DIOPHANTINE APPROXIMATION ON PLANAR CURVES: THE CONVERGENCE THEORY

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ABSTRACT. The convergence theory for the set of simultaneously ψ-approximable points lying on a planar curve is established. Our results complement the divergence theory developed in [1] and thereby completes the general metric theory for planar curves.

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Dedicated to Walter Hayman and Klaus Roth on their eightieth birthdays.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. **The motivation.** In this paper we establish variants of Conjecture 1 of Beresnevich *et al* [1] that are sufficient to establish Conjecture 2 and Conjecture H of [1]. Conjecture 1 is firmly rooted in replacing the upper bound in Huxley's theorem [3, Theorem 4.2.4] on rational points near planar curves by a bound which is essentially best possible. Establishing Conjecture 2 and Conjecture H completes the general metric theory (i.e. the Lebesgue and Hausdorff measure theories) for planar curves.

More precisely, let $\eta < \xi$, $I = [\eta, \xi]$ and $f : I \to \mathbb{R}$ be such that f'' is continuous on I and and bounded away from 0. For convenience we suppose that at the end points of I the appropriate one sided first and second derivatives exist. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be an *approximating function*, that is, a real, positive decreasing function with $\psi(t) \to 0$ as $t \to \infty$, and define, as in [1],

$$N_f(Q, \psi, I) \;:=\; \mathrm{card}\{\mathbf{p}/q \in \mathbb{Q}^2 \,:\, q \le Q,\, p_1/q \in I,\, |f(p_1/q) - p_2/q| < \psi(Q)/Q\}. \quad (1.1)$$

Here $\mathbf{p}/q := (p_1/q, p_2/q)$ with $\mathbf{p} = (p_1, p_2) \in \mathbb{Z}^2$ and $q \in \mathbb{N}$. In short, the function $N_f(Q, \psi, I)$ counts the number of rational points with bounded denominator lying within a specified neighbourhood of the curve parameterized by f; namely $\mathcal{C}_f := \{(x, f(x)) \in \mathbb{R}^2 : x \in I\}$. Then firstly we show that

$$N_f(Q, \psi, I) \ll \psi(Q)Q^2 \quad (1.2)$$

when $\psi(Q) \ge Q^{-\phi}$ and ϕ is any real number with $0 \le \phi \le \frac{2}{3}$ – see §1.2. Secondly with a further mild condition on f we show that the above holds when $\phi < 1$.

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Conjecture 1 of [1], states that (1.2) holds for any $f \in C^{(3)}(I)$ and any approximating function ψ such that $t\psi(t) \to \infty$ as $t \to \infty$. Essentially, for $f \in C^{(2)}(I)$ our first counting result requires that $t^{2/3}\psi(t) \to \infty$ as $t \to \infty$ and clearly falls well short of establishing the conjecture. Nevertheless, the result is more than adequate for establishing the stronger $C^{(2)}$ form of Conjecture 2 of [1] which states that any $C^{(3)}$ non-degenerate planar curve is of Khinchin type for convergence – see §1.3. On the other hand, our second counting result just falls short of establishing Conjecture 1 in that it essentially requires that $t^{1-\varepsilon}\psi(t) \to \infty$ as $t \to \infty$. However, it is strong enough to verify Conjecture H of [1] – the Hausdorff measure analogue of Conjecture 2 – see §1.4.

1.2. **The counting results.** Let η, ξ and f be as above. Furthermore, let $\delta > 0$ and consider the counting function

$$N(Q,\delta) := \text{card}\{(a,q) \in \mathbf{Z} \times \mathbb{N} : q \le Q, \eta q < a \le \xi q, \|qf(a/q)\| < \delta\} \ , \qquad (1.3)$$

where $\|x\| = \min\{|x-m| : m \in \mathbb{Z}\}$. The main results of this paper are

Theorem 1. *Suppose that* $Q \ge 1$ *and* $0 < \delta < \frac{1}{2}$. *Then*

$$N(Q,\delta) \ll \delta Q^2 + \delta^{-\frac{1}{2}} Q \ .$$

From this the next theorem is an easy deduction.

Theorem 2. *Suppose that* ψ *is an approximating function with* $\psi(Q) \ge Q^{-\phi}$ *where* ϕ *is any real number with* $\phi \le \frac{2}{3}$. *Then (1.2) holds.*

With a natural additional condition on f we are able to extend the validity of the bound in Theorem 1.

Theorem 3. *Suppose that* $0 < \theta < 1$ *and* $f'' \in \text{Lip}_\theta([\eta,\xi])$ *and that* $Q \ge 1$ *and* $0 < \delta < \frac{1}{2}$. *Then*

$$N(Q,\delta) \ll \delta Q^2 + \delta^{-\frac{1}{2}} Q^{\frac{1}{2}+\varepsilon} + \delta^{\frac{\theta-1}{2}} Q^{\frac{3-\theta}{2}}$$

When $\theta = 1$ the proof gives the above theorem with the term $\delta^{\frac{\theta-1}{2}} Q^{\frac{3-\theta}{2}}$ replaced by $Q \log(Q/\delta)$, and this is then always bounded by one of the other two terms.

We remark in passing that when $\delta > Q^{\varepsilon-1}$ our arguments can be extended to show that

$$N(Q,\delta) \sim (\xi - \eta)\delta Q^2$$

and this has relevance to the further development of the Khinchin theory. We intend to return to this in a future publication.

From Theorem 3, the next theorem is an easy deduction.

Theorem 4. *Suppose that* $0 < \theta < 1$ *and* $f'' \in \text{Lip}_\theta([\eta,\xi])$, *and suppose that* ψ *is an approximating function with* $\psi(Q) \ge Q^{-\phi}$ *where* ϕ *is any real number with* $\phi \le \frac{1+\theta}{3-\theta}$. *Then (1.2) holds.*

The following statement follows immediately from Theorem 4 and essentially verifies Conjecture 1 of [1].

Corollary 1. *Suppose that $f \in C^{(3)}([\eta,\xi])$, and suppose that ψ is an approximating function with $\psi(Q) \geq Q^{-\phi}$ where ϕ is any real number with $\phi < 1$. Then (1.2) holds.*

For approximating functions ψ satisfying $t^{2/3}\psi(t) \to \infty$ as $t \to \infty$, Theorem 2 removes the factor $\delta^{-\varepsilon}$ from Huxley's estimate (see [1, §1.4] and [3, Theorem 4.2.4, (4.2.20)]). With its slightly stronger hypothesis Theorem 4 also does this for approximating functions ψ satisfying $t^{1-\varepsilon}\psi(t) \to \infty$ as $t \to \infty$ and complements the lower bound estimate obtained in [1, Theorem 6]. Although apparently negligible, the extra factor $\delta^{-\varepsilon}$ in Huxley's estimate renders it inadequate for our purposes as it stands. However, it plays an important rôle in our proof. Moreover the duality described on page 72 of Huxley [3] is central to our argument. In Huxley's work the duality occurs in an elementary way. Here it arises as a consequence of the harmonic analysis, where it explicitly reverses the rôles of δ and Q.

1.3. **The Khinchin theory.** Given an approximating function ψ, a point $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ is called *simultaneously ψ–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\max_{1 \leq i \leq 2} \|q y_i\| < \psi(q) \ .$$

Let $\mathcal{S}(\psi)$ denote the set of simultaneously ψ–approximable points in $\mathbb{R}^2$. Khinchin's theorem provides a simple criteria for the 'size' of $\mathcal{S}(\psi)$ expressed in terms of two-dimensional Lebesgue measure $|\ \ |_{\mathbb{R}^2}$; namely

$$|\mathcal{S}(\psi)|_{\mathbb{R}^2} = \begin{cases} \textsc{Zero} & \text{if} \quad \sum \psi(t)^2 < \infty \\ \\ \textsc{Full} & \text{if} \quad \sum \psi(t)^2 = \infty \end{cases} ,$$

where 'full' simply means that the complement of the set under consideration is of zero measure. Now let $\mathcal{C}$ be a planar curve and consider the set $\mathcal{C} \cap \mathcal{S}(\psi)$ consisting of points $\mathbf{y}$ on $\mathcal{C}$ which are simultaneously ψ–approximable. The goal is to obtain an analogue of Khinchin's theorem for $\mathcal{C} \cap \mathcal{S}(\psi)$. Trivially, $|\mathcal{C} \cap \mathcal{S}(\psi)|_{\mathbb{R}^2} = 0$ irrespective of the approximating function ψ. Thus, when referring to the Lebesgue measure of the set $\mathcal{C} \cap \mathcal{S}(\psi)$ it is always with reference to the induced Lebesgue measure $|\ \ |_{\mathcal{C}}$ on $\mathcal{C}$. Now some useful terminology:

(1) $\mathcal{C}$ is of *Khinchin type for convergence* when $|\mathcal{C} \cap \mathcal{S}(\psi)|_{\mathcal{C}} = \textsc{Zero}$ for any approximating function ψ with $\sum \psi(t)^2 < \infty$.

