Diophantine approximation on planar curves and the distribution of rational points

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Abstract

Let $\mathcal{C}$ be a non–degenerate planar curve and for a real, positive decreasing function ψ let $\mathcal{C}(\psi)$ denote the set of simultaneously ψ–approximable points lying on $\mathcal{C}$. We show that $\mathcal{C}$ is of Khintchine type for divergence; i.e. if a certain sum diverges then the one-dimensional Lebesgue measure on $\mathcal{C}$ of $\mathcal{C}(\psi)$ is full. We also obtain the Hausdorff measure analogue of the divergent Khintchine type result. In the case that $\mathcal{C}$ is a rational quadric the convergence counterparts of the divergent results are also obtained. Furthermore, for functions ψ with lower order in a critical range we determine a general, exact formula for the Hausdorff dimension of $\mathcal{C}(\psi)$. These results constitute the first precise and general results in the theory of simultaneous Diophantine approximation on manifolds.

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Contents

1. Introduction

In n–dimensional Euclidean space there are two main types of Diophantine approximation which can be considered, namely simultaneous and dual. Briefly, the simultaneous case involves approximating points $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ by rational points $\{\mathbf{p}/q : (\mathbf{p}, q) \in \mathbb{Z}^n \times \mathbb{Z}\}$. On the other hand, the dual case involves approximating points $\mathbf{y}$ by rational hyperplanes $\{\mathbf{q}.\mathbf{x} = p : (p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^n\}$ where $\mathbf{x}.\mathbf{y} = x_1y_1 + \cdots + x_ny_n$ is the standard scalar product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. In both cases the 'rate' of approximation is governed by some given approximating function. In this paper we consider the general problem of simultaneous Diophantine approximation on manifolds. Thus, the points in $\mathbb{R}^n$ of interest are restricted to some manifold $\mathcal{M}$ embedded in $\mathbb{R}^n$. Over the past ten years or so, major advances have been made towards developing a complete 'metric' theory for the dual form of approximation. However, no such theory exists for the simultaneous case. To some extent this work is an attempt to address this in balance.

1.1 Background and the general problems

Simultaneous approximation in $\mathbb{R}^n$. In order to set the scene we recall two fundamental results in the theory of simultaneous Diophantine approximation in n–dimensional Euclidean space. Throughout $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a real, positive decreasing function and will be referred to as an *approximating function.* Given an approximating function ψ, a point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *simultaneously ψ–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\max_{1\leqslant i\leqslant n} \|qy_i\| < \psi(q)$$

where $\|x\| = \min\{|x-m| : m \in \mathbb{Z}\}$. In the case ψ is $\psi_v : h \to h^{-v}$ with $v > 0$ the point $\mathbf{y}$ is said to be *simultaneously v–approximable.* The set of simultaneously ψ–approximable points will be denoted by $\mathcal{S}_n(\psi)$ and similarly $\mathcal{S}_n(v)$ will denote the set of simultaneously v–approximable points in $\mathbb{R}^n$. Note that in view of Dirichlet's theorem (n-dimensional simultaneous version), $\mathcal{S}_n(v) = \mathbb{R}^n$ for any $v \leq 1/n$.

The following fundamental result provides a beautiful and simple criteria for the 'size' of the set $\mathcal{S}_n(\psi)$ expressed in terms of n–dimensional Lebesgue measure $|\ \ |_{\mathbb{R}^n}$.

Khintchine's Theorem (1924). *Let ψ be an approximating function. Then*

$$|\mathcal{S}_n(\psi)|_{\mathbb{R}^n} = \begin{cases} \text{Zero} & \text{if } \sum \psi(h)^n < \infty \\ \\ \text{Full} & \text{if } \sum \psi(h)^n = \infty \end{cases} .$$

Here 'full' simply means that the complement of the set under consideration is of zero measure. Thus the n–dimensional Lebesgue measure of the set of simultaneously ψ–approximable points in $\mathbb{R}^n$ satisfies a 'zero-full' law. The divergence part of the above statement constitutes the main substance of the theorem. The convergence part is a simple consequence of the Borel-Cantelli lemma from probability theory. Note that $|\mathcal{S}_n(v)|_{\mathbb{R}^n} = 0$ for $v > 1/n$ and so $\mathbb{R}^n$ is extremal – see below.

The next fundamental result is a Hausdorff measure version of the above theorem and shows that the s–dimensional Hausdorff measure $\mathcal{H}^s(\mathcal{S}_n(\psi))$ of the set $\mathcal{S}_n(\psi)$ satisfies an elegant 'zero-infinity' law.

Jarník's Theorem (1931). *Let $s \in (0,n)$ and ψ be an approximating function. Then*

$$\mathcal{H}^s\left(\mathcal{S}_n(\psi)\right) = \begin{cases} 0 & \text{if} \quad \sum h^{n-s}\,\psi(h)^s \ < \infty \\ \\ \infty & \text{if} \quad \sum h^{n-s}\,\psi(h)^s \ = \infty \end{cases} .$$

Furthermore

$$\dim \mathcal{S}_n(\psi) \ = \ \inf\{s : \textstyle\sum h^{n-s}\,\psi(h)^s < \infty\} \ .$$

The dimension part of the statement follows directly from the definition of Hausdorff dimension – see §2.2. In Jarník's original statement the additional hypotheses that $r\psi(r)^n \to 0$ as $r \to \infty$, $r\psi(r)^n$ is decreasing and that $r^{1+n-s}\psi(r)^s$ is decreasing were assumed. However, these are not necessary – see [6, §1.1 and §12.1]. Also, Jarník obtained his theorem for general Hausdorff measures $\mathcal{H}^h$ where h is a dimension function – see §8.1 and [6, §1.1 and §12.1]. However, for the sake of clarity and ease of discussion we have specialized to s-dimensional Hausdorff measure. Note that the above theorem implies that for $v > 1/n$

$$\mathcal{H}^d\left(\mathcal{S}_n(v)\right) \ = \ \infty \qquad \text{where} \qquad d := \dim \mathcal{S}_n(v) = \frac{1+n}{v+1} \ .$$

The two fundamental theorems stated above provide a complete measure theoretic description of $\mathcal{S}_n(\psi)$. For a more detailed discussion and various generalizations of these theorems see [6].

Simultaneous approximation restricted to manifolds. Let $\mathcal{M}$ be a manifold of dimension m embedded in $\mathbb{R}^n$. Given an approximating function ψ consider the set

$$\mathcal{M} \cap \mathcal{S}_n(\psi)$$

consisting of points $\mathbf{y}$ on $\mathcal{M}$ which are simultaneously ψ–approximable. Two natural problems now arise.

Problem 1. To develop a Khintchine type theory for $\mathcal{M} \cap \mathcal{S}_n(\psi)$.

Problem 2. To develop a Hausdorff measure/dimension theory for $\mathcal{M} \cap \mathcal{S}_n(\psi)$.

In short, the aim is to establish analogues of the two fundamental theorems described above and thereby provide a complete measure theoretic description of the sets $\mathcal{M} \cap \mathcal{S}_n(\psi)$. The fact that the points $\mathbf{y}$ of interest are of dependent variables, reflecting the fact that $\mathbf{y} \in \mathcal{M}$ introduces major difficulties in attempting to describe the measure theoretic structure of $\mathcal{M} \cap \mathcal{S}_n(\psi)$. This is true even in the specific case that $\mathcal{M}$ is a planar curve. More to the point, even for seemingly simple curves such as the unit circle or the parabola the problem is fraught with difficulties.

Non-degenerate manifolds. In order to make any reasonable progress with the above problems it is not unreasonable to assume that the manifolds $\mathcal{M}$ under consideration are **non-degenerate** [23]. Essentially, these are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane. Formally, a manifold $\mathcal{M}$ of dimension m embedded in $\mathbb{R}^n$ is said to be non-degenerate if it arises from a non–degenerate map $\mathbf{f} : U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^m$ and $\mathcal{M} := \mathbf{f}(U)$. The map $\mathbf{f} : U \to \mathbb{R}^n : \mathbf{u} \mapsto \mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_n(\mathbf{u}))$ is said to be *non–degenerate at* $\mathbf{u} \in U$ if there exists some $l \in \mathbb{N}$ such that $\mathbf{f}$ is l times continuously differentiable on some sufficiently small ball centred at $\mathbf{u}$ and the partial derivatives of $\mathbf{f}$ at $\mathbf{u}$ of orders up to l span $\mathbb{R}^n$. The map $\mathbf{f}$ is *non–degenerate* if it is non–degenerate at almost every (in terms of m–dimensional Lebesgue measure) point in U; in turn the manifold $\mathcal{M} = \mathbf{f}(U)$ is also said to be non–degenerate. Any real, connected analytic manifold not contained in any hyperplane of $\mathbb{R}^n$ is non–degenerate.

Note that in the case the manifold $\mathcal{M}$ is a planar curve $\mathcal{C}$, a point on $\mathcal{C}$ is non-degenerate if the curvature at that point is non-zero. Thus, $\mathcal{C}$ is a non-degenerate planar curve if the set of points on $\mathcal{C}$ at which the curvature vanishes is a set of one–dimensional Lebesgue measure zero. Moreover, it is not difficult to show that the set of points on a planar curve at which the curvature vanishes but the curve is non-degenerate is at most countable. In view of this, the curvature completely describes the non-degeneracy of planar curves. Clearly, a straight line is degenerate everywhere.

