, one needs to find out whether the invariants (14)–(17) of the  $\mathbf{m}$ -oscillator generate a set of transformations isomorphic to the group  $SU(2)$  [or  $U(2)$ ]. This is only possible if the associated vector fields are well-defined everywhere in the space where they act. The fields will be studied separately for functions  $f$  from the spaces  $\Gamma, \Gamma_m$ , or  $Y$ .

**A. Vector fields in the space of invariants**

The simplest case to look at is the orbits generated by the first component of  $J$ , which is a multiple of the Hamiltonian,  $J_0 = H/(2\omega)$ . Not surprisingly, one has

$$S_0[\tau](J) = J, \tag{39}$$

that is, all components of  $J$  are invariant under the action of  $J_0$ . Rotations about the 3-axis, i.e., with an axis passing through the poles  $J_3 = \pm J_0$ , are generated by the invariant  $J_3 = I_1/(2m_1) - I_2/(2m_2)$ ,

$$S_3[\tau](J) = (J_0, R_3(\tau)\vec{J}). \tag{40}$$

Each possible orbit in  $Y$  is generated by a linear combination of invariants  $\vec{n} \cdot \vec{J}$ ,

$$S_{\vec{n}}[\tau](J) = (J_0, R_{\vec{n}}(\tau)\vec{J}), \tag{41}$$

where the matrix  $R_{\vec{n}}(\tau)$  represents a rotation by an angle  $\tau$  about an axis parallel to the vector  $\vec{n}$ . In other words, every point of the sphere  $|\vec{J}|=J_0$  is mapped to another point of the same sphere, the energy  $E=2\omega J_0$  being conserved.

These results are conveniently summarized by a group theoretical statement. The set

$$\mathcal{R}_J = \{S_{\vec{n}}[\tau] \mid 0 \leq \tau < 2\pi, \vec{n} \in S^2\} \tag{42}$$

of maps acting in  $Y$  is a representation of the group  $SO(3)$ . In other words, there is a subset of all phase-space functions, such that its elements transform according to the group  $SO(3)$ . Mathematically, this group is the integrated form of the adjoint representation of the algebra (19). Consequently, one can attribute this group as a *symmetry group* to the reduced  $(m_1, m_2)$ -oscillator, for any frequency ratio. Note, however, that this symmetry does not act on points in phase space  $\Gamma$  but on points of the space of invariants  $Y$ .

### B. Vector fields in the reduced phase space

Again, the action of the generators  $J_0, J_3$ , and  $\vec{n} \cdot \vec{J}$  will be studied, now with respect to the variables  $\beta = (\beta_1, \beta_2)$ . It is straightforward to see that

$$S_0[\tau](\beta) = e^{-i\tau/2}\beta, \tag{43}$$

which is just the time evolution with  $\tau = 2\omega t$ . Similarly, the invariant  $J_3$  generates a flow

$$S_3[\tau](\beta) = \begin{pmatrix} e^{-i\tau/2} & 0 \\ 0 & e^{i\tau/2} \end{pmatrix} \beta. \tag{44}$$

Comparison with (20) shows that the function  $\chi = m_2\varphi_2 - m_1\varphi_1$  is left invariant. Transformation (44) is a special case of the map

$$S_{\vec{n}}[\tau](\beta) = (\sigma_0 \cos \tau/2 - i\vec{n} \cdot \vec{\sigma} \sin \tau/2)\beta \equiv \beta(\tau). \tag{45}$$

No ambiguities arise when mapping points  $\beta$  under  $S_{\vec{n}}[\tau]$ , for whatever values of the parameter  $\tau$  and the directions  $\vec{n}$ . Therefore, the set

$$\mathcal{R}_\beta = \{S_{\vec{n}}[\tau] \mid 0 \leq \tau < 4\pi, \vec{n} \in S^2\} \tag{46}$$

of maps faithfully represents the group  $SU(2)$  in  $\Gamma_{\mathbf{m}}$ . Consequently, an  $\mathbf{m}$ -oscillator admits as symmetry not only the three-dimensional rotation group  $SO(3)$  in  $Y$  but also the special unitary group  $SU(2)$  in  $\Gamma_{\mathbf{m}}$ .

In this restricted sense, and only in this one,  $(m_1, m_2)$ -oscillators are seen to possess both  $SO(3)$  and  $SU(2)$  as symmetry groups. This statement agrees with the fact that the algebras  $\mathfrak{so}(3)$  and  $\mathfrak{su}(2)$  are isomorphic. The next section deals with the question which groups, if any, are represented on the original phase space  $\Gamma$ .