(2) $\mathcal{C}$ is of *Khinchin type for divergence* when $|\mathcal{C} \cap \mathcal{S}(\psi)|_{\mathcal{C}} = \textsc{Full}$ for any approximating function ψ with $\sum \psi(t)^2 = \infty$.

To make any reasonable progress with developing a Khinchin theory for planar curves $\mathcal{C}$, it is reasonable to assume that the set of points on $\mathcal{C}$ at which the curvature vanishes is a set of one-dimensional Lebesgue measure zero, i.e. the curve is *non-degenerate* . In [1], the following result is established.

Theorem. *Any $C^{(3)}$ non-degenerate planar curve is of Khinchin type for divergence.*

To complete the Khinchin theory for $C^{(3)}$ non–degenerate planar curves we need to show that any such curve is of Khinchin type for convergence. A consequence of Theorem 1, or equivalently a slight variant of Theorem 2, is

Theorem 5. *Any $C^{(2)}$ non-degenerate planar curve is of Khinchin type for convergence.*

In the case $\psi : t \to t^{-v}$ with $v > 0$, let us write $\mathcal{S}(v)$ for $\mathcal{S}(\psi)$. Note that in view of Dirichlet's theorem (simultaneous version), $\mathcal{S}(v) = \mathbb{R}^2$ for any $v \leq 1/2$ and so $|\mathcal{C} \cap \mathcal{S}(v)|_{\mathcal{C}} = |\mathcal{C}|_{\mathcal{C}} :=$ Full for any $v \leq 1/2$. It is easily verified that Theorem 5 implies the following 'extremality' result due to Schmidt [4].

Corollary (Schmidt). *Let $\mathcal{C}$ be a $C^{(2)}$ non-degenerate planar curve. Then, for any $v > 1/2$*

$$|\mathcal{C} \cap \mathcal{S}(v)|_{\mathcal{C}} \ = \ 0 \ .$$

To be precise, Schmidt actually requires that $\mathcal{C}$ is a $C^{(3)}$ non-degenerate planar curve. For further background, including a comprehensive account of related works, we refer the reader to [1, §1].

1.4. **The Jarník theory.** Jarník's theorem is a Hausdorff measure version of Khinchin's theorem in that it provides a simple criteria for the 'size' of $\mathcal{S}(\psi)$ expressed in terms of s–dimensional Hausdorff measure $\mathcal{H}^s$. The Hausdorff measure and dimension of a set $X \in \mathbb{R}^2$ is defined as follows. For $\rho > 0$, a countable collection $\{B_i\}$ of Euclidean balls in $\mathbb{R}^2$ with diameter $\mathrm{diam}(B_i) \leq \rho$ for each i such that $X \subset \bigcup_i B_i$ is called a ρ-cover for X. Let s be a non-negative number and define $\mathcal{H}^s_\rho(X) \ = \ \inf\{\sum_i \mathrm{diam}(B_i)^s \ : \{B_i\}$ is a $\rho-$cover of $X\}$, where the infimum is taken over all possible ρ-covers of X. The *s-dimensional Hausdorff measure $\mathcal{H}^s(X)$* is defined by

$$\mathcal{H}^s(X) := \lim_{\rho \to 0} \mathcal{H}^s_\rho(X) = \sup_{\rho > 0} \mathcal{H}^s_\rho(X)$$

and the *Hausdorff dimension* $\dim X$ of X is defined by

$$\dim \ X := \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s(X) = \infty\} \ .$$

Jarník's theorem shows that the s–dimensional Hausdorff measure $\mathcal{H}^s(\mathcal{S}(\psi))$ of the set $\mathcal{S}(\psi)$ satisfies an elegant 'zero-infinity' law. Let $s \in (0,2)$ and ψ be an approximating function. Then

$$\mathcal{H}^s\left(\mathcal{S}(\psi)\right) = \begin{cases} 0 & \text{when} \quad \sum \ t^{2-s}\,\psi(t)^s \ < \infty \\ \\ \infty & \text{when} \quad \sum \ t^{2-s}\,\psi(t)^s \ = \infty \end{cases} .$$

Note that this trivially implies that $\dim \mathcal{S}(\psi) = \inf\{s : \sum t^{2-s}\,\psi(t)^s < \infty\}$.

Now let $\mathcal{C}$ be a planar curve. The goal is to obtain an analogue of Jarník's theorem for $\mathcal{C} \cap \mathcal{S}(\psi)$. In particular, our aim is to establish the following conjecture stated in [1].

Conjecture H *Let $s \in (1/2, 1)$ and ψ be an approximating function. Let $f \in C^{(3)}(I)$, where I is an interval and let $\mathcal{C}_f := \{(x, f(x)) : x \in I\}$. Assume that* $\dim\{x \in I : f''(x) =$

$0\} \leq 1/2$. *Then*

$$\mathcal{H}^s\left(\mathcal{C}_f \cap \mathcal{S}(\psi)\right) = \begin{cases} 0 & \text{when} \quad \sum t^{1-s}\,\psi(t)^{s+1} \quad < \infty \\ \\ \infty & \text{when} \quad \sum t^{1-s}\,\psi(t)^{s+1} \quad = \infty \end{cases} .$$

The divergent part of the above statement, namely that

$$\mathcal{H}^s\left(\mathcal{C}_f \cap \mathcal{S}(\psi)\right) = \infty \quad \text{when} \quad \sum t^{1-s}\,\psi(t)^{s+1} \quad = \infty,$$

is Theorem 3 in [1], and so the main substance of the conjecture is the convergence part. A consequence of Theorem 3 above, or equivalently a slight variant of Corollary 1, is the completion of the proof of Conjecture H.

Theorem 6. *Let* $s \in (1/2, 1)$ *and* ψ *be an approximating function. Let* $f \in C^{(3)}(I)$, *where* I *is an interval and let* $\mathcal{C}_f := \{(x, f(x)) : x \in I\}$. *Assume that* $\dim\{x \in I : f''(x) = 0\} \leq 1/2$. *Then*

$$\mathcal{H}^s\left(\mathcal{C}_f \cap \mathcal{S}(\psi)\right) \;=\; 0 \quad \text{when} \quad \sum t^{1-s}\,\psi(t)^{s+1} < \infty \;.$$

For further background, including an explanation of the conditions in Conjecture H and a comprehensive account of related works, we refer the reader to [1, §1].