1.2 The Khintchine type theory

The aim is to obtain an analogue of Khintchine's theorem for the set $\mathcal{M} \cap \mathcal{S}_n(\psi)$ of simultaneously ψ–approximable points lying on $\mathcal{M}$. First of all notice that if the dimension m of the manifold $\mathcal{M}$ is strictly less than n then $|\mathcal{M} \cap \mathcal{S}_n(\psi)|_{\mathbb{R}^n} = 0$ irrespective of the approximating function ψ. Thus, when referring to the Lebesgue measure of the set $\mathcal{M} \cap \mathcal{S}_n(\psi)$ it is always with reference to the induced Lebesgue measure on $\mathcal{M}$. More generally, given a subset S of $\mathcal{M}$ we shall write $|S|_{\mathcal{M}}$ for the measure of S with respect to the induced Lebesgue measure on $\mathcal{M}$. Notice that for $v \leq 1/n$, we have that $|\mathcal{M} \cap \mathcal{S}_n(v)|_{\mathcal{M}} = |\mathcal{M}|_{\mathcal{M}} :=$ FULL as it should be since $\mathcal{S}_n(v) = \mathbb{R}^n$.

To develop the Khintchine theory it is natural to consider the convergence and divergence cases separately and the following terminology is most useful.

Definition 1 *Let $\mathcal{M} \subset \mathbb{R}^n$ be a manifold. Then we say that*

1. *$\mathcal{M}$ is of Khintchine type for convergence if $|\mathcal{M} \cap \mathcal{S}_n(\psi)|_{\mathcal{M}} =$ ZERO for any approximating function ψ with $\sum_{h=1}^{\infty} \psi(h)^n < \infty$.*
2. *$\mathcal{M}$ is of Khintchine type for divergence if $|\mathcal{M} \cap \mathcal{S}_n(\psi)|_{\mathcal{M}} =$ FULL for any approximating function ψ with $\sum_{h=1}^{\infty} \psi(h)^n = \infty$.*

The set of manifolds which are of Khintchine type for convergence will be denoted by $\mathcal{K}_{<\infty}$. Similarly, the set of manifolds which are of Khintchine type for divergence will be denoted by $\mathcal{K}_{=\infty}$. Also, we define $\mathcal{K} := \mathcal{K}_{<\infty} \cap \mathcal{K}_{=\infty}$. By definition, if $\mathcal{M} \in \mathcal{K}$ then an analogue of Khintchine's theorem exists for $\mathcal{M} \cap \mathcal{S}_n(\psi)$ and $\mathcal{M}$ is simply said to be of Khintchine type. Thus Problem 1 mentioned above, is equivalent to describing the set of Khintchine type manifolds. Ideally, one would like to prove that any non-degenerate manifold is of Khintchine type. Similar terminology exists for the dual form of approximation in which 'Khintchine type' is replaced by 'Groshev type'; for further details see [11, pp. 29–30].

A weaker notion than 'Khintchine type for convergence' is that of extremality. A manifold $\mathcal{M}$ is said to be *extremal* if $|\mathcal{M} \cap \mathcal{S}_n(v)|_{\mathcal{M}} = 0$ for any $v > 1/n$. The set of extremal manifolds of $\mathbb{R}^n$ will be denoted by $\mathcal{E}$ and it is readily verified that $\mathcal{K}_{<\infty} \subset \mathcal{E}$. In 1932, Mahler made the conjecture that for any $n \in \mathbb{N}$ the Veronese curve $\mathcal{V}_n = \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$ is extremal. The conjecture was eventually settled in 1964 by Sprindzuk [28] – the special cases $n = 2$ and 3 had been done earlier. Essentially, it is this conjecture and its investigations which gave rise to the now flourishing area of 'Diophantine approximation on manifolds' within metric number theory. Up to 1998, manifolds satisfying a variety of analytic, arithmetic and geometric constraints had been shown to be extremal. For example, Schmidt in 1964 proved that any C^3 planar curve with non-zero curvature almost everywhere is extremal. However, Sprindzuk in the 1980's, had conjectured that any analytic manifold satisfying a necessary non–degeneracy condition is extremal. In 1998, Kleinbock and Margulis [23] showed that any non-degenerate manifold is extremal and thereby settled the conjecture of Sprindzuk.

Regarding the 'Khintchine theory' very little is known. The situation for the dual form of approximation is very different. For the dual case, it has recently been shown that any non–degenerate manifold is of Groshev type – the analogue of Khintchine type in the dual case (see [5], [12] and [6, §12.7]). For the simultaneous case, the current state of the Khintchine theory is somewhat ad-hoc. Either a specific manifold or a special class of manifolds satisfying various constraints is studied. For example it has been shown that (i) manifolds which are a topological product of at least four non–degenerate planar curves are in $\mathcal{K}$ [8]; (ii) the parabola $\mathcal{V}_2$ is in $\mathcal{K}_{<\infty}$ [9]; (iii) the so called 2–convex manifolds of dimension $m \geq 2$ are in $\mathcal{K}_{<\infty}$ [17] and (iv) straight lines through the origin satisfying a natural Diophantine condition are in $\mathcal{K}_{<\infty}$ [24]. Thus, even in the simplest geometric and arithmetic situation in which the manifold is a genuine curve in $\mathbb{R}^2$ the only known result to date is that of the parabola $\mathcal{V}_2$. To our knowledge, no curve has ever been shown to be in $\mathcal{K}_{=\infty}$.

In this paper we address the fundamental problems of §1.1 in the case that the manifold $\mathcal{M}$ is a planar curve (the specific case that $\mathcal{M}$ is a non-degenerate, rational quadric will be shown in full). Regarding Problem 1, our main result is the following. As usual, $C^{(n)}(I)$ will denote the set of n–times continuously differentiable functions defined on some interval I of $\mathbb{R}$.

Theorem 1 *Let ψ be an approximating function with $\sum_{h=1}^{\infty} \psi(h)^2 = \infty$. Let $f \in C^{(3)}(I_0)$, where I_0 is an interval, and $f''(x) \neq 0$ for almost all $x \in I_0$. Then for almost all $x \in I_0$ the point $(x, f(x))$ is simultaneously ψ–approximable.*

Corollary 1 *Any $C^{(3)}$ non–degenerate planar curve is of Khintchine type for divergence.*

To complete the 'Khintchine theory' for $C^{(3)}$ non–degenerate planar curves we need to show that any such curve is of Khintchine type for convergence. We are currently able to prove this in the special case that the planar curve is a non-degenerate, rational quadric. However, the truth of Conjecture 1 in §1.4 regarding the distribution of rational points 'near' planar curves would yield the complete convergence theory.

1.2.1. The Khintchine theory for rational quadrics

As above, let $\mathcal{V}_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^2\}$ denote the standard parabola and let $\mathcal{C}_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ and $\mathcal{C}_1^* := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = 1\}$ denote the unit circle and standard hyperbola respectively. Next, let $\mathcal{Q}$ denote a non-degenerate, rational quadric in the plane. By this we mean that $\mathcal{Q}$ is the image of either the circle $\mathcal{C}_1$, the hyperbola $\mathcal{C}_1^*$ or the parabola $\mathcal{V}_2$ under a rational affine transformation of the plane. Furthermore, for an approximating function ψ let

$$\mathcal{Q}(\psi) \ := \ \mathcal{Q} \cap \mathcal{S}_2(\psi)$$

In view of Corollary 1 we have that $\mathcal{Q}$ is in $\mathcal{K}_{=\infty}$. The following result shows that any non-degenerate, rational quadric is in fact in $\mathcal{K}$ and provides a complete criteria for the size of

$\mathcal{Q}(\psi)$ expressed in terms of Lebesgue measure. Clearly, it contains the only previously known result that the parabola is in $\mathcal{K}_{<\infty}$.

Theorem 2 *Let ψ be an approximating function. Then*

$$\left|\mathcal{Q}(\psi)\right|_{\mathcal{Q}} = \begin{cases} \text{ZERO} & \text{if} \ \sum \psi(h)^2 \ < \infty \\ \text{FULL} & \text{if} \ \sum \psi(h)^2 \ = \infty \end{cases} .$$

1.3 The Hausdorff measure/dimension theory

The aim is to obtain an analogue of Jarník's theorem for the set $\mathcal{M} \cap \mathcal{S}_n(\psi)$ of simultaneously ψ–approximable points lying on $\mathcal{M}$. In the dual case, the analogue of the divergent part of Jarník's theorem has recently been established for any non-degenerate manifold [6, §12.7]. Prior to this, a general lower bound for the Hausdorff dimension of the dual set of v–approximable points lying on any extremal manifold had been obtained [13]. Also in the dual case, exact formulae for the dimension of the dual v-approximating sets are known for the case of the Veronese curve [2, 10] and for any planar curve with curvature non-zero except for a set of dimension zero [1].

As with the Khintchine theory, very little is currently known regarding the Hausdorff measure/dimension theory for the simultaneous case. Contrary to the dual case, $\dim \mathcal{M} \cap \mathcal{S}_n(v)$ behaves in a rather complicated way and appears to depend on the arithmetic properties of $\mathcal{M}$. For example, let $\mathcal{C}_R = \{x^2 + y^2 = R^2\}$ be the circle of radius R centered at the origin. It is easy to verify that $\mathcal{C}_{\sqrt{3}}$ contains no rational points $(s/q, t/q)$. On the other hand, any Pythagorean triple (s, t, q) gives rise to a rational point on the unit circle $\mathcal{C}_1$ and so there are plenty of rational points on $\mathcal{C}_1$. For $v > 1$, these facts regarding the distribution of rational points on the circle under consideration lead to $\dim \mathcal{C}_{\sqrt{3}} \cap \mathcal{S}_2(v) = 0$ whereas $\dim \mathcal{C}_1 \cap \mathcal{S}_2(v) = 1/(1+v)$ [6, 14]. The point is that for $v > 1$, the rational points of interest must lie on the associated circle. Further evidence for the complicated behavior of the dimension can be found in [26]. Recently, $\dim \mathcal{M} \cap \mathcal{S}_n(v)$ has been calculated for large values of v when the manifold $\mathcal{M}$ is parameterized by polynomials with integer coefficients [15] and for $v > 1$ when the manifold is a non-degenerate, rational quadric in $\mathbb{R}^n$ [18]. Also, as a consequence of Wiles' theorem [30], $\dim \mathcal{M} \cap \mathcal{S}_2(v) = 0$ for the curve $x^k + y^k = 1$ with $k > 2$ and $v > k - 1$ [11, p. 94].