### C. Vector fields acting in phase space

It will be shown in this section that the vector fields associated with the invariants  $J$  are not defined globally when they act on the variables  $\alpha$  which span phase space  $\Gamma$ . Consequently, it is *not* possible to implement the group  $SU(2)$  on phase space  $\Gamma$ . More explicitly, it will be shown that the action  $S_{\vec{n}}[\tau](\alpha) = (\alpha_1(\tau), \alpha_2(\tau))$  on  $\Gamma$  is nonlinear, and that it is inevitably singular for some parameters  $(\vec{n}, \tau)$  and initial points  $\alpha$ . Contrary to one's intuition the flows can be defined only locally, and they cannot be extended to define a *group* of symmetry transformations.

To begin with, consider the flows generated by  $J_0$  and  $J_3$ , respectively. The resulting orbits *are* well-defined for all initial points: they are given by

$$S_0[\tau](\alpha) = \begin{pmatrix} e^{-i\tau/(2m_1)} & 0 \\ 0 & e^{-i\tau/(2m_2)} \end{pmatrix} \alpha, \tag{47}$$

and by

$$S_3[\tau](\alpha) = \begin{pmatrix} e^{-i\tau/(2m_1)} & 0 \\ 0 & e^{i\tau/(2m_2)} \end{pmatrix} \alpha, \tag{48}$$

respectively. Equation (47) describes the time evolution of the point  $\alpha \in \Gamma$ , hence both the energy  $E = 2\omega J_0$  and the torus  $\mathcal{T}(I_1, I_2)$  are left invariant. Since the values of the actions change according to  $I_n(0) \rightarrow I_n(\tau) = |\alpha_n(\tau)|^2$ , the flow in (48) also conserves the energy while it maps a torus  $\mathcal{T}(I_1, I_2)$  to a different one,  $\mathcal{T}(I_1(\tau), I_2(\tau))$ .

Now consider fields which are generated by *arbitrary* linear combinations of the invariants,  $\vec{n} \cdot \vec{J}$ . Denote potential solutions of the differential equation

$$\frac{d\alpha}{d\tau} = \{\alpha, \vec{n} \cdot \vec{J}\} \equiv V_{\vec{n}}(\alpha) \tag{49}$$

by  $\alpha_{\vec{n}}(\tau) = S_{\vec{n}}[\tau](\alpha(0))$ , with some initial point  $\alpha(0) \in \Gamma$ . Explicitly, the complex two-component field  $V_{\vec{n}}$  reads

$$V_{\vec{n}} = \frac{1}{2i} \left( \frac{(\vec{n} \cdot \vec{J} + im_1(\vec{n} \wedge \vec{J})_3 + n_3 J_0) / \bar{\alpha}_1}{(\vec{n} \cdot \vec{J} + im_2(\vec{n} \wedge \vec{J})_3 - n_3 J_0) / \bar{\alpha}_2} \right). \tag{50}$$

It is finite but ill-defined on the hyperplanes  $\mathcal{P}_1 = \{\alpha | \alpha_1 = 0, \alpha_2 \neq 0\}$  and  $\mathcal{P}_2 = \{\alpha | \alpha_1 \neq 0, \alpha_2 = 0\}$ . There are points which, when transported by the flow  $S_{\vec{n}}[\tau]$ , hit the planes  $\mathcal{P}_1$  or  $\mathcal{P}_2$  for some value of  $\tau$ . The associated orbits will be called *singular* since they cannot be continued unambiguously across the planes. This is due to the terms in (50) which contain  $J_1$  and  $J_2$ ,

$$J_1 \pm iJ_2 = \frac{|\alpha_1| |\alpha_2|}{\sqrt{m_1 m_2}} \exp[\pm i\chi], \tag{51}$$

while all other terms are zero on  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Here is a toy example to illustrate the underlying problem. Consider a one-dimensional system with variable  $\alpha = \sqrt{I} \exp[i\varphi]$ , satisfying  $\{I, \varphi\} = 1$ . The flow generated by  $\sqrt{I}$  is ill-defined at the origin,

$$\frac{d\alpha}{d\tau} = \{\sqrt{I}, \alpha\} = \frac{i}{2} \exp[i\varphi], \tag{52}$$

as its value depends on the way the point  $\alpha$  is approached. If a trajectory were reaching the origin, it would be impossible to continue it unambiguously beyond this point. It is important to realize that this singularity as well as the one encountered in the singular planes is not due to a choice of coordinates but an intrinsic property of the flow.