2. The proof of Theorem 1

It clearly suffices to prove Theorem 1 and indeed Theorem 3 with $N(Q, \delta)$ replaced by

$$\widetilde{N}(Q, \delta) := \operatorname{card}\{(a, q) \in \mathbf{Z} \times \mathbb{N} : Q < q \leq 2Q, \eta q < a \leq \xi q, \|qf(a/q)\| < \delta\} \;.$$

Let

$$J = \left\lfloor \frac{1}{2\delta} \right\rfloor \tag{2.1}$$

and consider the Fejér kernel

$$\mathcal{K}_J(\alpha) = J^{-2}\left|\sum_{h=1}^{J} e(h\alpha)\right|^2 = \left(\frac{\sin \pi J\alpha}{J \sin \pi\alpha}\right)^2 .$$

When $\|\alpha\| \leq \delta$ we have $|\sin \pi J\alpha| = \sin \pi\|J\alpha\| \geq 2\|J\alpha\| = 2\|\,J\|\alpha\|\,\| = 2J\|\alpha\|$, since $J\|\alpha\| \leq \delta \left\lfloor \frac{1}{2\delta} \right\rfloor \leq \frac{1}{2}$. Hence, when $\|\alpha\| \leq \delta$, we have

$$\mathcal{K}_J(\alpha) \geq \frac{2\|\alpha\| J}{J\pi\|\alpha\|} = \frac{2}{\pi}.$$

Thus

$$\widetilde{N}(Q, \delta) \leq \frac{\pi}{2} \sum_{Q<q\leq 2Q} \; \sum_{\eta q<a\leq \xi q} \mathcal{K}_J\big(qf(a/q)\big).$$

Since

$$\mathcal{K}_J(\alpha) = \sum_{j=-J}^{J} \frac{J - |j|}{J^2}\, e(j\alpha)$$

we have

$$\widetilde{N}(Q, \delta) \leq \pi\delta(\xi - \eta)Q^2 + N_1 + O(\delta Q) = N_1 + O(\delta Q^2)$$

where
$$N_1 = \frac{\pi}{2} \sum_{0<|j|\le J} \frac{J-|j|}{J^2} \sum_{Q<q\le 2Q} \sum_{\eta q<a\le \xi q} e\big(jqf(a/q)\big).$$
We observe that the function $F(\alpha) = jqf(\alpha/q)$ has derivative $jf'(\alpha/q)$. Given j with $0<|j|\le J$ we define
$$H_- = \lfloor \inf jf'(\beta)\rfloor - 1, \quad H_+ = \lceil \sup jf'(\beta)\rceil + 1,$$
$$h_- = \lceil \inf jf'(\beta)\rceil + 1, \quad h_+ = \lfloor \sup jf'(\beta)\rfloor - 1$$
where the extrema are over the interval $[\eta,\xi]$. Then, by Lemma 4.2 of Vaughan [6],
$$\sum_{\eta q<a\le \xi q} e\big(jqf(a/q)\big) = \sum_{H_-\le h\le H_+} \int_{\eta q}^{\xi q} e\big(jqf(\alpha/q) - h\alpha\big)d\alpha + O\big(\log(2+H)\big)$$
where $H = \max(|H_-|,|H_+|)$. Clearly $H \ll |j| \le J$ and so
$$N_1 = N_2 + O\big(Q\log\tfrac{1}{\delta}\big)$$
where
$$N_2 = \frac{\pi}{2} \sum_{0<|j|\le J} \frac{J-|j|}{J^2} \sum_{Q<q\le 2Q} \sum_{H_-\le h\le H_+} \int_{\eta q}^{\xi q} e\big(jqf(\alpha/q) - h\alpha\big)d\alpha.$$
The integral here is
$$q\int_{\eta}^{\xi} e\big(q(jf(\beta) - h\beta)\big)d\beta.$$
The function $g(\beta) = q(jf(\beta) - h\beta)$ has second derivative $qjf''(\beta)$ whose modulus lies between constant multiples of $q|j|$. Hence, by Lemma 4.4 of Titchmarsh [5], for any subinterval $\mathcal{I}$ of $[\eta,\xi]$,
$$\int_{\mathcal{I}} e\big(q(jf(\beta) - h\beta)\big)d\beta \ll \frac{1}{\sqrt{q|j|}}. \tag{2.2}$$
Thus the contribution to N_2 from any h with $H_- \le h \le h_-$ or $h_+ \le h \le H_+$ is
$$\ll J^{-1} \sum_{j=1}^{J} j^{-1/2} \sum_{Q<q\le 2Q} q^{1/2}.$$
Therefore
$$N_2 = N_3 + O\left(\delta^{\frac12} Q^{\frac32}\right).$$
where
$$N_3 = \frac{\pi}{2} \sum_{0<|j|\le J} \frac{J-|j|}{J^2} \sum_{Q<q\le 2Q} q \sum_{h_-<h<h_+} \int_{\eta}^{\xi} e\big(q(jf(\beta) - h\beta)\big)d\beta. \tag{2.3}$$
The sum over h here is taken to be empty when $h_+ \le h_- + 1$.

We have
$$\delta^{\frac12}Q^{\frac32} = \big(\delta Q^2\big)^{\frac12}\big(Q\big)^{\frac12} \le \delta Q^2 + Q.$$
Thus it remains to treat N_3.

Since f' is continuous and $\inf jf'(\beta) < h_- < h < h_+ < \sup jf'(\beta)$ it follows that there is a $\beta_h = \beta_{j,h} \in [\eta,\xi]$ such that $jf'(\beta_h) = h$. Let
$$\lambda_h = \lambda_{j,h} = \|jf(\beta_h) - h\beta_h\| \tag{2.4}$$
We need to bound various sums involving λ_h. To that end the following lemma is very useful.

Lemma 2.1. *Suppose that ϕ has a continuous second derivative on $[\Upsilon, \Xi]$ which is bounded away from 0, and suppose that Ψ is real and satisfies $0 < \Psi < \frac{1}{4}$. Then for any fixed $\varepsilon > 0$ and $R \geq 1$, the number M of triples of integers r, b, c such that $(r, b, c) = 1$, $R \leq r < 2R$, $\Upsilon r < b \leq \Xi r$ and $|r\phi(b/r) - c| \leq \Psi$ satisfies*

$$M \ll_\varepsilon \Psi^{1-\varepsilon} R^2 + R.$$

Proof. If $\Upsilon < 0 < \Xi$, then we split $[\Upsilon, \Xi]$ into two subintervals $[\Upsilon, 0]$, $[0, \Xi]$ and consider them separately. Thus we may suppose $0 \notin (\Upsilon, \Xi)$. If $\Xi \leq 0$, then by replacing b/r by $-b/r$ and $\Psi(\alpha)$ by $\Psi(-\alpha)$ we can transfer our attention to the interval $[-\Xi, -\Upsilon]$. Thus it always suffices to consider intervals $[\Upsilon, \Xi]$ with $0 \leq \Upsilon \leq \Xi$. Now choose $K \in \mathbb{N}$ so that $K > \Xi$, say $K = \lfloor \Xi \rfloor + 1$. We extend the definition of ϕ so that ϕ is twice differentiable with a continuous second derivative and bounded away from 0 on the whole of $[0, 1]$. For example, if $\Upsilon/K > 0$, then for $0 \leq \alpha < \Upsilon/K$ we can take $\phi(\alpha) = \frac{1}{2}(\alpha - \Upsilon)^2\phi''(\Upsilon) + (\alpha - \Upsilon)\phi'(\Upsilon) + \phi(\Upsilon)$, and likewise when $\Xi < \alpha \leq K$. If we now define $F(x)$ on $[0, 1]$ by $F(x) = \phi(xK)/K$, then F will satisfy the hypothesis of Theorem 4.2.4 of Huxley [3]. The condition $(r, b, c) = 1$ ensures that the rational number points $(b/r, c/r)$ are counted uniquely. Thus M is bounded by the number of r, b, c with $R \leq r < 2R$, $0 < b < rK$, $|\phi(b/r) - c/r| \leq R^{-1}\Psi$. We take $T = K$, $M = K$, $\Delta = K^{-1}$, $Q = R$, $\delta = \Psi$ and apply the conclusion (4.2.20), *ibidem*, to obtain the desired result. □

Lemma 2.2. *Suppose that ϕ has a continuous second derivative on $[\Upsilon, \Xi]$ which is bounded away from 0, and suppose that Ψ is real and satisfies $0 < \Psi < \frac{1}{4}$. Then for any $\varepsilon > 0$ and $R \geq 1$,*

$$\sum_{R \leq r < 2R} \sum_{\substack{\Upsilon r < b \leq \Xi r \\ \|r\phi(b/r)\| \leq \Psi}} 1 \ll_\varepsilon \Psi^{1-\varepsilon} R^2 + R \log 2R.$$

Proof. For a given pair r, b counted in the double sum let c be the unique integer with $|r\phi(b/r) - c| \leq \Psi$. We sort the triples according to the value of the greatest common divisor $(r, b, c) = d$, say. Then the double sum does not exceed

$$\sum_{d \leq 2R} M(d)$$

where $M(d)$ is the number of triples of integers s, g, h such that $(s, g, h) = 1$, $R/d \leq s < 2R/d$, $\Upsilon s < g \leq \Xi s$ and $|s\phi(g/s) - h| \leq \Psi d^{-1}$. By Lemma 2.1,

$$M(d) \ll_\varepsilon \left(\Psi d^{-1}\right)^{1-\varepsilon} (R/d)^2 + R/d.$$

Summing over the $d \leq 2R$ gives the lemma. □

We apply this through the next lemma.