The above examples illustrate that in the simultaneous case there is no hope of establishing a single, general formula for $\dim \mathcal{M} \cap \mathcal{S}_n(v)$. Recall, that for $v = 1/n$ we have that $\dim \mathcal{M} \cap \mathcal{S}_n(v) = \dim \mathcal{M} := m$ for any manifold embedded in $\mathbb{R}^n$ since $\mathcal{S}_n(v) = \mathbb{R}^n$ by Dirichlet's theorem. Now notice that in the various examples considered above the varying behaviour of $\dim \mathcal{M} \cap \mathcal{S}_n(v)$ is exhibited for values of v bounded away from the Dirichlet exponent $1/n$. Nevertheless, it is believed that when v lies in a critical range near the Dirichlet exponent $1/n$ then, for a wide class of manifolds (including non–degenerate manifolds), the behaviour of $\dim \mathcal{M} \cap \mathcal{S}_n(v)$ can be captured by a single, general formula. That is to say, that $\dim \mathcal{M} \cap \mathcal{S}_n(v)$ is independent of the arithmetic properties of $\mathcal{M}$ for v close to $1/n$. We shall prove that this is indeed the case for planar curves. Note that for planar curves the Dirichlet exponent is $1/2$ and that the above 'circles example' shows that any critical range for v is a subset of $[1/2, 1]$. In general, the critical range is governed by the dimension of the ambient space and the dimension of the manifold.

Before stating our results we introduce the notion of lower order. Given an approximating function ψ, the *lower order* λ_ψ of $1/\psi$ is defined by

$$\lambda_\psi \ := \ \liminf_{h\to\infty} \ \frac{-\log\psi(h)}{\log h} \ ,$$

and indicates the growth of the function $1/\psi$ 'near' infinity. Note that λ_ψ is non-negative since ψ is a decreasing function. Regarding Problem 2, our main results are as follows.

Theorem 3 *Let $f \in C^{(3)}(I_0)$, where I_0 is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Assume that there exists at least one point on the curve $\mathcal{C}_f$ which is non-degenerate. Let $s \in (1/2, 1)$ and ψ be an approximating function. Then*

$$\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(\psi)) \ = \ \infty \qquad \text{if} \qquad \sum_{h=1}^{\infty} h^{1-s}\,\psi(h)^{s+1} \ = \ \infty \ .$$

Theorem 4 *Let $f \in C^{(3)}(I_0)$, where I_0 is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Let ψ be an approximating function with $\lambda_\psi \in [1/2, 1)$. Assume that*

$$\dim\big\{x \in I_0 : f''(x) = 0\big\} \leqslant \frac{2-\lambda_\psi}{1+\lambda_\psi} \ . \tag{1}$$

Then

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi) \ = \ d \ := \ \frac{2-\lambda_\psi}{1+\lambda_\psi} \ .$$

Furthermore, suppose that $\lambda_\psi \in (1/2, 1)$. Then

$$\mathcal{H}^d(\mathcal{C}_f \cap \mathcal{S}_2(\psi)) \ = \ \infty \qquad \text{if} \qquad \limsup_{h\to\infty} h^{2-s}\psi(h)^{s+1} > 0 \ .$$

By considering the function $\psi : h \to h^{-v}$, an immediate consequence of the theorems is the following corollary.

Corollary 2 *Let $f \in C^{(3)}(I_0)$, where I_o is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Let $v \in [1/2, 1)$ and assume that* $\dim\{x \in I_0 : f''(x) = 0\} \leqslant (2-v)/(1+v)$. *Then*

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(v) \ = \ d \ := \ \frac{2-v}{1+v} \ .$$

Moreover, if $v \in (1/2, 1)$ then $\mathcal{H}^d(\mathcal{C}_f \cap \mathcal{S}_2(v)) = \infty$.

Remark. Regarding Theorem 4, the hypothesis (1) on the set $\{x \in I_0 : f''(x) = 0\}$ is stronger than simply assuming that the curve $\mathcal{C}_f$ is non-degenerate. It requires the curve to be non–degenerate everywhere except on a set of Hausdorff dimension no larger than $(2-\lambda_\psi)/(1+\lambda_\psi)$ – rather than just measure zero. Note that the hypothesis can be made independent of the lower order λ_ψ (or indeed of v in the case of the corollary) by assuming that $\dim\{x \in I_0 : f''(x) = 0\} \le 1/2$. The proof of Theorem 4 follows on establishing the upper and lower bounds for $\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi)$ separately. Regarding the lower bound statement, all that is required is that there exists at least one point on the curve $\mathcal{C}_f$ which is non-degenerate. This is not at all surprising since the lower bound statement can be viewed as a simple consequence of Theorem

3. The hypothesis (1) is required to obtain the upper bound dimension statement. Even for non-degenerate curves, without such a hypothesis the statement of Theorem 4 is clearly false as the following example shows.

Example: *The Cantor curve.* Let K denote the standard middle third Cantor set obtained by removing the middle third of the unit interval $[0,1]$ and then inductively repeating the process on each of the remaining intervals. For our purpose, a convenient expression for K is the following:

$$\bigcap_{i=1}^{\infty}([0,1] \setminus \bigcup_{j=1}^{2^{i-1}} I_{i,j}) = [0,1] \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} I_{i,j} \quad ,$$

where $I_{i,j}$ is the j^{th} interval of the 2^{i-1} open intervals of length 3^{-i} removed at the i^{th}-level of the Cantor construction. Note that the intervals $I_{i,j}$ are pair wise disjoint. Give a pair (i,j), define the function

$$f_{i,j} \,:\, x \to f_{i,j}(x) \;:=\; \begin{cases} e^{-i - \frac{1}{(x-a)(b-x)}} & \text{if} \quad x \in I_{i,j} \\ 0 & \text{if} \quad x \in [0,1] \setminus I_{i,j} \end{cases} ,$$

where a and b are the end points of the interval $I_{i,j}$. Now set

$$f : x \to f(x) \;:=\; \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} f_{i,j}(x) \quad .$$

Note that the function f is obviously $C^{(\infty)}$ as the sum converges uniformly. Also, for $x \in K$ and $m \in \mathbb{N}$ we have that $f_{i,j}^{(m)}(x) = 0$ and so

$$f^{(m)}(x) \;=\; \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} f_{i,j}^{(m)}(x) \;=\; 0 \quad .$$

On the other hand, for $x \in [0,1] \setminus K$ we have that $f^{(m)}(x) > 0$. Thus the curve $C_K = \{(x, f(x)) : x \in (0,1)\}$ is exactly degenerate on K and non-degenerate elsewhere. Note that C_K is a non-degenerate curve since K is of Lebesgue measure zero. The upshot of this is that for any $x \in K$ the point $(x, f(x))$ is 1-approximable; i.e. there exists infinitely many $q \in \mathbb{N}$ such that

$$\|qx\| < q^{-1} \qquad \text{and} \qquad \|qf(x)\| < q^{-1} \quad .$$

The second inequality is trivial as $f(x) = 0$ and the first inequality is a consequence of Dirichlet's theorem. Thus,

$$\dim \mathcal{C}_K \cap \mathcal{S}_2(v) \;\geq\; \dim K \;=\; \log 2 / \log 3$$

irrespective of $v \in (1/2, 1)$. Obviously, by choosing Cantor sets K with dimension close to one, we can ensure that $\dim \mathcal{C}_K \cap \mathcal{S}_2(v)$ is close to one irrespective of $v \in (1/2, 1)$.

For simultaneous Diophantine approximation on planar curves, Theorem 3 is the precise analogue of the divergent part of Jarník's theorem and Theorem 4 establishes a complete Hausdorff dimension theory.

Note that the measure part of Theorem 4 is substantially weaker than Theorem 3 – the general measure statement. For example, with $v \in (1/2, 1)$ and $\alpha = 1/(d+1)$ consider the approximating function ψ given by

$$\psi : h \to h^{-v} (\log h)^{-\alpha} \quad .$$

Then $\lambda_\psi = v$ and assuming that (1) is satisfied, the dimension part of Theorem 4 implies that

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi) \;=\; d \;:=\; \frac{2-v}{1+v} \;.$$

However,

$$\limsup_{h\to\infty} h^{2-d}\psi(h)^{d+1} \;=\; \lim_{h\to\infty} (\log h)^{-1} \;=\; 0$$

and so the measure part of Theorem 4 is not applicable. Nevertheless,

$$\sum h^{1-d}\psi(h)^{d+1} = \sum (h\log h)^{-1} = \infty$$

and Theorem 3 implies that $\mathcal{H}^d(\mathcal{C}_f \cap \mathcal{S}_2(\psi)) \;=\; \infty$.

Theorem 3 falls short of establishing a complete Hausdorff measure theory for simultaneous Diophantine approximation on planar curves. In its simplest form, it should be possible to summarize the Hausdorff measure theory by a clear cut statement of the following type.