To visualize the entire set of singular orbits, look at their images in  $Y$ , that is, the orbits  $S_{\vec{n}}[\tau](\vec{J})$ ,  $\tau \in \mathbb{R}$ . For given energy  $E = 2\omega J_0$ , the points of  $\mathcal{P}_1$  correspond to the north pole  $(0, 0, J_0 = |\alpha_1|^2/2m_2)$  of the sphere  $S^2(J_0)$ , while those of  $\mathcal{P}_2$  are mapped to its south pole,  $(0, 0, -J_0 = -|\alpha_2|^2/(2m_1))$ . By  $\psi \circ \phi$ , an orbit  $S_{\vec{n}}[\tau](\alpha)$  goes to a circle  $R_{\vec{n}}(\tau)\vec{J}(\alpha)$ . *Singular* orbits thus correspond to circles going through either one or both poles of the sphere, while *regular* orbits hit neither of them: for almost all flows, associated with a given vector  $\vec{n}$ , there exist two ‘‘critical’’ circles passing through the north pole and the south pole, respectively. These circles

coalesce into a single one passing through *both* poles if the axis of rotation is in the equatorial plane,  $\vec{n}=(n_1, n_2, 0)$ . They degenerate to points located at the poles if  $\vec{n}=\vec{e}_3=(0, 0, \pm 1)$ . Two conclusions can be drawn from this picture:

- (1) for any given unit vector  $\vec{n} \neq \pm \vec{e}_3$ , the map  $S_{\vec{n}}[\tau]$  has at least one singular orbit in  $\Gamma$ ;
- (2) any point  $\alpha \in \Gamma$  can be sent to a singular hyperplane by a map  $S_{\vec{n}}[\tau]$  with an appropriately chosen vector  $\vec{n}_{\mathcal{P}}$ .

In fact, the vectors  $\vec{n}_{\mathcal{P}}$  can be chosen from *two* continuous sets: they only need to be in a plane (passing through the origin) which is perpendicular to either of the vectors  $\vec{J}(\alpha) \pm J_0(\alpha)\vec{e}_3$ , or, explicitly,

$$\vec{n}_{\mathcal{P}}^{\pm} = \frac{c_1(\vec{J}(\alpha) \pm J_0(\alpha)\vec{e}_3) \pm c_2\vec{J}(\alpha) \wedge \vec{e}_3}{|c_1(\vec{J}(\alpha) \pm J_0(\alpha)\vec{e}_3) \pm c_2\vec{J}(\alpha) \wedge \vec{e}_3|}. \tag{53}$$

*Regular* orbits of  $S_{\vec{n}}[\tau]$  are easily computed without solving the differential equation (49). One needs to determine modulus and phase of the variables  $\alpha_n, n=1, 2$ , as a function of  $\tau$ . It is useful to write down the orbits in the reduced phase space and in the space of the invariants. According to (45), the reduced variables evolve linearly,

$$\beta_1(\tau) = (\cos \tau/2 - in_3 \sin \tau/2)\beta_1 - (in_1 + n_2)\sin(\tau/2)\beta_2, \tag{54}$$

$$\beta_2(\tau) = (\cos \tau/2 + in_3 \sin \tau/2)\beta_2 - (in_1 - n_2)\sin(\tau/2)\beta_1, \tag{55}$$

while the invariants  $\vec{J}=\vec{J}(\beta)$  evolve in  $Y$  as

$$\vec{J}(\tau) = \cos \tau \vec{J} + (1 - \cos \tau)\vec{n} \circ \vec{n} \cdot \vec{J} - \sin \tau \vec{n} \wedge \vec{J}. \tag{56}$$

Using  $|\alpha_n|^2 = m_n |\beta_n|^2, n=1, 2$ , the  $\tau$ -dependence of the moduli is simply

$$|\alpha_1(\tau)|^2 = m_1(j_0 + j_3(\tau)), \quad |\alpha_2(\tau)|^2 = m_2(j_0 - j_3(\tau)). \tag{57}$$

For the evolution of the phases, plug Eqs. (54) into

$$\exp[im_1\varphi_1(\tau)] = \left( \frac{\alpha_n(\tau)}{|\alpha_n(\tau)|} \right)^{m_n} = \frac{\beta_n(\tau)}{|\beta_n(\tau)|}, \quad n=1, 2, \tag{58}$$

giving

$$\exp[im_1\varphi_1(\tau)] = \frac{(\cos \tau/2 - in_3 \sin \tau/2)\beta_1 - ((in_1 + n_2)\sin \tau/2)\beta_2}{|(\cos \tau/2 - in_3 \sin \tau/2)\beta_1 - ((in_1 + n_2)\sin \tau/2)\beta_2|} \equiv \exp[i\Phi_1(\tau)], \tag{59}$$