Lemma 2.3. *We have*

$$\sum_{0 < |j| \leq J} \sum_{\substack{h_- < h < h_+ \\ \lambda_h > Q^{-1}}} |j|^{-\frac{1}{2}} \lambda_h^{-\frac{1}{2}} \ll J^{\frac{3}{2}} + J^{\frac{1}{2}} (\log J) Q^{\frac{1}{2}}, \tag{2.5}$$

$$\sum_{0 < |j| \leq J} \sum_{\substack{h_- < h < h_+ \\ \lambda_h > Q^{-1}}} |j|^{-\frac{1}{2}} \lambda_h^{-1} \ll J^{\frac{3}{2}} Q^{\varepsilon} + J^{\frac{1}{2}} (\log J) Q \tag{2.6}$$

and

$$\sum_{0<|j|\leq J} \sum_{\substack{h_-<h<h_+\\ \lambda_h\leq Q^{-1}}} |j|^{-\frac12} \ll J^{\frac32}Q^{\varepsilon-1} + J^{\frac12}\log J. \tag{2.7}$$

Proof. Clearly in (2.5) and (2.6) we can restrict our attention to terms with $\lambda_h < \frac14$, since those terms with $\lambda_h \geq \frac14$ contribute $\ll J^{\frac32}$ to the total. Let

$$\Upsilon = \inf f'(\beta),\ \Xi = \sup f'(\beta)$$

where the extrema are taken over $[\eta,\xi]$. When $j<0$ we replace j by $-j$ and h by $-h$ in each of the sums in question, and write β_h for β_{-h} and λ_h for λ_{-h} to see that the sums are bounded by

$$\sum_{j=1}^{J} \sum_{\substack{\Upsilon j<h<\Xi j\\ \lambda_h>Q^{-1}}} j^{-\frac12}\lambda_h^{-\frac12},$$

$$\sum_{j=1}^{J} \sum_{\substack{\Upsilon j<h<\Xi j\\ \lambda_h>Q^{-1}}} j^{-\frac12}\lambda_h^{-1},$$

and

$$\sum_{j=1}^{J} \sum_{\substack{\Upsilon j<h<\Xi j\\ \lambda_h\leq Q^{-1}}} j^{-\frac12}$$

respectively.

Let g denote the inverse function of f', so that g is defined on $[\Upsilon,\Xi]$ and $\beta_h = g(h/j)$. Let $F(\alpha) = \alpha g(\alpha) - f\big(g(\alpha)\big)$. Then

$$F'(\alpha) = \alpha g'(\alpha) + g(\alpha) - f'\big(g(\alpha)\big)g'(\alpha) = g(\alpha)$$

and

$$F''(\alpha) = g'(\alpha) = \frac{1}{f''(g(\alpha))}$$

and so, in particular, F'' is bounded away from 0. Thus

$$\lambda_h = \|jF(h/j)\|$$

and F satisfies the conditions on ϕ in Lemma 2.2. The desired bounds now follow by partial summation. □

We now return to the estimation of N_3, defined by (2.3). By (2.2), the terms in N_3 with $\lambda_h \leq Q^{-1}$ contribute

$$\ll J^{-1} \sum_{0<|j|\leq J} \sum_{\substack{h_-<h<h_+\\ \lambda_h\leq Q^{-1}}} \sum_{Q<q\leq 2Q} q^{\frac12}|j|^{-\frac12}$$

and by (2.7) this is

$$\ll J^{\frac12}Q^{\frac12+\varepsilon} + J^{-\frac12}(\log J)Q^{\frac32}.$$

Hence

$$N_3 = N_4 + O\big(\delta^{-\frac12}Q^{\frac12+\varepsilon} + \delta^{\frac12}\big(\log\tfrac1\delta\big)Q^{\frac32}\big) \tag{2.8}$$

where

$$N_4 = \frac{\pi}{2} \sum_{0<|j|\le J} \frac{J-|j|}{J^2} \sum_{Q<q\le 2Q} q \sum_{\substack{h_-<h<h_+\\ \lambda_h>Q^{-1}}} \int_\eta^\xi e\big(q(jf(\beta)-h\beta)\big)d\beta.$$

Since

$$\delta^{\frac12}\left(\log\tfrac{1}{\delta}\right)Q^{\frac32} = \left(\delta Q^2\right)^{\frac12}\left(\left(\log\tfrac{1}{\delta}\right)^2 Q\right)^{\frac12} \le \delta Q^2 + \delta^{-\frac12}Q \tag{2.9}$$

this gives

$$N_3 = N_4 + O\big(\delta Q^2 + \delta^{-\frac12}Q\big).$$

Let $c = \big(\sup|f''(\beta)|\big)^{-1/2}$ where the supremum is taken over $[\eta,\xi]$. The set $\mathcal{A}(j,h)$ of those β in $[\eta,\xi]$ for which $|\beta-\beta_h| > c\sqrt{\lambda_h/|j|}$ consists of at most two intervals, and may be empty. By the mean value theorem, for such β we have

$$jf'(\beta) - h = (\beta-\beta_h)jf''(\beta^*)$$

for some $\beta^*\in[\eta,\xi]$. Thus

$$|jf'(\beta)-h| \gg \sqrt{|j|\lambda_h}.$$

Hence, by integration by parts, we have

$$\int_{\mathcal{A}(j,h)} e\big(q(jf(\beta)-h\beta)\big)d\beta \ll \frac{1}{q\sqrt{|j|\lambda_h}}.$$

Therefore the total contribution to N_4 from the $\mathcal{A}(j,h)$ is

$$\ll J^{-1}Q \sum_{0<|j|\le J} \sum_{\substack{h_-<h<h_+\\ \lambda_h>Q^{-1}}} \frac{1}{\sqrt{|j|\lambda_h}}$$

and by (2.5) this is

$$\ll J^{\frac12}Q + J^{-\frac12}(\log J)Q^{\frac32}.$$

Thus, by (2.9),

$$N_4 = N_5 + O\big(\delta Q^2 + \delta^{-\frac12}Q\big)$$

where

$$N_5 = \frac{\pi}{2} \sum_{0<|j|\le J} \frac{J-|j|}{J^2} \sum_{Q<q\le 2Q} q \sum_{\substack{h_-<h<h_+\\ \lambda_h>Q^{-1}}} \int_{\mathcal{B}(j,h)} e\big(q(jf(\beta)-h\beta)\big)d\beta$$

and $\mathcal{B}(j,h)$ denotes the set of $\beta\in[\eta,\xi]$ with $|\beta-\beta_h| \le c\sqrt{\lambda_h/|j|}$.