Conjecture H *Let $s \in (1/2, 1)$ and ψ be an approximating function. Let $f \in C^{(3)}(I_0)$, where I_o is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Assume that* $\dim\{x \in I_0 : f''(x) = 0\} \le 1/2$. *Then*

$$\mathcal{H}^s\left(\mathcal{C}_f \cap \mathcal{S}_2(\psi)\right) = \begin{cases} 0 & \text{if } \sum h^{1-s}\,\psi(h)^{s+1} < \infty \\[2ex] \infty & \text{if } \sum h^{1-s}\,\psi(h)^{s+1} = \infty \end{cases} .$$

The divergent part of the above statement is Theorem 3. As with the 'Khintchine theory', the above convergent part would follow on proving Conjecture 1 of §1.4. However, for rational quadrics we are able to prove the convergent result independently of any conjecture.

Theorem 5 *Let $s \in (1/2, 1)$ and ψ be an approximating function. Then for any non-degenerate, rational quadric $\mathcal{Q}$ we have that*

$$\mathcal{H}^s\left(\mathcal{Q} \cap \mathcal{S}_2(\psi)\right) = 0 \qquad \text{if } \sum h^{1-s}\,\psi(h)^{s+1} < \infty \;.$$

1.4 Rational points close to a curve

First some useful notation. For any point $\mathbf{r} \in \mathbb{Q}^n$ there exists the smallest $q \in \mathbb{N}$ such that $q\mathbf{r} \in \mathbb{Z}^n$. Thus, every point $\mathbf{r} \in \mathbb{Q}^n$ has a unique representation in the form

$$\frac{\mathbf{p}}{q} = \frac{(p_1, \dots, p_n)}{q} = \left(\frac{p_1}{q}, \dots, \frac{p_n}{q}\right)$$

with $(p_1, \dots, p_n) \in \mathbb{Z}^n$. Henceforth, we will only consider points of $\mathbb{Q}^n$ in this form.

Understanding the distribution of rational points close to a reasonably defined curve is absolutely crucial towards making any progress with the main problems considered in this paper. More precisely, the behaviour of the following counting function will play a central role.

The function $N_f(Q, \psi, I)$. Let I_0 denote a finite, open interval of $\mathbb{R}$ and let f be a function in $C^{(3)}(I_0)$ such that

$$0 \;<\; c_1 \;:=\; \inf_{x\in I_0} |f''(x)| \;\le\; c_2 \;:=\; \sup_{x\in I_0} |f''(x)| \;<\; \infty \;. \tag{2}$$

Given an interval $I \subseteq I_0$, an approximating function ψ and $Q \in \mathbb{R}^+$ consider the counting function $N_f(Q, \psi, I)$ given by

$$N_f(Q, \psi, I) \ := \ \#\{\mathbf{p}/q \in \mathbb{Q}^2 \, : \, q \leqslant Q, \, p_1/q \in I, \, |f(p_1/q) - p_2/q| < \psi(Q)/Q\}.$$

In short, the function $N_f(Q, \psi, I)$ counts 'locally' the number of rational points with bounded denominator lying within a specified neighbourhood of the curve parameterized by f. In [20], Huxley obtains a reasonably sharp upper bound for $N_f(Q, \psi, I)$. We will obtain an exact lower bound and also prove that the rational points under consideration are 'evenly' distributed. The proofs of the Khintchine type and Hausdorff measure/dimension theorems stated in this paper rely heavily on this information. In particular, the exact upper bound in Theorem 4 is easily established in view of Huxley's result [20, Theorem 4.2.4] which we state in a simplified form.

Huxley's estimate: *Let ψ be an approximating function such that $t\psi(t) \to \infty$ as $t \to \infty$. For $\varepsilon > 0$ and Q sufficiently large*

$$N_f(Q, \psi, I_0) \ \leqslant \ \psi(Q) \, Q^{2+\varepsilon}. \tag{3}$$

The complementary lower bound is the substance of our next result.

Theorem 6 *Let ψ be an approximating function satisfying*

$$\lim_{t \to +\infty} \psi(t) = \lim_{t \to +\infty} \frac{1}{t\psi(t)} = 0. \tag{4}$$

There exists a constant $c > 0$, depending on I, such that for Q sufficiently large

$$N_f(Q, \psi, I) \ \geqslant \ c \, Q^2 \, \psi(Q) \, |I| \quad .$$

We suspect that the lower bound given by Theorem 6 is best possible up to a constant multiple. It is plausible that for compact curves, the constant c is independent of I.

Regarding Huxley's estimate, the presence of the 'ε' factor prevents us from proving the desired 'convergent' measure theoretic results. We suspect that a result of the following type is in fact true – proving it is another matter.

Conjecture 1 *Let ψ be an approximating function such that $t\psi(t) \to \infty$ as $t \to \infty$. There exists a constant $\hat{c} > 0$ such that for Q sufficiently large*

$$N_f(Q, \psi, I_0) \ \leqslant \ \hat{c} \, Q^2 \, \psi(Q) \ .$$

Conjecture 1 has immediate consequences for the main problems considered in this paper. In particular, it would imply the following.

Conjecture 2 *Any $C^{(3)}$ non–degenerate planar curve is of Khintchine type for convergence.*

Conjecture 2 would naturally complement Theorem 1 of this paper. The implication Conjecture 1 $\Longrightarrow$ Conjecture 2 is reasonably straightforward – simply modify the argument set out in the proof of Theorem 2. Also, it is not difficult to verify that Conjecture 1 implies the 'convergent' part of Conjecture H – simply modify the argument set out in the proof of Theorem 5. An intriguing problem is to determine whether or not the two conjectures stated above are in fact equivalent.

2. Proof of the rational quadric statements

2.1 Proof of Theorem 2

The divergence part of the theorem is a trivial consequence of Corollary 1 to Theorem 1. To establish the convergence part we proceed as follows.

Let ψ be an approximating function such that $\sum \psi(h)^2 < \infty$. The claim is that $\left|\mathcal{Q}(\psi)\right|_{\mathcal{Q}} = 0$. We begin by introducing an auxiliary function Ψ given by

$$\Psi(h) \ := \ \max\left\{\psi(h),\ h^{-\frac{1}{2}}\,(\log h)^{-1}\right\} \ .$$

Clearly, Ψ is an approximating function and furthermore

$$\sum \Psi(h)^2 < \infty \qquad \text{and} \qquad \Psi(h) \ \geq \ \psi(h) \ .$$

Thus $\mathcal{Q}(\psi) \subset \mathcal{Q}(\Psi)$ and the claim will follow on showing that $\left|\mathcal{Q}(\Psi)\right|_{\mathcal{Q}} = 0$. It is easily verified that such a 'zero' statement is invariant under rational affine transformations of the plane. In view of this, it suffices to consider the curves $\mathcal{C}_1$, $\mathcal{C}_1^*$ and $\mathcal{V}_2$ – see §1.2.1.

In the following, $C(q;s,t)$ will denote the square with centre at the rational point $(s/q,t/q)$ and of side length $2\Psi(q)/q$.

Case (a): $\mathcal{Q} = \mathcal{C}_1$. For $m \in \mathbb{N}$, let

$$W_m(\Psi;\mathcal{C}_1) \ := \ \bigcup_{2^m < q \leq 2^{m+1}} \ \bigcup_{(s,t)\in\mathbb{Z}^2} \mathcal{C}_1 \cap C(q;s,t) \ .$$

Then $\mathcal{C}_1(\Psi) = \limsup_{m\to\infty} W_m(\Psi;\mathcal{C}_1)$ and in view of the Borel-Cantelli lemma $\left|\mathcal{C}_1(\Psi)\right|_{\mathcal{C}_1} = 0$ if $\sum \left|W_m(\Psi;\mathcal{C}_1)\right|_{\mathcal{C}_1} < \infty$. Next, note that if $\mathcal{C}_1 \cap C(q;s,t) \neq \emptyset$ then $(q-2\sqrt{2}\Psi(q))^2 \ \leq \ s^2+t^2 \ \leq \ (q+2\sqrt{2}\Psi(q))^2$ and $\left|\mathcal{C}_1 \cap C(q;s,t)\right|_{\mathcal{C}_1} \ll \Psi(q)/q$. It follows that

$$\begin{aligned}
\left|W_m(\Psi;\mathcal{C}_1)\right|_{\mathcal{C}_1} \ &\ll \ \sum_{2^m<q\leq 2^{m+1}} \ \sum_{\substack{(s,t)\in\mathbb{Z}^2: \\ (q-2\sqrt{2}\Psi(q))^2 \leq s^2+t^2 \leq (q+2\sqrt{2}\Psi(q))^2}} \left|\mathcal{C}_1 \cap C(q;s,t)\right|_{\mathcal{C}_1} \\
&\ll \ \frac{\Psi(2^m)}{2^m} \sum_{2^m<q\leq 2^{m+1}} \ \sum_{\substack{n: \\ |q-\sqrt{n}|<4\Psi(q)}} r(n) \quad , \qquad (5)
\end{aligned}$$

where $r(n)$ denotes the number of representations of n as the sum of two squares.

With reference to Theorem A of Appendix II, with $\psi := 4\Psi$, $Q := 2^m$ and $N := [Q/\Psi(Q)]$ it is easily verified that the error term associated with $\sum_{Q<q\leqslant 2Q}\sum_n' r(n)$ is

$$\ll Q^{\frac{15}{8}}(\log Q)^{65}\Psi(Q) \ .$$

Here we use the trivial fact that $\Psi(Q^*) := \Psi(Q+1) \leq \Psi(Q)$ since Ψ is decreasing. On the other hand, for the main term we have that

$$Q^2\Psi(2Q) \ \ll \ \sum_{Q<q\leqslant 2Q} q\Psi(q) \ \ll \ Q^2\Psi(Q) \ .$$

Thus, Theorem A implies that

$$\sum_{2^m<q\leq 2^{m+1}} \ \sum_{\substack{n: \\ |q-\sqrt{n}|<4\Psi(q)}} r(n) \ \ll \ 2^{2m} \ \Psi(2^m) \quad . \tag{6}$$

This estimate together with (5) implies that $\big|W_m(\Psi;\mathcal{C}_1)\big|_{\mathcal{C}_1} \ll 2^m \ \Psi(2^m)^2$. In turn, we obtain that

$$\sum_{m\in\mathbb{N}} \big|W_m(\Psi;\mathcal{C}_1)\big|_{\mathcal{C}_1} \ \ll \ \sum_{m\in\mathbb{N}} 2^m \ \Psi(2^m)^2 \ \asymp \ \sum_{h\in\mathbb{N}} \Psi(h)^2 \ < \ \infty \ .$$

This completes the proof of the theorem in the case that $\mathcal{Q}$ is the image of the unit circle $\mathcal{C}_1$ under a rational affine transformation of the plane. The other two cases are similar. The key is to bring (6) into play.