and a similar equation for  $\exp[im_2\varphi_2(\tau)]$ . The two phases  $\varphi_n(\tau) = \Phi_n(\tau)/m_n, n=1, 2$ , must be continuous whenever  $\tau$  reaches the value  $4\pi$ . They will both have a value which is a multiple of  $2\pi$  when the parameter  $\tau$  takes the value  $4\pi m_1 m_2$ . This result seems to suggest that  $S[\tau\vec{n} \cdot \vec{J}](\alpha)$  might be an  $m_1 m_2$ -fold covering of the subgroup  $S[\tau\vec{n} \cdot \vec{J}](\beta), 0 \leq \tau < 4\pi, \vec{n} \in S^2$ , of the special unitary group,  $SU(2)$ . Due to the existence of singular orbits, however, this is not possible. Further, it is well-known that the only universal covering of  $SU(2)$  is this group itself. Nevertheless, one might describe the situation as a *ramified covering* of  $SU(2)$  since the maps  $S[\tau\vec{n} \cdot \vec{J}]$  combine according to a group product law.

To visualize the obstruction of a global action of the group  $SU(2)$  differently, recall that a given map  $S[\tau\vec{n} \cdot \vec{J}]$  sends a torus  $\mathcal{T}(I_1, I_2) \in \Gamma$  to a torus  $\mathcal{T}(I'_1, I'_2)$  such that  $m_1 I_1 + m_2 I_2 = m_1 I'_1 + m_2 I'_2$  holds. For some  $\vec{n}$  and  $\tau_0$  it happens that one of the actions vanishes,  $I'_1$ , say. This means that the initial two-dimensional torus ( $S^1 \times S^1$ ) is mapped to a *one-dimensional torus*, i.e., a circle

$S^1$ , and, therefore, one of the angle variables has lost its meaning. Once this has happened, it is impossible to unambiguously continue the trajectory which has hit the singular plane, as the missing angle could take any value. The phenomenon is similar to the passage of a spherical wave through a focus.

It will be useful to give a name to the situation encountered here. A system with phase space  $\Gamma$  will be said to have a *faint G symmetry* if it admits a set of globally defined invariants which form an algebra  $\mathcal{A}$  while the group  $G$  associated with it cannot be realized on  $\Gamma$  but only on a smaller part of it. Thus, all two-dimensional commensurate  $\mathbf{m}$ -oscillators have a faint  $SU(2)$  symmetry.

**V. THE N-DIMENSIONAL COMMENSURATE OSCILLATOR**

To describe a commensurate harmonic oscillator in  $N$  dimensions, the present notation is straightforward to adapt. Let the label  $n$  run from 1 to  $N$ : the Hamiltonian of a commensurate  $\mathbf{m}$ -oscillator with  $\mathbf{m}=(m_1, \dots, m_N)$ ,  $m_n \in \mathbb{N}_+$ , reads

$$H(q, p) = \frac{\omega}{2} \sum_{n=1}^N \frac{1}{m_n} (p_n^2 + q_n^2) = \frac{\omega}{2} \sum_{n=1}^N \frac{1}{m_n} \bar{\alpha}_n \alpha_n = \frac{\omega}{2} \sum_{n=1}^N \frac{1}{m_n} I_n. \tag{60}$$

The complex canonical variables are given by  $\alpha_n = (q_n + ip_n)/\sqrt{2}$ ,  $n = 1, \dots, N$ , while actions  $I_n$  and angles  $\varphi_n$  are defined through  $\alpha_n = \sqrt{I_n} \exp[i\varphi_n]$ . Thus there are three sets of  $N$  pairs of canonical variables to choose from, with brackets

$$\{q_n, p_{n'}\} = \frac{1}{i} \{\bar{\alpha}_n, \alpha_{n'}\} = \{I_n, \varphi_{n'}\} = \delta_{nn'}, \quad n, n' = 1, \dots, N. \tag{61}$$

It will be assumed that the positive integer numbers  $m_n$  do not have an overall common divisor. For the discussion to follow, two cases will be distinguished: a commensurate oscillator is said to be *canonical* if *no* pair of numbers  $m_n$  and  $m_{n'}$ ,  $n \neq n'$ , admits a common divisor but one. This class will be studied first. The presence of common divisors among subsets of the frequencies  $\omega_n$  gives rise to interesting additional complications which will be considered later on.