Given j and h included in the sum, choose $n = n(j,h)$ so that $\lambda_h = |jf(\beta_h) - h\beta_h - n|$. For $\beta \in B(j,h)$ we have

$$jf(\beta) - h\beta - n = jf(\beta_h) - h\beta_h - n + \tfrac12(\beta-\beta_h)^2 jf''(\beta^\flat) \tag{2.10}$$

where $\beta^\flat\in[\eta,\xi]$. When $\frac14 \le \lambda_h$ we have

$$\tfrac18 \le \tfrac12\lambda_h \le |jf(\beta)-h\beta-n| \le \tfrac32\lambda_h \le \tfrac34.$$

Thus $\|jf(\beta)-h\beta\| = |jf(\beta)-h\beta-m|$ with $m=n$ or $m=n\pm1$, and so

$$\frac18 \le \|jf(\beta)-h\beta\|.$$

On the other hand, when $\lambda_h < \frac14$ the identity (2.10) shows that

$$\tfrac12\lambda_h \le \|jf(\beta)-h\beta\| \le \tfrac32\lambda_h$$

and so generally

$$\|jf(\beta) - h\beta\| \asymp \lambda_h.$$

Therefore for j and h included in the sum we have

$$\int_{\mathcal{B}(j,h)} \sum_{Q<q\le 2Q} qe\big(q(jf(\beta)-h\beta)\big)d\beta \ll Q\lambda_h^{-1}\text{meas}\mathcal{B}(j,h) \ll Q\lambda_h^{-1}\sqrt{\lambda_h/|j|}.$$

and hence

$$N_5 \ll J^{-1}Q \sum_{0<|j|\le J} \sum_{\substack{h_-<h<h_+\\ \lambda_h>Q^{-1}}} |j|^{-\frac12}\lambda_h^{-\frac12}.$$

Thus, by (2.5),

$$N_5 \ll J^{-1}Q\big(J^{\frac32} + Q^{\frac12}J^{\frac12}\log J\big) \ll \delta^{-\frac12}Q + \delta^{\frac12}\big(\log\tfrac{1}{\delta}\big)Q^{\frac32}.$$

This with (2.9) completes the proof of Theorem 1.

3. The proof of Theorem 2

By (1.1), when $\psi(Q) \le \frac12$, $N_f(Q,\psi,I)$ is

$$\le \text{card}\{a,q \,:\, q\le Q,\ a\in qI,\ \|qf(a/q)\| < q\psi(Q)/Q\}$$

and this is bounded by

$$\text{card}\{a,q \,:\, q\le Q,\ a\in qI,\ \|qf(a/q)\| < \psi(Q)\}\ .$$

Now the conclusion is immediate from Theorem 1.

4. The proof of Theorem 3

For convenience we extend the definition of f to $\mathbb{R}$ by defining $f(\beta)$ to be $\frac12(\beta-\xi)^2f''(\xi) + (\beta-\xi)f'(\xi)+f(\xi)$ when $\beta>\xi$ and to be $\frac12(\beta-\eta)^2f''(\eta)+(\beta-\eta)f'(\eta)+f(\eta)$ when $\beta<\eta$. Note that then $f''\in \text{Lip}_\theta(\mathbb{R})$ and f'' is still bounded away from 0 and is bounded .

We follow the proof of Theorem 1 as far as (2.8). We note that the complete error term here is in fact

$$\delta Q^2 + Q\log\tfrac{1}{\delta} + \delta^{-\frac12}Q^{\frac12+\varepsilon} + \delta^{\frac12}(\log\tfrac{1}{\delta})Q^{\frac32}$$

Thus

$$\widetilde{N}(Q,\delta) \ll N_4 + \delta Q^2 + Q\log\tfrac{1}{\delta} + \delta^{-\frac12}Q^{\frac12+\varepsilon} + \delta^{\frac12}(\log\tfrac{1}{\delta})Q^{\frac32}$$

where

$$N_4 = \frac{\pi}{2}\sum_{0<|j|\le J}\frac{J-|j|}{J^2}\sum_{Q<q\le 2Q} q \sum_{\substack{h_-<h<h_+\\ \lambda_h>Q^{-1}}} \int_\eta^\xi e\big(q(jf(\beta)-h\beta)\big)d\beta.$$

Moreover, given j and h included in the sums there is unique $\beta_h = \beta_{j,h}$ such that

$$f'(\beta_h) = h/j.$$

Let

$$\mu = \frac{\xi-\eta}{2}.$$

Then in the integral above we replace the interval $[\eta, \xi]$ by $[\beta_h - \mu, \beta_h + \mu]$. For any β not in both intervals we have $|\beta - \beta_h| \geq \mu$, $\beta \leq \eta$, or $\beta \geq \xi$. For some $\beta^* \in [\eta, \xi]$ we have $(\beta_h - \eta) j f''(\beta^*) = j f'(\beta_h) - j f'(\eta) \geq h - h_-$ so $\beta_h - \eta \gg (h - h_-)/|j|$ and likewise $\xi - \beta_h \gg (h_+ - h)/|j|$. Hence, if $\beta \leq \eta$, then $\beta_h - \beta \gg (h - h_-)/|j|$, and if $\beta \geq \xi$, then $\beta - \beta_h \gg (h_+ - h)/|j|$. Moreover, as $\mu \gg (h - h_-)/|j|$ and $\mu \gg (h_+ - h)/|j|$ it follows that whenever β is not in both intervals we have either $|\beta - \beta_h| \gg (h - h_-)/|j|$ or $|\beta - \beta_h| \gg (h_+ - h)/|j|$. For any such β there is a $\beta^\flat$ such that $j f'(\beta) - h = j(f'(\beta) - f'(\beta_h)) = j(\beta - \beta_h) f''(\beta^\flat)$, whence $|j f'(\beta) - h| \gg h - h_-$ or $|j f'(\beta) - h| \gg h_+ - h$. It then follows by integration by parts that if $\mathcal{A} = [\eta, \xi] \backslash [\beta_h - \mu, \beta_h + \mu]$ or $\mathcal{A} = [\beta_h - \mu, \beta_h + \mu] \backslash [\eta, \xi]$, then

$$\int_{\mathcal{A}} e\big(q(j f(\beta) - h\beta)\big) d\beta \ll \frac{1}{q(h - h_-)} + \frac{1}{q(h_+ - h)}.$$

Thus

$$N_4 = N_5 + O\left(\sum_{0<|j|\leq J} \sum_{h_- < h < h_+} \frac{Q/J}{h - h_-} + \frac{Q/J}{h_+ - h} \right)$$

where

$$N_5 = \frac{\pi}{2} \sum_{0<|j|\leq J} \frac{J - |j|}{J^2} \sum_{\substack{h_- < h < h_+ \\ \lambda_h > Q^{-1}}} \sum_{Q<q\leq 2Q} q \int_{\beta_h - \mu}^{\beta_h + \mu} e\big(q(j f(\beta) - h\beta)\big) d\beta.$$

Thus

$$N_4 = N_5 + O\big(Q \log \tfrac{1}{\delta}\big).$$

For convenience we write

$$F(\alpha) = F(\alpha; j, h) = \big(f(\alpha + \beta_h) - f(\beta_h)\big) - h\alpha/j.$$

Then

$$F(0) = 0, \quad F'(\alpha) = f'(\alpha + \beta_h) - h/j, \quad F'(0) = 0, \quad F''(\alpha) = f''(\alpha + \beta_h),$$

and

$$\int_{\beta_h - \mu}^{\beta_h + \mu} e\big(q(j f(\beta) - h\beta)\big) d\beta = e(q\phi_h) \int_{-\mu}^{\mu} e\big(qjF(\alpha)\big) d\alpha$$

where

$$\phi_h = \phi_{j,h} = j f(\beta_h) - h\beta_h$$

so that

$$\lambda_h = \|\phi_h\|.$$

Since $f'' \in \mathrm{Lip}_\theta(\mathbb{R})$, we have $F'' \in \mathrm{Lip}_\theta(\mathbb{R})$ and so, in particular,

$$F''(\alpha) = F''(0) + O\big(|\alpha|^\theta\big) = f''(\beta_h) + O\big(|\alpha|^\theta\big),$$

and thus

$$F'(\alpha) = \alpha f''(\beta_h) + O\big(|\alpha|^{1+\theta}\big), \quad F(\alpha) = \tfrac{1}{2}\alpha^2 f''(\beta_h) + O\big(|\alpha|^{2+\theta}\big).$$

For brevity write $c_2 = f''(\beta_h)$.