Case (b): $\mathcal{Q} = \mathcal{C}_1^*$. For $k \in \mathbb{N}$, let $\mathcal{C}_{1;k}^* := \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = 1 \text{ with } |x_1| \leq 2^k\}$. Thus, $\mathcal{C}_{1;k}^*$ is the hyperbola $\mathcal{C}_1^*$ with the first co-ordinate bounded above by 2^k. For $m \in \mathbb{N}$, let

$$W_m(\Psi;\mathcal{C}_{1;k}^*) \ := \ \bigcup_{2^m<q\leq 2^{m+1}} \ \bigcup_{(s,t)\in\mathbb{Z}^2} \mathcal{C}_{1;k}^* \cap C(q;s,t)$$

and let $\mathcal{C}_{1;k}^*(\Psi) := \limsup_{m\to\infty} W_m(\Psi;\mathcal{C}_{1;k}^*)$. Clearly, $\mathcal{C}_1^*(\Psi) = \bigcup_{k=1}^{\infty} \mathcal{C}_{1;k}^*(\Psi)$ and so $\big|\mathcal{C}_1^*(\Psi)\big|_{\mathcal{C}_1^*} = 0$ if $\big|\mathcal{C}_{1;k}^*(\Psi)\big|_{\mathcal{C}_1^*} = 0$ for each $k \in \mathbb{N}$. The latter follows on showing that $\sum \big|W_m(\Psi;\mathcal{C}_{1;k}^*)\big|_{\mathcal{C}_1^*} < \infty$. It is easily verified that if $\mathcal{C}_{1;k}^* \cap C(q;s,t) \neq \emptyset$ then $1/2 < |s|/q < a := 2^{k+1}$, $|t| < |s|$ and

$$|q^2 + t^2 - s^2| \ < \ 8\,|s|\,\Psi(q) + 8\,\Psi(q)^2 \ < \ 8\,|s|\,\Psi(|s|/a) + 8\,\Psi(|s|/a)^2 \ .$$

Here we have used that fact that the function Ψ is decreasing. It follows via (6), that for m sufficiently large

$$\begin{aligned}
\big|W_m(\Psi;\mathcal{C}_{1;k}^*)\big|_{\mathcal{C}_1} \ &\ll \ \frac{\Psi(2^m)}{2^m} \sum_{2^m<q\leq 2^{m+1}} \ \sum_{\substack{(s,t)\in\mathbb{Z}^2 \,:\, q/2<s<aq \\ (s-8\Psi(s/a))^2 \leq q^2+t^2 \leq (s+8\Psi(s/a))^2}} 1 \\
&\leq \ \frac{\Psi(2^m)}{2^m} \sum_{2^{m-1}<s\leq a2^{m+1}} \ \sum_{\substack{n: \\ |s-\sqrt{n}|<8\Psi(s/a)}} r(n) \\
&\leq \ \frac{\Psi(2^m)}{2^m} \sum_{i=0}^{k+2} \ \sum_{2^{m+i-1}<s\leq 2^{m+i}} \ \sum_{\substack{n: \\ |s-\sqrt{n}|<8\Psi(s/a)}} r(n) \\
&\ll \ k \ \frac{\Psi(2^m)}{2^m} \ 2^{2(m+k+1)} \ \Psi(2^{m-k-2}) \ \ll \ k \ 2^{3k} \ 2^{m-k-2} \ \Psi(2^{m-k-2})^2 \ .
\end{aligned}$$

Thus, $\sum \big|W_m(\Psi;\mathcal{C}_{1;k}^*)\big|_{\mathcal{C}_1^*} \ll \sum 2^m \ \Psi(2^m)^2 \ \asymp \ \sum \Psi(h)^2 \ < \infty$ and we are done.

Case (c): $\mathcal{Q} = \mathcal{V}_2$. For $k \in \mathbb{N}$, let $\mathcal{V}_{2;k} := \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = x_1^2 \text{ with } |x_1| \leq 2^k\}$. For $m \in \mathbb{N}$, let

$$W_m(\Psi;\mathcal{V}_{2;k}) \ := \ \bigcup_{2^m<q\leq 2^{m+1}} \ \bigcup_{(s,t)\in\mathbb{Z}^2} \mathcal{V}_{2;k} \cap C(q;s,t) \quad .$$

We need to show that $\sum \left|W_m(\Psi;\mathcal{V}_{2;k})\right|_{\mathcal{V}_2} < \infty$. It is easily verified that if $\mathcal{V}_{2;k} \cap C(q;s,t) \neq \emptyset$ then $0 \leq |s|/q < a := 2^{k+1}$, $-1 < t/q < a^2$ and $|s^2 - tq| < 2\,\Psi(q)(2\,|s| + |t|) + 4\,\Psi(q)^2 < 6\,a^2 q\Psi(q) + 4\,\Psi(q)^2$; that is,

$$|(2s)^2 - 4tq| < 24\,a^2 q\Psi(q) + 16\,\Psi(q)^2 \ . \tag{7}$$

Let $w := q + t$ and $z := q - t$. Then, $2q = w + z$, $2t = w - z$ and $q - 1 < w < q(a^2 + 1)$. Furthermore, (7) becomes

$$|(2s)^2 + z^2 - w^2| < 24\,a^2 q\Psi(q) + 16\,\Psi(q)^2 < 48\,a^2\,w\,\Psi\left(\tfrac{w}{(a^2+1)}\right) + 16\,\Psi\left(\tfrac{w}{(a^2+1)}\right)^2 \ . \tag{8}$$

It follows, that for m sufficiently large

$$\begin{aligned}
\left|W_m(\Psi;\mathcal{V}_{2;k})\right|_{\mathcal{V}_2} &\ll \frac{\Psi(2^m)}{2^m} \sum_{2^m < q \leq 2^{m+1}} \ \sum_{\substack{(s,t)\in\mathbb{Z}^2 \,:\, -q<t<a^2 q \\ (7)\ \text{holds}}} 1 \\
&\leq \frac{\Psi(2^m)}{2^m} \sum_{2^{m-1} < w \leq (a^2+1)2^{m+1}} \ \sum_{(s,z)\in\mathbb{Z}^2 \,:\, (8)\ \text{holds}} 1 \\
&\leq \frac{\Psi(2^m)}{2^m} \sum_{2^{m-1} < w \leq a^2 2^{m+2}} \ \sum_{\substack{n: \\ |w-\sqrt{n}|<48\Psi(w/(2a^2))}} r(n) \ .
\end{aligned}$$

As in case (b), the desired statement now follows on using (6) to estimate the double sum. ♠

Before moving onto the proof of Theorem 5, we define Hausdorff measure and dimension for the sake of completeness and in order to establish some notation.

2.2 Hausdorff measure and dimension

The Hausdorff dimension of a non–empty subset X of n–dimensional Euclidean space $\mathbb{R}^n$, is an aspect of the size of X that can discriminate between sets of Lebesgue measure zero.

For $\rho > 0$, a countable collection $\{C_i\}$ of Euclidean cubes in $\mathbb{R}^n$ with side length $l(C_i) \leq \rho$ for each i such that $X \subset \bigcup_i C_i$ is called a ρ-cover for X. Let s be a non-negative number and define

$$\mathcal{H}^s_\rho(X) = \inf\left\{\sum_i l^s(C_i) \ : \{C_i\} \text{ is a } \rho\text{–cover of } X\right\} ,$$

where the infimum is taken over all possible ρ-covers of X. The *s-dimensional Hausdorff measure* $\mathcal{H}^s(X)$ of X is defined by

$$\mathcal{H}^s(X) = \lim_{\rho\to 0}\mathcal{H}^s_\rho(X) = \sup_{\rho>0}\mathcal{H}^s_\rho(X)$$

and the *Hausdorff dimension* $\dim X$ of X by

$$\dim X = \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s(X) = \infty\} \ .$$

Strictly speaking, in the standard definition of Hausdorff measure the ρ–cover by cubes is replaced by non–empty subsets in $\mathbb{R}^n$ with diameter at most ρ. It is easy to check that the resulting measure is comparable to $\mathcal{H}^s$ defined above and thus the Hausdorff dimension is the same in both cases. For our purpose using cubes is just more convenient. Moreover, if $\mathcal{H}^s$ is zero or infinity then there is no loss of generality by restricting to cubes. Further details and alternative definitions of Hausdorff measure and dimension can be found in [19, 25].

2.3 Proof of Theorem 5

To a certain degree the proof follows the same line of argument as the proof of the convergent part of Theorem 2. In particular, it suffices to consider the rational quadrics $\mathcal{C}_1$, $\mathcal{C}_1^*$ and $\mathcal{V}_2$. Below, we consider the case of the unit circle $\mathcal{C}_1$ and leave the hyperbola $\mathcal{C}_1^*$ and parabola $\mathcal{V}_2$ to the reader. The required modifications are obvious.