**A. Constants of motion and Lie algebras**

In analogy to Eq. (10), each function

$$K_{nn'} = \alpha_n^{m_n} (\bar{\alpha}_{n'})^{m_{n'}}, \quad n, n' = 1, \dots, N, \tag{62}$$

is seen to be an invariant for the commensurate  $N$ -oscillator,  $\{H, K_{nn'}\} = 0$ . These  $N^2$  constants of motion depend on only  $N(N+1)/2$  real invariants, namely  $N$  independent actions  $I_n, n = 1, \dots, N$ , and  $N(N-1)/2$  relative angles

$$\chi_{nn'} = m_n \varphi_n - m_{n'} \varphi_{n'}, \quad 1 \leq n < n' \leq N. \tag{63}$$

As in the two-dimensional case, the range of the functions  $\chi_{nn'}$  must be restricted to the interval  $[0, 2\pi)$  because two values  $\chi_{nn'}$  and  $(\chi_{nn'} + 2\pi)$ , respectively, correspond to the *same* orbit. The angles  $\chi_{nn'}$  satisfy  $(N-1)(N-2)/2$  linear relations,

$$\chi_{nn'} + \chi_{n'n''} + \chi_{n''n} \equiv 0, \quad n, n', n'' \text{ all different.} \tag{64}$$

Therefore, there are no more than  $(2N-1)$  functionally independent constants of motion, the maximum number of possibly independent invariants. As independent invariants, one may choose, for example, the  $N$  actions  $I_n$  and  $(N-1)$  relative angles  $\chi_{n, n+1}, n = 1, \dots, N-1$ .

The  $(2N-1)$ - dimensional surface of constant energy  $H=E$  is an ellipsoid  $\mathcal{E}(E)$  in phase space  $\Gamma$ . It contains the  $N$ -dimensional torus  $\mathcal{T}(I_1, \dots, I_N)$  of constant actions  $I_n$  as a submanifold. Lines of constant actions and angles are the *orbits* of the motion, winding around a torus  $\mathcal{T}$ . Each orbit is a one-dimensional closed loop given by

$$\alpha_n(t) = \sqrt{I_n} \exp(-i\omega t/m_n + \varphi_n(0)), \tag{65}$$

where  $m_n\varphi_n(0) - m_{n+1}\varphi_{n+1}(0) = \chi_{n,n+1}(0)$ . One revolution is completed after a time  $t = 2\pi M/\omega$ , with the number  $M$  taking a value such that the winding numbers  $w_n = M/m_n$  of each subsystem are integer without overall common divisor. In the canonical case,  $M$  is equal to  $\prod_1^N m_n$ . Here is an example for  $N=3$  which illustrates the noncanonical case: let  $\mathbf{m} = (km'_1, km'_2, m_3)$ . The number  $M$  would then take the value  $km'_1 m'_2 m_3 = m_1 m_2 m_3 / k$ .

It is important to note that in a canonical (but not isotropic)  $\mathbf{m}$ -oscillator (i.e., all  $m_n \neq 1$ ), there exist orbits with  $(2^N - 1)$  different periods. There are  $N$  orbits corresponding to motion of a single oscillator only; there are  $N(N-1)/2$  orbits winding around two-dimensional tori with frequencies  $1/m_n$  and  $1/m_{n'}$ ,  $1 \leq n < n' \leq N$ , etc.

As in the two-dimensional case, the maps  $R_n \alpha = (\alpha_1, \dots, e^{i2\pi/m_n} \alpha_n, \dots, \alpha_N)$  generate a cyclic group  $C_m = \{R_1^{r_1} \dots R_N^{r_N} | r_n \in \mathbb{Z}\}$ , the ambiguity group of the map  $\psi$ :

$$\psi(R_1^{r_1} \dots R_N^{r_N} \alpha) = \psi(\alpha). \tag{66}$$

### B. Reduced phase space and space of invariants

The  $(2N-1)$  phase-space functions  $I_n$  and  $\chi_{n,n+1}$  form a basis of a Lie algebra commuting with the Hamiltonian  $H$ . Since the functions  $\chi_{nn'}$  are not continuous on phase space  $\Gamma$ , it is natural to look at appropriate periodic functions of them. Introduce, in analogy to Eq. (20), the set of invariants

$$\beta_n = \frac{|\alpha_n|}{\sqrt{m_n}} \left( \frac{\alpha_n}{|\alpha_n|} \right)^{m_n} = \sqrt{\frac{I_n}{m_n}} \exp[im_n \varphi_n], \quad n = 1, \dots, N. \tag{67}$$

They provide canonical coordinates on the  $2N$ -dimensional reduced phase space  $\Gamma_{\mathbf{m}}$ , now with  $\mathbf{m} = (m_1, \dots, m_N)$ ,

$$\{\bar{\beta}_n, \beta_{n'}\} = i \delta_{nn'}. \tag{68}$$

As before, (67) is a non-bijective map  $\psi: \alpha \rightarrow \beta(\alpha)$ . It is not a projection of the phase space on a subspace but should be thought of as a ramified cover of the reduced space  $\Gamma_{\mathbf{m}}$ .