Since f'', and hence F'', is bounded and bounded away from 0, and f'' is continuous it follows that F' is strictly monotonic and so can only change sign once. But $F'(0) = 0$. We suppose for the time being that $c_2 > 0$. Now F' is strictly increasing, and hence positive when $\alpha > 0$. Thus F is strictly increasing for $\alpha \geq 0$ and positive for $\alpha > 0$. Let G be the inverse function of F on $[0,\infty)$. Then G' exists on $(0,\infty)$ and $G'(\beta)=1/F'\big(G(\beta)\big)$. Thus for any ν with

$$0<\nu<\mu$$

we have

$$\int_\nu^\mu e\big(qjF(\alpha)\big)d\alpha=\int_{F(\nu)}^{F(\mu)} e(qj\beta)G'(\beta)d\beta.$$

Note that we will eventually choose ν to be judicially small in terms of q and j. Since F' is non-zero for $\alpha>0$ it follows that G'' exists on $(0,\infty)$, and is continuous, and so by integration by parts we have

$$\int_{F(\nu)}^{F(\mu)} e(qj\beta)G'(\beta)d\beta=\left[\frac{e(qj\beta)G'(\beta)}{2\pi iqj}\right]_{F(\nu)}^{F(\mu)}-\int_{F(\nu)}^{F(\mu)}\frac{e(qj\beta)}{2\pi iqj}G''(\beta)d\beta.$$

Moreover

$$G''(\beta)=-\frac{F''(G(\beta))G'(\beta)}{F'(G(\beta))^2}=-\frac{F''(G(\beta))}{G'(\beta)^3}.$$

We also have, for $\alpha>0$

$$\beta=F(\alpha)=\tfrac12c_2\alpha^2+O\big(\alpha^{2+\theta}\big).$$

Since $\mu\ll1$ it follows that for $0<\alpha\le\mu$ we have

$$G(\beta)=\alpha=\sqrt{\frac{2\beta}{c_2}}\Big(1+O\big(\beta^{\theta/2}\big)\Big)=\sqrt{\frac{2\beta}{c_2}}+O\big(\beta^{(1+\theta)/2}\big).$$

We further have

$$F'\big(G(\beta)\big)=F'(\alpha)=\sqrt{2c_2\beta}+O\big(\beta^{(1+\theta)/2}\big).$$

and

$$F''\big(G(\beta)\big)=c_2+O\big(\alpha^\theta\big)=c_2+O\big(\beta^{\theta/2}\big).$$

Hence

$$G'(\beta)=\frac{1}{F'\big(G(\beta)\big)}=(2c_2\beta)^{-\frac12}+O\big(\beta^{(\theta-1)/2}\big)$$

and

$$G''(\beta)=-\Big(c_2+O\big(\beta^{\theta/2}\big)\Big)\Big((2c_2\beta)^{-\frac12}+O\big(\beta^{(\theta-1)/2}\big)\Big)^3=-\frac{c_2}{(2c_2\beta)^{3/2}}+O\big(\beta^{(\theta-3)/2}\big).$$

Substituting the above approximations we have

$$\left[\frac{e(qj\beta)G'(\beta)}{2\pi iqj}\right]_{F(\nu)}^{F(\mu)}-\int_{F(\nu)}^{F(\mu)}\frac{e(qj\beta)}{2\pi iqj}G''(\beta)d\beta$$

$$=-\frac{e(qjF(\nu))G'(F(\nu))}{2\pi iqj}+\int_{F(\nu)}^{\infty}\frac{e(qj\beta)}{2\pi iqj}\frac{c_2}{(2c_2\beta)^{3/2}}d\beta+E$$

where

$$E\ll\frac{1}{q|j|}+\int_{F(\nu)}^{F(\mu)}\frac{\beta^{(\theta-3)/2}}{q|j|}d\beta\ll\frac{F(\nu)^{(\theta-1)/2}+1}{q|j|}\ll\frac{\nu^{\theta-1}+1}{q|j|}.$$

We also have

$$G'(F(\nu))=(2c_2F(\nu))^{-\frac12}+O\big(F(\nu)^{(\theta-1)/2}\big).$$

Hence, by substitution and integration by parts,

$$\left[\frac{e(qj\beta)G'(\beta)}{2\pi iqj}\right]_{F(\nu)}^{F(\mu)} - \int_{F(\nu)}^{F(\mu)} \frac{e(qj\beta)}{2\pi iqj} G''(\beta)d\beta$$

$$= \int_{F(\nu)}^{\infty} \frac{e(qj\beta)}{\sqrt{2c_2\beta}} d\beta + O\left(\frac{\nu^{\theta-1}+1}{q|j|}\right).$$

We now turn to

$$\int_0^\nu e\big(qjF(\alpha)\big)d\alpha.$$

This differs from

$$\int_0^\nu e\big(qj\tfrac{1}{2}c_2\alpha^2)\big)d\alpha = \int_0^{\frac{1}{2}c_2\nu^2} \frac{e(qj\beta)}{\sqrt{2c_2\beta}} d\beta$$

by

$$\ll \int_0^\nu q|j|\alpha^{2+\theta}d\alpha \ll q|j|\nu^{3+\theta}.$$

Now $F(\nu) = \frac{1}{2}c_2\nu^2 + O\big(\nu^{2+\theta}\big)$ and so

$$\int_{\frac{1}{2}c_2\nu^2}^{F(\nu)} \frac{e(qj\beta)}{\sqrt{2c_2\beta}} d\beta \ll \nu^{1+\theta}.$$

The choice $\nu = c/\sqrt{q|j|}$, where the positive constant c is chosen to ensure that $\nu < \mu$, gives

$$\int_0^\mu e\big(qjF(\alpha)\big)d\alpha = \int_0^\infty \frac{e(qj\beta)}{\sqrt{2c_2\beta}} d\beta + O\big((q|j|)^{(-1-\theta)/2}\big).$$

Hence

$$\int_0^\mu e\big(qjF(\alpha)\big)d\alpha = \frac{W_{\mathrm{sgn}(j)}}{\sqrt{qc_2|j|}} + O\big((q|j|)^{(-1-\theta)/2}\big)$$

where

$$W_\pm = \int_0^\infty \frac{e(\pm\gamma)}{\sqrt{2\gamma}} d\gamma.$$

A cognate argument shows that also

$$\int_{-\mu}^0 e\big(qjF(\alpha)\big)d\alpha = \frac{W_{\mathrm{sgn}(j)}}{\sqrt{qc_2|j|}} + O\big((q|j|)^{(-1-\theta)/2}\big).$$

When $c_2 < 0$ perhaps the simplest thing is to observe that this case is formally equivalent to taking complex conjugates. Thus, in general, we have

$$\int_{-\mu}^\mu e\big(qjF(\alpha)\big)d\alpha = \frac{2W_{\mathrm{sgn}(c_2j)}}{\sqrt{q|c_2j|}} + O\big((q|j|)^{(-1-\theta)/2}\big).$$

Hence

$$\sum_{Q<q\le 2Q} q \int_{\beta_h-\mu}^{\beta_h+\mu} e\big(q(jf(\beta)-h\beta)\big)d\beta$$

$$= \sum_{Q<q\le 2Q} q^{\frac{1}{2}} e(q\phi_h) \frac{2W_{\mathrm{sgn}(c_2j)}}{\sqrt{|c_2j|}} + O\big(Q^{(3-\theta)/2}|j|^{(-1-\theta)/2}\big).$$

Thus

$$\sum_{Q<q\le 2Q} q\int_{\beta_h-\mu}^{\beta_h+\mu} e\big(q(jf(\beta)-h\beta)\big)d\beta \ll Q^{\frac12}\lambda_h^{-1}|j|^{-\frac12}+Q^{(3-\theta)/2}|j|^{(-1-\theta)/2}\ .$$

Hence, by (2.6),

$$N_5 \ll J^{\frac12}Q^{\frac12+\varepsilon}+J^{-\frac12}(\log J)Q^{\frac32}+J^{(1-\theta)/2}Q^{(3-\theta)/2}.$$

Thus we have established that

$$\widetilde{N}(Q,\delta) \ll \delta Q^2+\delta^{-\frac12}Q^{\frac12+\varepsilon}+\delta^{\frac12}Q^{\frac32}\log\tfrac1\delta+Q\log\tfrac1\delta+\delta^{\frac{\theta-1}{2}}Q^{\frac{3-\theta}{2}}.$$

When $\frac1\delta \le Q^{1-\varepsilon}\log Q$ we have

$$\delta^{\frac12}\big(\log\tfrac1\delta\big)Q^{\frac32} \le \delta Q^{2-\frac12\varepsilon}(\log Q)^{\frac32} \ll \delta Q^2$$

and when $\frac1\delta > Q^{1-\varepsilon}\log Q$ we have

$$\delta^{\frac12}\big(\log\tfrac1\delta\big)Q^{\frac32} \ll \delta^{-\frac12}Q^{\frac12+\varepsilon}.$$

Moreover, when $\frac1\delta \le Q^{1-2\varepsilon}\log^2 Q$ we have

$$\big(\log\tfrac1\delta\big)Q \ll \delta Q^2$$

and when $\frac1\delta > Q^{1-2\varepsilon}\log^2 Q$ we have

$$\big(\log\tfrac1\delta\big)Q \ll \delta^{-\frac12}Q^{\frac12+\varepsilon}.$$

Therefore

$$\widetilde{N}(Q,\delta) \ll \delta Q^2+\delta^{-\frac12}Q^{\frac12+\varepsilon}+\delta^{\frac{\theta-1}{2}}Q^{\frac{3-\theta}{2}}.$$

This completes the proof of Theorem 3.