Let ψ be an approximating function such that $\sum h^{1-s}\,\psi(h)^{s+1} < \infty$ and consider the auxiliary function Ψ given by

$$\Psi(h) \ := \ \max\left\{\psi(h),\ h^{-1}\,(\log h)^{260}\right\} \quad .$$

Clearly, Ψ is an approximating function and since $s > 1/2$ we have that $\sum h^{1-s}\,\Psi(h)^{s+1} < \infty$. With the same notation as in the proof of Theorem 2, for each $l \in \mathbb{N}$

$$\{W_m(\Psi, \mathcal{C}_1) : m = l,\, l+1, \dots\}$$

is a cover for $\mathcal{C}_1(\Psi) := \mathcal{C}_1 \cap \mathcal{S}_2(\psi)$ by squares $C(q; s, t)$ of maximal side length $2\Psi(2^l)/2^l$. It follows from the definition of s–dimensional Hausdorff measure that with $\rho := 2\Psi(2^l)/2^l$

$$\begin{aligned}
\mathcal{H}^s_\rho(\mathcal{C}_1(\Psi)) \ &\leq \ \sum_{m=l}^{\infty} \ \sum_{2^m < q \leq 2^{m+1}} \ \sum_{\substack{(s,t)\in\mathbb{Z}^2: \\ (q-2\sqrt{2}\Psi(q))^2 \leq s^2+t^2\ (q+2\sqrt{2}\Psi(q))^2}} \left(\frac{2\Psi(2^m)}{2^m}\right)^s \\
&\ll \ \sum_{m=l}^{\infty} \left(\frac{\Psi(2^m)}{2^m}\right)^s \sum_{2^m < q \leq 2^{m+1}} \ \sum_{\substack{n: \\ |q-\sqrt{n}|<4\Psi(q)}} r(n) \quad .
\end{aligned}$$

In view of Theorem A of Appendix II, the contribution from the two inner sums is $\ll 2^{2m}\Psi(2^m)$. Thus,

$$\mathcal{H}^s_\rho(\mathcal{C}_1(\Psi)) \ \ll \ \sum_{m=l}^{\infty} 2^{m(2-s)}\ \Psi(2^m)^{1+s} \ \to \ 0$$

as $\rho \to 0$; or equivalently at $l \to \infty$. Hence, $\mathcal{H}^s(\mathcal{C}_1(\psi)) \leq \mathcal{H}^s(\mathcal{C}_1(\Psi)) = 0$ as required. ♠

3. Ubiquitous systems

In [6], a general framework is developed for establishing divergent results analogous to those of Khintchine and Jarník (see §1.1) for a natural class of lim sup sets. The framework is based on the notion of 'ubiquity', which goes back to [2] and [16] and captures the key measure theoretic structure necessary to prove such measure theoretic laws. The 'ubiquity' introduced below is a much simplified version of that in [6] and takes into consideration the specific applications that we have in mind.

3.1 Ubiquitous systems in $\mathbb{R}$

Let I_0 be an interval in $\mathbb{R}$ and $\mathcal{R} := (R_\alpha)_{\alpha\in\mathcal{J}}$ be a family of *resonant points* R_α of I_0 indexed by an infinite, countable set $\mathcal{J}$. Next let $\beta : \mathcal{J} \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on $\mathcal{J}$. Thus, the function β attaches a 'weight' β_α to the resonant point R_α. Also, for $t \in \mathbb{N}$ let $J(t) := \{\alpha \in \mathcal{J} : \beta_\alpha \leqslant 2^t\}$ and assume that $\#J(t)$ is always finite. Given an approximating function Ψ let

$$\Lambda(\mathcal{R},\beta,\Psi) \;:=\; \{x \in I_0 : |x - R_\alpha| < \Psi(\beta_\alpha) \text{ for infinitely many } \alpha \in \mathcal{J}\} \;.$$

The set $\Lambda(\mathcal{R},\beta,\Psi)$ is easily seen to be a lim sup set. The general theory of ubiquitous systems developed in [6], provides a natural measure theoretic condition for establishing divergent results analogous to those of Khintchine and Jarník for $\Lambda(\mathcal{R},\beta,\Psi)$. Since $\Lambda(\mathcal{R},\beta,\Psi)$ is a subset of I_0, any Khintchine type result would naturally be with respect to one-dimensional Lebesgue measure $|\,.\,|$.

Throughout, $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a function satisfying $\lim_{t\to\infty} \rho(t) = 0$ and is usually referred to as the *ubiquitous function.* Also $B(x,r)$ will denote the ball (or rather the interval) centred at x or radius r.

Definition 2 (Ubiquitous systems on the real line) *Suppose there exists a function ρ and an absolute constant $\kappa > 0$ such that for any interval $I \subseteq I_0$*

$$\liminf_{t\to\infty} \left| \bigcup_{\alpha\in J(t)} \left(B(R_\alpha, \rho(2^t)) \cap I\right)\right| \;\geqslant\; \kappa\,|I| \;.$$

Then the system $(\mathcal{R};\beta)$ is called locally ubiquitous in I_0 with respect to ρ*.*

The consequences of this definition of ubiquity are the following key results.

Lemma 1 *Suppose that $(\mathcal{R},\beta)$ is a local ubiquitous system in I_0 with respect to ρ and let Ψ be an approximating function such that $\Psi(2^{t+1}) \leqslant \frac{1}{2}\Psi(2^t)$ for t sufficiently large. Then*

$$|\Lambda(\mathcal{R},\beta,\Psi)| \;=\; \text{FULL} \;:=\; |I_0| \qquad \text{if} \qquad \sum_{t=1}^{\infty} \frac{\Psi(2^t)}{\rho(2^t)} = \infty \;\;.$$

Lemma 2 *Suppose that $(\mathcal{R},\beta)$ is a local ubiquitous system in I_0 with respect to ρ and let Ψ be an approximating function. Let $s \in (0,1)$ and let*

$$G \;:=\; \limsup_{t\to\infty} \frac{\Psi(2^t)^s}{\rho(2^t)} \;\;.$$

(i) *Suppose that $G = 0$ and that $\Psi(2^{t+1}) \leqslant \frac{1}{2}\Psi(2^t)$ for t sufficiently large. Then,*

$$\mathcal{H}^s(\Lambda(\mathcal{R},\beta,\Psi)) \;=\; \infty \qquad \text{if} \qquad \sum_{t=1}^{\infty} \frac{\Psi(2^t)^s}{\rho(2^t)} \;=\; \infty \;\;.$$

(ii) *Suppose that $G > 0$. Then, $\mathcal{H}^s(\Lambda(\mathcal{R},\beta,\Psi)) \;=\; \infty$.*

Corollary 3 *Suppose that $(\mathcal{R},\beta)$ is a local ubiquitous system in I_0 with respect to ρ and let Ψ be an approximating function. Then*

$$\dim(\Lambda(\mathcal{R},\beta,\Psi)) \ \geqslant \ d \ := \ \min\left\{1, \left|\limsup_{t\to\infty} \frac{\log\rho(2^t)}{\log\Psi(2^t)}\right|\right\}.$$

Moreover, if $d<1$ and $\limsup_{t\to\infty}\Psi(2^t)^d/\rho(2^t)>0$, then $\mathcal{H}^d(\Lambda(\mathcal{R},\beta,\Psi))=\infty$.

The concept of ubiquity was originally formulated by Dodson, Rynne & Vickers [16] to obtain lower bounds for the Hausdorff dimension of lim sup sets. In the one-dimensional setting considered here, their 'ubiquity result' essentially corresponds to Corollary 3 above. Furthermore, the ubiquitous systems of [16] essentially coincide with the regular systems of Baker & Schmidt [2] and both have proved very useful in obtaining lower bounds for the Hausdorff dimension of lim sup sets. However, unlike the framework developed in [6], both [2] and [16] fail to shed any light on establishing the more desirable divergent Khintchine and Jarník type results. The latter, clearly implies lower bounds for the Hausdorff dimension. For further details regarding regular systems and the original formulation of ubiquitous systems see [6, 11].

Lemmas 1 and 2 follow directly from Corollaries 2 and 4 in [6]. Note that in Lemma 2, if $G>0$ then the divergent sum condition of part (i) is trivially satisfied. The dimension statement (Corollary 3) is a consequence of part (ii) of Lemma 2 and so the regularity condition $2\,\Psi(2^{t+1})\leqslant\Psi(2^t)$ on the function Ψ is not necessary – see [6, Corollary 6].

The framework and results of [6] are abstract and general unlike the concrete situation described above. In view of this and for the sake of completeness we retraced the argument of [6] in the above simple setting at the end of the paper §A-C. This has the effect of making the paper self-contained and more importantly should help the interested reader with understanding the abstract approach undertaken in [6]. The direct proofs of Lemmas 1 and 2 are substantially easier (both technically and conceptionally) than the general statements of [6].

3.2 Ubiquitous systems close to a curve in $\mathbb{R}^n$

In this section we develop the theory of ubiquity to incorporate the situation in which the resonant points of interest lie within some specified neighborhood of a given curve in $\mathbb{R}^n$.