Not surprisingly, Eqs. (26) have a straightforward generalization. With  $\beta = (\beta_1, \dots, \beta_N)$ , one defines  $(N^2 - 1)$  invariants sesquilinear in the coordinates  $\beta_n$  by

$$J_{nn'}^s = \frac{1}{2} \bar{\beta} \cdot (E_{nn'} + E_{n'n}) \beta = \frac{1}{2} (\bar{\beta}_n \beta_{n'} + \bar{\beta}_{n'} \beta_n), \quad 1 \leq n < n' \leq N, \tag{69}$$

$$J_{nn'}^a = \frac{1}{2} \bar{\beta} \cdot \frac{1}{i} (E_{nn'} - E_{n'n}) \beta = \frac{1}{2i} (\bar{\beta}_n \beta_{n'} - \bar{\beta}_{n'} \beta_n), \quad 1 \leq n < n' \leq N, \tag{70}$$

$$J_{nn}^d = \frac{1}{2} \bar{\beta} \cdot (E_{nn} - E_{n+1 n+1}) \beta = \frac{1}{2} (\bar{\beta}_n \beta_n - \bar{\beta}_{n+1} \beta_{n+1}), \quad n = 1, \dots, N-1. \tag{71}$$

The matrices  $E_{nn'}$  are of size  $N \times N$  with elements

$$(E_{nn'})_{kk'} = \delta_{nk} \delta_{n'k'}, \quad n, n', k, k' = 1, \dots, N, \tag{72}$$

i.e., the only nonzero elements are equal to one at position  $(n, n')$ , and they generate the Lie algebra  $u(n)$  with respect to the matrix commutator.<sup>21</sup> This property is inherited by the  $N^2$  phase-space functions

$$J_{nn'} = \bar{\beta} \cdot E_{nn'} \beta \equiv \bar{\beta}_n \beta_{n'}; \tag{73}$$

their Poisson brackets,



$$\{J_{nn'}, J_{kk'}\} = i(\delta_{nk'}J_{kn'} - \delta_{n'k}J_{nk'}), \tag{74}$$

also realize the algebraic relations of  $u(N)$ .

It is possible to find  $(N^2 - 1)$  linear combinations of the matrices  $E_{nn'}$ , which are traceless and Hermitian—hence they provide a basis of the algebra  $su(N)$ . In fact, these combinations have been introduced already in Eq. (69) when defining  $J_{nn'}^s, J_{nn'}^a$ , and  $J_{nn}^d$ . Therefore, these functions form a basis of the algebra  $su(N)$  with respect to the Poisson bracket. When supplemented by (a multiple of) the Hamiltonian

$$J_0 = \frac{1}{2}\bar{\beta} \cdot \mathbf{1}_N \beta = \frac{1}{2} \sum_{n=1}^N \bar{\beta}_n \beta_n \equiv \frac{2}{\omega} H, \tag{75}$$

where  $\mathbf{1}_N$  is the  $N$ -dimensional unit matrix, the algebra  $u(n)$  can be recovered.

**C. Vector fields**

There is a first group of transformations which acts in the space of invariants  $Y$ . As before, it is the set of finite transformations on the space generated by the real invariants (69)–(71). In other words, it arises from integrating the adjoint representation of the algebra formed by the invariants. As this group will play no role in the following, its discussion is suppressed.

Next, the invariants  $J_{nn'}^s$ , and  $J_{nn'}^a$ , generate canonical linear maps in the reduced space  $\Gamma_{\mathbf{m}}$ ,

$$\frac{d\beta_k}{d\tau} = \{\beta_k, J_{nn'}^s\} = \frac{i}{2}(\delta_{kn}\beta_{n'} + \delta_{kn'}\beta_n) \equiv (J_{nn'}^s \beta)_k, \tag{76}$$

$$\frac{d\beta_k}{d\tau} = \{\beta_k, J_{nn'}^a\} = \frac{1}{2}(\delta_{kn}\beta_{n'} - \delta_{kn'}\beta_n) \equiv (J_{nn'}^a \beta)_k, \tag{77}$$

and similar ones follow when taking  $J_{nn}^d$  as generator. These linear equations can be integrated in the space  $\Gamma_{\mathbf{m}}$  for arbitrary initial values  $\beta(0) = \beta_0 \in \Gamma_{\mathbf{m}}$ ,

$$\beta(\tau) = \exp(\tau J_{nn'}^\varepsilon) \beta_0, \quad \varepsilon = a, s. \tag{78}$$

The solutions are unitary maps of  $\Gamma_{\mathbf{m}}$  to itself. In analogy to the two-dimensional case, they will be denoted by

$$\beta(\tau) = \text{Exp}[\tau J_{nn'}^\varepsilon](\beta), \quad \varepsilon = a, s, \tag{79}$$

and similarly for finite transformations generated by the invariants  $J_{nn}^d$ . Due to the linearity of the equations, no ambiguities arise upon integration. Therefore, the set of transformations in the reduced space  $\Gamma_{\mathbf{m}}$  is isomorphic to the group  $SU(N)$ . In this restricted sense, the  $\mathbf{m}$ -oscillator has the special unitary group in  $N$  dimensions as a symmetry group. This group of symmetry transformations is *not* defined in the phase space  $\Gamma$  of the  $\mathbf{m}$ -oscillator but only in  $\Gamma_{\mathbf{m}}$ .