5. The proof of Theorem 4

This is easily deduced from Theorem 3 in the same manner that Theorem 2 is deduced from Theorem 1.

6. The proof of Theorem 5

We are given that $\mathcal{C}$ is a $C^{(2)}$ non-degenerate planar curve. Thus, $\mathcal{C}=\mathcal{C}_f := \{(x,f(x))\in\mathbb{R}^2 : x\in I\}$ for some interval I of $\mathbb{R}$ and $f\in C^{(2)}(I)$. Also, since $\mathcal{C}_f$ is non-degenerate we have that $f''(x)\neq 0$ for almost all $x\in I$. Throughout, ψ is an approximating function such that

$$\textstyle\sum_{t=1}^{\infty}\psi(t)^2 \ < \ \infty\ .$$

The claim is that $|\mathcal{C}_f\cap\mathcal{S}(\psi)|_{\mathcal{C}_f}=0$.

Step 1. We show that there is no loss of generality in assuming that

$$\psi(t) \ \ge\ t^{-\frac12}(\log t)^{-1} \quad \text{for all } t. \tag{6.1}$$

To this end, define $\Psi : t\to\Psi(t) := \max\{\psi(t),\, t^{-\frac12}(\log t)^{-1}\}$. Clearly, Ψ is an approximating function and furthermore $\sum\Psi(t)^2<\infty$. By definition, $\mathcal{S}(\psi)\subset\mathcal{S}(\Psi)$ and so it suffices to establish the claim with ψ replaced by Ψ. Hence, without loss of generality, (6.1) can be assumed.

Step 2. Let $\Omega_{f,\psi}$ be the set of $x \in I$ such that the system of inequalities

$$\begin{cases} \left|x - \frac{p_1}{q}\right| < \frac{\psi(q)}{q} \\ \left|f(x) - \frac{p_2}{q}\right| < \frac{\psi(q)}{q} \end{cases}, \tag{6.2}$$

is satisfied for infinitely many $\mathbf{p}/q \in \mathbb{Q}^2$ with $p_1/q \in I$. Notice that since f is continuously differentiable, the map $x \mapsto (x, f(x))$ is locally bi-Lipshitz and so

$$|\mathcal{C}_f \cap \mathcal{S}(\psi)|_{\mathcal{C}_f} = 0 \quad \Longleftrightarrow \quad |\Omega_{f,\psi}|_{\mathbb{R}} = 0 \ .$$

Hence, it suffices to show that

$$|\Omega_{f,\psi}|_{\mathbb{R}} = 0 \ . \tag{6.3}$$

Step 3. Next, without loss of generality, we can assume that I is open in $\mathbb{R}$. Notice that the set $B := \{x \in I : |f''(x)| = 0\}$ is closed in I. Thus the set $G := I \setminus B$ is open and a standard argument allows one to write G as a countable union of bounded intervals I_i on which f satisfies

$$0 \ < \ c_1 \ := \ \inf_{x \in I_0} |f''(x)| \ \leq \ c_2 \ := \ \sup_{x \in I_0} |f''(x)| \ < \ \infty \ . \tag{6.4}$$

The constants c_1, c_2 depend on the particular choice of interval I_i. For the moment, assume that $|\Omega_{f,\psi} \cap I_i|_{\mathbb{R}} = 0$ for any $i \in \mathbb{N}$. On using the fact that $|B|_{\mathbb{R}} = 0$, we have that

$$|\Omega_{f,\psi}|_{\mathbb{R}} \ \leq \ |B \cup \bigcup_{i=1}^{\infty} (\Omega_{f,\psi} \cap I_i)|_{\mathbb{R}} \ \leq \ |B|_{\mathbb{R}} + \ \sum_{i=1}^{\infty} |\Omega_{f,\psi} \cap I_i|_{\mathbb{R}} = 0$$

and this establishes (6.3). Thus, without loss of generality, and for the sake of clarity we assume that f satisfies (6.4) on I and that I is bounded. The upshot of this is that f satisfies the conditions imposed in Theorem 1.

Step 4. For a point $\mathbf{p}/q \in \mathbb{Q}^2$, denote by $\sigma(\mathbf{p}/q)$ the set of $x \in I$ satisfying (6.2). Trivially,

$$|\sigma(\mathbf{p}/q)|_{\mathbb{R}} \ \leq \ 2\psi(q)/q \ . \tag{6.5}$$

Assume that $\sigma(\mathbf{p}/q) \neq \emptyset$ and let $x \in \sigma(\mathbf{p}/q)$. By the mean value theorem, $f(x) = f(p_1/q) + f'(\tilde{x})(x - p_1/q)$ for some $\tilde{x} \in I$. We can assume that f' is bounded on I since f'' is bounded and I is a bounded interval. Suppose $2^n \leq q < 2^{n+1}$. By (6.2),

$$\left|f(\tfrac{p_1}{q}) - \tfrac{p_2}{q}\right| \ \leq \ \left|f(x) - \tfrac{p_2}{q}\right| \ + \ \left|f'(\tilde{x})\big(x - \tfrac{p_1}{q}\big)\right| \ < \ c_3\,\psi(q)/q \ \leq \ c_3\,\psi(2^n)/2^n \ ,$$

where $c_3 > 0$ is a constant. Thus,

$$\begin{aligned} &\text{card}\{\mathbf{p}/q \in \mathbb{Q}^2 \ : \ 2^n \leq q < 2^{n+1}, \, \sigma(\mathbf{p}/q) \neq \emptyset\} \\ &\qquad \leq \ \text{card}\left\{\mathbf{p}/q \in < \mathbb{Q}^2 : q \leq 2^{n+1}, \, p_1/q \in I, \, \left|f(\tfrac{p_1}{q}) - \tfrac{p_2}{q}\right| < c_3\,\psi(2^n)/2^n\right\} \\ &\qquad \leq \ \text{card}\left\{a/q \in \mathbb{Q} : q \leq 2^{n+1}, \, a/q \in I, \, \left\|qf(\tfrac{a}{q})\right\| < 2c_3\,\psi(2^n)\right\} \ . \end{aligned}$$

In view of (6.1), Theorem 1 implies that

$$\text{card}\{\mathbf{p}/q \in \mathbb{Q}^2 : 2^n \leq q < 2^{n+1}, \, \sigma(\mathbf{p}/q) \neq \emptyset\} \ll \psi(2^n)\, 2^{2n} \ . \tag{6.6}$$

Step 5. For $n \geq 0$, let

$$\Omega_{f,\psi}(n) \ := \bigcup_{\mathbf{p}/q \in \mathbb{Q}^2,\, \sigma(\mathbf{p}/q)\neq\emptyset,\, 2^n \leq q < 2^{n+1}} \sigma(\mathbf{p}/q) \ .$$

Then $|\Omega_{f,\psi}|_{\mathbb{R}} = |\limsup_{n\to\infty} \Omega_{f,\psi}(n)|_{\mathbb{R}}$ and the Borel-Cantelli Lemma implies (6.3) if $\sum_{n=0}^{\infty} |\Omega_{f,\psi}(n)|_{\mathbb{R}} < \infty$. In view of (6.5) and (6.6), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} |\Omega_{f,\psi}(n)|_{\mathbb{R}} \ &= \ \sum_{n=0}^{\infty} \ \sum_{\mathbf{p}/q \in \mathbb{Q}^2,\, \sigma(\mathbf{p}/q)\neq\emptyset,\, 2^n \leq q < 2^{n+1}} |\sigma(\mathbf{p}/q)|_{\mathbb{R}} \\ &\ll \ \sum_{n=0}^{\infty} \psi(2^n)/2^n \times \psi(2^n)\, 2^{2n} \ \asymp \ \sum_{t=1}^{\infty} \psi(t)^2 \ < \ \infty \ .\end{aligned}$$

This completes the proof of Theorem 5.