With $n\geq 2$, let $\mathcal{R}:=(R_\alpha)_{\alpha\in\mathcal{J}}$ be a family of *resonant points* R_α of $\mathbb{R}^n$ indexed by an infinite set $\mathcal{J}$. As before, $\beta:\mathcal{J}\to\mathbb{R}^+:\alpha\mapsto\beta_\alpha$ is a positive function on $\mathcal{J}$. For a point R_α in $\mathcal{R}$, let $R_{\alpha,k}$ represent the k'th coordinate of R_α. Thus, $R_\alpha:=(R_{\alpha,1},R_{\alpha,2},\dots,R_{\alpha,n})$. Throughout this section and the remainder of the paper we will use the notation $\mathcal{R}_\mathcal{C}(\Phi)$ to denote the sub-family of resonant points R_α in $\mathcal{R}$ which are "Φ–close" to the curve $\mathcal{C}=\mathcal{C}_\mathbf{f}:=\{(x,f_2(x),\dots,f_n(x)):x\in I_0\}$ where Φ is an approximating function, $\mathbf{f}=(f_1,\dots,f_n):I_0\to\mathbb{R}^n$ is a continuous map with $f_1(x)=x$ and I_0 is an interval in $\mathbb{R}$. Formally, and more precisely

$$\mathcal{R}_\mathcal{C}(\Phi):=(R_\alpha)_{\alpha\in\mathcal{J}_\mathcal{C}(\Phi)} \qquad \text{where} \qquad \mathcal{J}_\mathcal{C}(\Phi):=\{\alpha\in\mathcal{J}:\max_{1\leqslant k\leqslant n}|f_k(R_{\alpha,1})-R_{\alpha,k}|<\Phi(\beta_\alpha)\}\ .$$

Finally, we will denote by $\mathcal{R}_1$ the family of first co-ordinates of the points in $\mathcal{R}_\mathcal{C}(\Phi)$; that is

$$\mathcal{R}_1 \ := \ (R_{\alpha,1})_{\alpha\in\mathcal{J}_\mathcal{C}(\Phi)} \ .$$

By definition, $\mathcal{R}_1$ is a subset of the interval I_0 and can therefore be regarded as a set of resonant points for the theory of ubiquitous systems in $\mathbb{R}$. This leads us naturally to the following definition in which the ubiquity function ρ is as in §3.1.

Definition 3 (Ubiquitous systems near curves) *The system $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is called locally ubiquitous with respect to ρ if the system $(\mathcal{R}_1, \beta)$ is locally ubiquitous in I_0 with respect to ρ.*

Next, given an approximating function Ψ let $\Lambda(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)$ denote the the set $x \in I_0$ for which the system of inequalities

$$\left\{ \begin{array}{rcl} |x - R_{\alpha,1}| & < & \Psi(\beta_\alpha) \\ \max\limits_{2 \leqslant k \leqslant n} |f_k(x) - R_{\alpha,k}| & < & \Psi(\beta_\alpha) + \Phi(\beta_\alpha) \end{array} \right. ,$$

is simultaneously satisfied for infinitely many $\alpha \in \mathcal{J}$. The following two lemmas are the analogues of Lemmas 1 and 2 for the case of ubiquitous systems close to a curve. Similarly, Corollary 4 is the analogue of Corollary 3.

Lemma 3 *Consider the curve $\mathcal{C} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$, where $f_2, \ldots, f_n$ are locally Lipshitz in a finite interval I_0. Let Φ and Ψ be approximating functions. Suppose that $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is a locally ubiquitous system with respect to ρ. If Ψ and ρ satisfy the conditions of Lemma 1 then*

$$|\,\Lambda\,(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)\,| \;=\; |I_0| \;.$$

Lemma 4 *Consider the curve $\mathcal{C} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$, where $f_2, \ldots, f_n$ are locally Lipshitz in a finite interval I_0. Let Φ and Ψ be approximating functions. Suppose that $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is a locally ubiquitous system with respect to ρ. Let $s \in (0,1)$ and let*

$$G \;:=\; \limsup_{t \to \infty} \frac{\Psi(2^t)^s}{\rho(2^t)} \;.$$

(i) *Suppose that $G = 0$ and that $\Psi(2^{t+1}) \leqslant \frac{1}{2}\Psi(2^t)$ for t sufficiently large. Then,*

$$\mathcal{H}^s(\Lambda(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)) \;=\; \infty \qquad \text{if} \qquad \sum_{t=1}^{\infty} \frac{\Psi(2^t)^s}{\rho(2^t)} \;=\; \infty \;.$$

(ii) *Suppose that $G > 0$. Then, $\mathcal{H}^s(\Lambda(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)) \;=\; \infty$.*

Corollary 4 *Consider the curve $\mathcal{C} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$, where $f_2, \ldots, f_n$ are locally Lipshitz in a finite interval I_0. Let Φ and Ψ be approximating functions. Suppose that $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is a locally ubiquitous system with respect to ρ. Then*

$$\dim \Lambda\,(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi) \;\geqslant\; d \;:=\; \min\left\{1, \left|\limsup_{t \to \infty} \frac{\log \rho(2^t)}{\log \Psi(2^t)}\right|\right\}.$$

Moreover, if $d < 1$ and $\limsup_{t \to \infty} \Psi(2^t)^d / \rho(2^t) > 0$, then $\mathcal{H}^d(\Lambda\,(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)) = \infty$.

Proof of Lemmas 3 & 4 and Corollary 4. It suffices to prove the lemmas for a sufficiently small neighborhood of a fixed point in I_0. Therefore, there is no loss of generality in assuming that $f_2, \ldots, f_n$ satisfy the Lipshitz condition on I_0. Thus, we can fix a constant $c_3 \geqslant 1$ such that for $k \in \{2, \ldots, n\}$ and $x, y \in I_0$

$$|f_k(x) - f_k(y)| \leqslant c_3 |x - y|. \tag{9}$$

Since $(\mathcal{R}_\mathcal{C}(\Phi),\beta)$ is a locally ubiquitous system with respect to ρ, by definition $(\mathcal{R}_1,\beta)$ is a locally ubiquitous system in I_0 with respect to ρ. The set $\Lambda(\mathcal{R}_1,\beta,\Psi/c_3)$ consists of $x\in I_0$ for which the inequality

$$|x-R_{\alpha,1}|<\Psi(\beta_\alpha)/c_3\leqslant\Psi(\beta_\alpha) \tag{10}$$

is satisfied for infinitely many $\alpha\in\mathcal{J}_\mathcal{C}(\Phi)$. Suppose x satisfies (10) for some $\alpha\in\mathcal{J}_\mathcal{C}(\Phi)$. In view of (9), $|f_k(x)-f_k(R_{\alpha,1})|\leqslant c_3|x-R_{\alpha,1}|$ which implies that

$$\begin{aligned}
|f_k(x)-R_{\alpha,k}| &= |f_k(x)-f_k(R_{\alpha,1})+f_k(R_{\alpha,1})-R_{\alpha,k}|\\
&\leqslant |f_k(x)-f_k(R_{\alpha,1})|+|f_k(R_{\alpha,1})-R_{\alpha,k}|\\
&\leqslant c_3|x-R_{\alpha,1}|+\Phi(\beta_\alpha)\\
&< c_3\cdot\Psi(\beta_\alpha)/c_3+\Phi(\beta_\alpha)=\Psi(\beta_\alpha)+\Phi(\beta_\alpha).
\end{aligned}$$

Thus $\Lambda(\mathcal{R}_1,\beta,\Psi/c_3)\subset\Lambda(\mathcal{R},\beta,\Psi)$. Applying Lemmas 1 & 2 and Corollary 3 to the set $\Lambda(\mathcal{R}_1,\beta,\Psi/c_3)$ gives the desired statements concerning the set $\Lambda(\mathcal{R}_\mathcal{C}(\Phi),\beta,\Psi)$. ♠

4. Proof of Theorem 6

We begin by stating a key result which not only implies Theorem 6 but gives rise to a ubiquitous system that will be required in proving Theorems 1 and 4.

4.1 The ubiquity version of Theorem 6

Theorem 7 *Let I_0 denote a finite, open interval of $\mathbb{R}$ and let f be a function in $C^{(3)}(I_0)$ satisfying (2). Let ψ be an approximating function satisfing* (4). *Then for any interval $I\subseteq I_0$ there exist constants $\delta_0,C_1>0$ such that for Q sufficiently large*

$$\left|\bigcup_{\mathbf{p}/q\in A_Q(I)}\left(B\left(\frac{p_1}{q},\frac{C_1}{Q^2\psi(Q)}\right)\cap I\right)\right|\ \geqslant\ \frac{1}{2}\,|I|\quad ,$$

where

$$A_Q(I)\ :=\ \left\{\mathbf{p}/q\in\mathbb{Q}^2\,:\,\delta_0Q<q\leqslant Q,\ p_1/q\in I\,,\ |f(p_1/q)-p_2/q|<\psi(Q)/Q\right\}\quad .$$

Proof of Theorem 6 modulo Thorem 7. This is trivial. Given the hypotheses of Theorem 7, the hypotheses of Theorem 6 are clearly satisfied. Fix an interval $I\subseteq I_0$. By Theorem 7, there exist constants δ_0 and C_1 so that for all Q sufficiently large

$$\#A_Q(I)\cdot\frac{2C_1}{Q^2\psi(Q)}\geq\sum_{\mathbf{p}/q\in A_Q(I)}\left|B\left(\frac{p_1}{q},\frac{C_1}{Q^2\psi(Q)}\right)\right|\geq\left|\bigcup_{\mathbf{p}/q\in A_Q(I)}\left(B\left(\frac{p_1}{q},\frac{C_1}{Q^2\psi(Q)}\right)\cap I\right)\right|\geqslant\frac{|I|}{2}.$$

We have that $N_f(Q,\psi,I)\geqslant\#A_Q(I)$ and Theorem 6 follows. ♠

The following corollary of Theorem 7 is crucial for proving Theorems 1 and 4.

Corollary 5 *Let ψ and f be as in Theorem 7 and $\mathcal{C} := \{(x, f(x)) : x \in I_0\}$. With reference to the ubiquitous framework of §3.2, set*

$$\beta : \ \mathcal{J} := \mathbb{Z}^2 \times \mathbb{N} \to \mathbb{N} : (\mathbf{p}, q) \to q \ , \qquad \Phi : t \to t^{-1}\psi(t) \qquad \text{and} \qquad \rho : t \to u(t)/(t^2\psi(t)) \tag{11}$$

where $u : \mathbb{R}^+ \to \mathbb{R}^+$ is any function such that $\lim_{t\to\infty} u(t) = \infty$. Then the system $(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta)$ is locally ubiquitous with respect to ρ.