Finally, a genuine “pullback” of  $SU(N)$  in phase space  $\Gamma$  does not exist, for the same reasons as in the case  $N=2$ . In fact, it is sufficient to consider a pair of oscillators with frequencies  $\omega/m_n$  and  $\omega/m_{n'}$ , say, in order to see that there are obstructions which prevent the existence of a *global* symmetry group in phase space  $\Gamma$ . This pair of degrees of freedom is equivalent to a two-dimensional  $(m_n, m_{n'})$ -oscillator, and no set of transformations acting on it can be found which would be isomorphic to  $SU(2)$ . If, however, the  $N$ -dimensional oscillator would have the full symmetry  $SU(N)$ , a subgroup  $SU(2)$  should be associated with this pair of oscillators. Consequently, the group  $SU(N)$  cannot be identified as a symmetry group of the canonical  $N$ -dimensional commensurate oscillator. In analogy to the two-dimensional commensurate oscillator it is seen to have a *faint*  $SU(N)$  symmetry only.

**D. The  $\mathbf{m}$ -oscillator with common divisors**

The canonical  $\mathbf{m}$ -oscillator has been shown to be invariant under transformations isomorphic to the group  $SU(N)$  in the reduced space  $\Gamma_{\mathbf{m}}$ . For canonical and isotropic  $N$ -dimensional oscil-

lators, subsystems of dimension  $N' < N$  are invariant only with respect to a subalgebra  $\mathcal{A}_{\mathbf{m}'}$  of the algebra  $\mathcal{A}_{\mathbf{m}} = \text{su}(N)$ . If the oscillator is neither isotropic nor canonical, other possibilities arise.

A noncanonical oscillator is characterized by frequencies  $\mathbf{m} = (m_1, \dots, m_N)$  with at least one pair  $(m_k, m_l)$  having a common integer divisor different from one. Let  $N' < N$  frequencies have a common divisor. Then, for the  $\mathbf{m}'$ -oscillator corresponding to these frequencies, constants of motion do exist which form an algebra  $\mathcal{A}_{\mathbf{m}'} = \text{su}(N')$ . This algebra, however, is *not* a subalgebra of  $\mathcal{A}_{\mathbf{m}}$  as follows immediately from considering the  $\mathbf{m}'$ -oscillator as an  $N'$ -dimensional commensurate oscillator in its own right. Suppose that, after removing the common divisor, the resulting oscillator, characterized by  $\mathbf{m}' = (m'_1, \dots, m'_{N'})$ , is a canonical one. Then one can construct a group of symmetry transformations  $\text{SU}(N')$  in the reduced phase space  $\Gamma_{\mathbf{m}'}$ , and  $\Gamma_{\mathbf{m}'}$  is *not* a subspace of  $\Gamma_{\mathbf{m}}$ . The Poisson brackets of the generators of  $\text{SU}(N')$  acting in  $\Gamma_{\mathbf{m}'}$  and those of  $\text{SU}(N)$  acting in  $\Gamma_{\mathbf{m}}$  will not be linear combinations of the initial ones. Hence, the combination of these two algebras will not close under the Lie product—the resulting algebra will be *infinite*. This property will be important for quantum mechanical commensurate oscillators since it entails additional degeneracies of energy which otherwise appear to be accidental.

Turn these results around: there is no finite algebra to account for all the symmetries of a noncanonical  $\mathbf{m}$ -oscillator. Obviously, this situation can arise only if  $N \geq 3$  (if  $N = 2$  any common divisor can be factored out immediately). In fact, if  $m'_n = 1$ ,  $n' = 1, \dots, N'$ , the subsystem is even an isotropic oscillator, and it has a group  $\text{SU}(N')$  of symmetry transformations on phase space  $\Gamma$ .

It is helpful to illustrate this discussion by an exhaustive list of “classes” for small values of  $N$ .

◇  $N = 2$ : A commensurate oscillator is either isotropic or canonical (a common divisor of the frequencies  $m_1 \neq m_2$  can be factored out).