7. The proof of Theorem 6

In spirit, the proof of Theorem 6 follows the same line of argument as the proof of Theorem 5. Throughout, $s \in (1/2, 1)$ and ψ is an approximating function such that

$$\textstyle\sum_{t=1}^{\infty} t^{1-s} \psi(t)^{s+1} \ < \ \infty \ .$$

Step 1. Choose $\eta > 0$ such that $\eta < (2s-1)/(s+1)$. Note that $(2s-1)/(s+1)$ is strictly positive since $s > 1/2$. By considering the auxiliary function $\Psi : t \to \Psi(t) := \max\{\psi(t),\, t^{-1+\eta}\}$, it is easily verified that there is no loss of generality in assuming that

$$\psi(t) \ \geq \ t^{-1+\eta} \quad \text{for all } t. \tag{7.1}$$

Step 2. Let $\Omega_{f,\psi}$ be defined via the system of inequalities (6.2) as in Step 2 of §6. On making use of the fact that the map $x \mapsto (x, f(x))$ is locally bi-Lipshitz we have that

$$\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}(\psi)) = 0 \quad \Longleftrightarrow \quad \mathcal{H}^s(\Omega_{f,\psi}) = 0 \ .$$

Hence, it suffices to show that $\mathcal{H}^s(\Omega_{f,\psi}) = 0$.

Step 3. Let $B := \{x \in I : |f''(x)| = 0\}$. Since $\dim B \leq 1/2$ and $s > 1/2$, it follows from the definition of $\mathcal{H}^s$ that $\mathcal{H}^s(B) = 0$. As in Step 3 of §6, the set $G := I \setminus B$ can be written as a countable union of bounded intervals I_i on which f satisfies (6.4) and moreover we can assume that $|I_i|_{\mathbb{R}} \leq 1$. Since $f \in C^{(3)}(I)$, it follows that $|f''(x) - f''(y)| \ll |x-y| \leq |x-y|^{\theta}$ for any $x, y \in I_i$ and $0 \leq \theta \leq 1$; i.e. $f'' \in \mathrm{Lip}_{\theta}(I_i)$. In particular, with Theorem 3 in mind, we may take

$$1 > \theta > \tfrac{2-3\eta}{2-\eta} \ .$$

Now the same argument as in Step 3 of §6 with Lebesgue measure $|\ \ |_{\mathbb{R}}$ replaced by Hausdorff measure $\mathcal{H}^s$, enables us to conclude that f satisfies (6.4) on I and moreover the conditions imposed in Theorem 3 are satisfied.

Step 4. This is exactly as in Step 4 of §6 apart from the fact that the conclusion (6.6) follows as a consequence of (7.1) and Theorem 3.

Step 5. With $\Omega_{f,\psi}(n)$ as in Step 5 of §6, we have that for each $l \in \mathbb{N}$,

$$\{\Omega_{f,\psi}(n) : n = l,\, l+1, \dots\}$$

is a cover for $\Omega_{f,\psi}$ by sets $\sigma(\mathbf{p}/q)$ of maximal diameter $2\psi(2^l)/2^l$. This makes use of the trivial fact that each set $\sigma(\mathbf{p}/q)$ is contained in an interval of length at most $2\psi(q)/q$. It follows from the definition of Hausdorff measure that with $\rho := 2\psi(2^l)/2^l$,

$$\begin{aligned} \mathcal{H}^s_\rho(\Omega_{f,\psi}) \;&\leq\; \sum_{n=l}^{\infty} \; \sum_{\mathbf{p}/q\in\mathbb{Q}^2,\, \sigma(\mathbf{p}/q)\neq\emptyset,\, 2^n\leq q<2^{n+1}} (2\,\psi(2^n)/2^n)^s \\ &\ll\; \sum_{n=l}^{\infty} (\psi(2^n)/2^n)^s \times \psi(2^n)\, 2^{2n} \;\longrightarrow\; 0 \end{aligned}$$

as $\rho \to 0$; or equivalently at $l \to \infty$. Hence, $\mathcal{H}^s(\Omega_{f,\psi}) = 0$ and this completes the proof of Theorem 6.

8. Various generalizations: the multiplicative setup

For the sake of brevity, we shall restrict our attention to the Lebesgue theory only.

Given approximating functions ψ_1, ψ_2, a point $\mathbf{y} \in \mathbb{R}^2$ is said to be *simultaneously* (ψ_1,ψ_2)*–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\|qy_i\| < \psi_i(q) \qquad\qquad 1 \leq i \leq 2\ .$$

Let $\mathcal{S}(\psi_1,\psi_2)$ denote the set of simultaneously (ψ_1,ψ_2)–approximable points in $\mathbb{R}^2$. This set is clearly a generalization of $\mathcal{S}(\psi)$ in which $\psi = \psi_1 = \psi_2$. The following statement is a natural generalization of Khinchin's theorem:

$$|\mathcal{S}(\psi_1,\psi_2)|_{\mathbb{R}^2} = \begin{cases} \text{Zero} & \text{if } \sum \psi_1(t)\,\psi_2(t) < \infty \\ \text{Full} & \text{if } \sum \psi_1(t)\,\psi_2(t) = \infty \end{cases}.$$

Next, given an approximating function ψ, a point $\mathbf{y} \in \mathbb{R}^2$ is said to be *multiplicatively* ψ*–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\prod_{i=1}^{2} \|qy_i\| \;<\; \psi(q)\ .$$

Let $\mathcal{S}^*(\psi)$ denote the set of multiplicatively ψ–approximable points in $\mathbb{R}^2$. In view of Gallagher's theorem we have that:

$$|\mathcal{S}^*(\psi)|_{\mathbb{R}^2} = \begin{cases} \text{Zero} & \text{if } \sum \psi(t)^2\ \log t \;<\; \infty \\ \text{Full} & \text{if } \sum \psi(t)^2\ \log t \;=\; \infty \end{cases}.$$

Now let $\mathcal{C}$ be a $C^{(3)}$ non-degenerate planar curve. The goal is to obtain the analogues of the above 'zero-full' statements for the sets $\mathcal{C} \cap \mathcal{S}(\psi_1,\psi_2)$ and $\mathcal{C} \cap \mathcal{S}^*(\psi)$. It is highly likely that the counting results obtained in this paper, in particular Theorem 3, together with the ideas developed in [2] will yield the following convergence statements.

Claim 1. $|\mathcal{C} \cap \mathcal{S}(\psi_1,\psi_2)|_{\mathcal{C}} = 0 \quad \text{if} \quad \sum \psi_1(t)\psi_2(t) < \infty.$

Claim 2. $\quad |\mathcal{C} \cap \mathcal{S}^*(\psi)|_{\mathcal{C}} = 0 \quad \text{if} \quad \sum \psi(t) \log t < \infty.$

In the case that the planar curve $\mathcal{C}$ belongs to a special class of rational quadrics, both these claims have been established in [2]. Furthermore, in [2] the divergent analogue of Claim 1 has been established. Thus, establishing Claim 1 would complete the Lebesgue theory for simultaneously (ψ_1, ψ_2)–approximable points on planar curves.

Currently, D. Badziahin is attempting to establish the above claims and is also investigating the Hausdorff measure theory.

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