◇  $N = 3$ : Five classes of commensurate oscillators can be identified. An  $\mathbf{m}$ -oscillator is either isotropic or canonical, or it belongs to one of the three following classes:

- (1) a single pair of two frequencies have a common divisor,  $\mathbf{m} = (jm'_1, jm'_2, m_3)$ , say;
- (2) two pairs have common but different divisors,  $\mathbf{m} = (jm'_1, jkm'_2, km'_3)$ , say;
- (3) all three pairs have common but different divisors,  $\mathbf{m} = (jkm'_1, klm'_2, ljm'_3)$ , say.

For  $N > 3$  the number of different classes increases rapidly with  $N$ .

Consider an example of type 1 for  $N = 3$  in detail. The three coordinates  $\beta_n$  of the space  $\Gamma_{\mathbf{m}}$  allow one to define eight constants of motion  $J$ . In addition, introduce coordinates of the reduced phase space  $\Gamma_{\mathbf{m}'}$ ,

$$\beta'_n = \frac{|\alpha_n|}{\sqrt{m'_n}} \left( \frac{\alpha_n}{|\alpha_n|} \right)^{m'_n}, \quad n = 1, 2. \tag{80}$$

The four functions

$$J'_{nn'} = \bar{\beta}'_n \beta'_{n'}, \quad n, n' = 1, 2, \tag{81}$$

are a different set of constants of motion because the Hamiltonian of the subsystem (1,2) has an overall factor  $1/k$ . The constants  $J'$  are the basis of a Lie algebra  $\mathcal{A}_{\mathbf{m}'}$  isomorphic to  $\text{su}(2)$  (setting aside the fourth commuting invariant), as the subsystem is an  $\mathbf{m}'$ -oscillator with  $N = 2$ . The resulting algebra  $\mathcal{A}_{\mathbf{m}'}$  gives rise to another faint  $\text{SU}(2)$  symmetry. It is, however, *neither* a subalgebra of the faint  $\text{SU}(N)$  symmetry (as it is implemented on a different reduced phase space  $\Gamma_{\mathbf{m}'}$ ) *nor* do the generators of  $\mathcal{A}_{\mathbf{m}}$  and  $\mathcal{A}_{\mathbf{m}'}$  commute. Consequently, the union of both algebras gives rise to an infinite algebra. Finally, if  $m'_1 = m'_2 = 1$ , three of the functions  $J'$  would generate the group  $\text{SU}(2)$  on the original phase space  $\Gamma$ . In other words, the faint  $\text{SU}(N)$  symmetry of an  $\mathbf{m}$ -oscillator with common divisors is compatible with the existence of smaller groups acting globally in phase space  $\Gamma$ .

## VI. SUMMARY AND OUTLOOK

This article deals with the problem which symmetry group to associate with an  $N$ -dimensional *commensurate* harmonic oscillator. Historically, structural similarities to the isotopic oscillator seemed to indicate that the introduction of rational frequency ratios  $m_n/m_{n'}$  would not affect the existence of the group  $SU(N)$  of symmetry transformations. This suggestion was based on the following observations. Arbitrary rational frequency ratios  $m_n/m_{n'}$ , are still compatible with the existence of  $(2N-1)$  globally defined invariants. In both cases, the invariants confine trajectories to a one-dimensional manifold in phase space, the orbit. Furthermore, the invariants form an algebra  $\mathfrak{su}(N)$  with respect to the Poisson bracket. There is, however, a subtle difference between an isotropic and a commensurate oscillator: isotropy forces all orbits to have the *same* period whereas commensurate frequencies allow for orbits with *different* periods. Consequently, these system *are* distinguishable from an experimental point of view.

It has been shown that the algebra  $\mathfrak{su}(N)$  of the commensurate oscillator cannot be extended globally to a representation of the group  $SU(N)$  in phase space. Strictly speaking, it is thus not possible to attribute this group as a symmetry group to the commensurate harmonic oscillator. The group  $SU(N)$  is associated with commensurate oscillators in a restricted sense only: to do so, the action of the invariants must be considered in a *reduced* phase space the points of which are no longer in a one-to-one correspondence with the states of the system. The commensurate oscillator is said to have a *faint*  $SU(N)$  symmetry. Furthermore, if the rationally related frequencies have common divisors, additional sets of symmetry transformations can be found. They are not subgroups of the faint group  $SU(N)$ , which acts in reduced phase, but they act in different reduced phase spaces.

To conclude, it has been shown that the symmetries of commensurate harmonic oscillators come in a surprisingly rich variety and depend in a subtle way on the frequency ratios. Classical and quantum mechanical oscillators are closely related. Therefore, it will be promising to study the impact of faint symmetries on the Hilbert-space structure of quantum mechanical commensurate oscillators.<sup>3</sup> In particular, a systematic group-theoretical account of their degenerate energy levels is expected to benefit from the concept of faint symmetry.

## ACKNOWLEDGMENT

S.W. is grateful for support by the Swiss National Science Foundation.

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