Remark. Given $\alpha = (\mathbf{p}, q) \in \mathcal{J}$, the associated resonant point R_α in the above ubiquitous system is simply the rational point $\mathbf{p}/q$ in the plane. Furthermore, $\mathcal{R} := \mathbb{Q}^2$.

Proof of Corollary 5. For an interval $I \subseteq I_0$, let

$$A^*_Q(I) \ := \ \{\mathbf{p}/q \in \mathbb{Q}^2 \ : \ Q/u(Q) < q \leqslant Q \ , \ p_1/q \in I \ , \ |f(p_1/q) - p_2/q| < \psi(Q)/Q\} \ \ .$$

For any $\delta_0 \in (0, 1)$, we have that $1/u(Q) < \delta_0$ for Q sufficiently large since $\lim_{t\to\infty} u(t) = \infty$. Thus, for Q sufficiently large, $A_Q(I) \subset A^*_Q(I)$ and Theorem 7 implies that

$$\left| \bigcup_{\mathbf{p}/q \in A^*_Q(I)} \left(B\left(\frac{p_1}{q}, \frac{u(Q)}{Q^2\psi(Q)}\right) \cap I \right) \right| \geqslant \left| \bigcup_{\mathbf{p}/q \in A_Q(I)} \left(B\left(\frac{p_1}{q}, \frac{C_1}{Q^2\psi(Q)}\right) \cap I \right) \right| \geqslant \frac{|I|}{2}.$$

This establishes the corollary. ♠

4.2 An auxiluary lemma

The following lemma is an immediate consequence of Theorem 1.4 in [12].

Lemma 5 *Let $\mathbf{g} := (g_1, g_2) : I_0 \to \mathbb{R}^2$ be a $C^{(2)}$ map such that $(g_1' g_2'' - g_2' g_1'')(x_0) \neq 0$ for some point $x_0 \in I_0$. Given positive real numbers δ, K, T and an interval $I \subseteq I_0$, let $B(I, \delta, K, T)$ denote the set of $x \in I$ for which there exists $(q, p_1, p_2) \in \mathbb{Z}^3 \setminus \{0\}$ satisfying the following system of inequalities:*

$$\begin{cases} |q\, g_1(x) \ + \ p_1\, g_2(x) + p_2| \ \leqslant \ \delta \\[2ex] |q\, g_1'(x) \ + \ p_1\, g_2'(x)| \ \leqslant \ K \\[2ex] |q| \ \leqslant \ T \ \ . \end{cases}$$

Then there is a sufficiently small $\eta = \eta(x_0) > 0$ so that for any interval $I \subset (x_0 - \eta, x_0 + \eta)$ there exists a constant $C > 0$ such that for

$$0 < \delta \leqslant 1, \quad T \geqslant 1, \quad K > 0 \quad \text{and} \quad \delta K T \leqslant 1 \tag{12}$$

one has

$$|B(I, \delta, K, T)| \leqslant C \max\left(\delta^{1/3}, (\delta K T)^{1/9}\right) |I|. \tag{13}$$

Note that the constant C depends on the interval I. We now show that under the assumption that $\mathbf{g}$ is non-degenerate everywhere, the above lemma can be extended to a global statement in which I is any sub-interval of I_0.

Lemma 6 *Assume that the conditions of Lemma 5 are satisfied and that* $(g_1'g_2''-g_2'g_1'')(x)\neq 0$ *for all* $x\in I_0$*. Then for any finite interval* $I\subseteq I_0$ *there is a constant* $C>0$ *such that for any* δ,K,T *satisfying* (12) *one has the estimate* (13).

Proof of Lemma 6. As I is a finite interval, its closure $\overline{I}$ is compact. By Lemma 5, for every point $x\in\overline{I}$ there is an interval $B(x,\eta(x))$ centred at x such that for any sub-interval J of $B(x,\eta(x))$ there is a constant $C=C_J$ (dependent on J) satisfying (13) with δ,K,T satisfying (12). Since $\overline{I}$ is compact, there is a finite cover $\{I_i:=B(x_i,\eta(x_i)):i=1,\dots,n\}$ of $\overline{I}$. Choose this cover so that n is minimal. Then any interval in this cover is not contained in the union of the others. Otherwise, we would be able to choose another cover with smaller n. We show that any three intervals of this minimal cover do not intersect. Assume the contrary. So there is an $x\in(a_1,b_2)\cap(a_2,b_2)\cap(a_3,b_3)$, where (a_i,b_i), $i=1,2,3$ are intervals of the minimal cover. Then $a_i<x<b_i$ for each i. Without loss of generality, assume that $a_1\leqslant a_2\leqslant a_3$. If $b_2<b_3$ then $(a_2,b_2)\subset(a_1,b_3)=(a_1,b_1)\cap(a_3,b_3)$, which contradicts the minimality of the cover. Similarly, if $b_3\leqslant b_2$ then $(a_3,b_3)\subset(a_1,b_2)=(a_1,b_1)\cap(a_2,b_2)$, a contradiction. This means that the multiplicity of the cover is at most 2. Hence $\sum_{i=1}^n|I_i|\leqslant 2|I|$, where $I_i:=B(x_i,\eta(x_i)$. This together with Lemma 5 implies that

$$\begin{aligned}
|B(I,\delta,K,T)| &= \left|\bigcup_{i=1}^n B(I_i,\delta,K,T)\right| \ \leq\ \textstyle\sum_{i=1}^n|B(I_i,\delta,K,T)|\\
&\leqslant \textstyle\sum_{i=1}^n C_{I_i}\ \max\left(\delta^{1/3},(\delta KT)^{1/9}\right)|I_i|\\
&\leqslant \max_{i=1,\dots,n} C_{I_i}\cdot\max\left(\delta^{1/3},(\delta KT)^{1/9}\right)\ \textstyle\sum_{i=1}^n|I_i|\\
&\leqslant 2\max_{i=1,\dots,n} C_{I_i}\cdot\max\left(\delta^{1/3},(\delta KT)^{1/9}\right)|I|\ ,
\end{aligned}$$

as required. ♠

4.3 Proof of Theorem 7

Define $\mathbf{g}(x):=(g_1(x),g_2(x))$ by setting $g_1(x):=xf'(x)-f(x)$ and $g_2(x):=-f'(x)$. Then $\mathbf{g}\in C^{(2)}$. Also, note that

$$\mathbf{g}'(x)=(xf''(x),\,-f''(x))\ ,\qquad \mathbf{g}''(x)=(f''(x)+xf'''(x),\,-f'''(x))\tag{14}$$

and

$$(g_1'g_2''-g_2'g_1'')(x)=f''(x)^2\ .$$

As $f''(x)\neq 0$ everywhere, Lemma 6 is applicable to this $\mathbf{g}$. In view of the conditions on the theorem,

$$\sup_{x\in I_0}|g_2'(x)|=\sup_{x\in I_0}|f''(x)|\leqslant c_2.\tag{15}$$

Define $\delta_0:=\min\{1,(2^{19}c_2C^9)^{-1}\}$, where C is the constant appearing in Lemma 6. Without loss of generality, assume that $C>1$.

Next, fix an interval $I\subseteq I_0$. By Minkowski's linear forms theorem in the geometry of numbers, for any $x\in I$ and $Q\in\mathbb{N}$ there is a solution $(q,p_1,p_2)\in\mathbb{Z}^3\smallsetminus\{0\}$ to the system

$$\left\{\begin{array}{l}
|qg_1(x)+p_1g_2(x)+p_2|\leqslant\delta_0\psi(Q)\\[2ex]
|qg_1'(x)+p_1g_2'(x)|\leqslant c_2(\delta_0Q\psi(Q))^{-1}\\[2ex]
0\leqslant q\leqslant Q\ .
\end{array}\right.\tag{16}$$

By definition, the set $B(I,\delta,K,T)$ with

$$\delta := \delta_0\psi(Q), \quad K := c_2(\delta_0 Q\psi(Q))^{-1}, \quad T := 2\delta_0 Q \tag{17}$$

consists of points $x \in I$ such that there exists a non-zero integer solution (q,p_1,p_2) to the system (16) with $q \leqslant 2\delta_0 Q$. By Lemma 6, for sufficiently large Q we have that

$$\begin{aligned} |B(I,\delta,K,T)| &\leqslant C\,|I|\,\max\left\{(\delta_0\psi(Q))^{1/3},\ \left(\delta_0\psi(Q)c_2(\delta_0 Q\psi(Q))^{-1}2\delta_0 Q\right)^{1/9}\right\} \\ &= C\,(2c_2\delta_0)^{1/9}|I| \leqslant |I|/4 \ . \end{aligned}$$

Therefore, with δ, K, T given by (17) and Q sufficiently large

$$|\tfrac{3}{4}I \setminus B(I,\delta,K,T)| \geqslant |I|/2\ , \tag{18}$$

where $\frac{3}{4}I$ is the interval I scaled by $\frac{3}{4}$. Notice, that for $x \in \frac{3}{4}I \setminus B(I,\delta,K,T)$ we have that

$$q > 2\delta_0 Q \tag{19}$$

for any solution (q,p_1,p_2) of (16). From now on, assume that $x \in \frac{3}{4}I \setminus B(I,\delta,K,T)$. In view of (14) and the second inequality of (16) we have that

$$|qxf''(x) - p_1 f''(x)| < c_2(\delta_0 Q\psi(Q))^{-1}.$$

This together with (19) and the fact that $|f''(x)| > c_1$, implies that

$$\left|x - \frac{p_1}{q}\right| \leqslant \frac{c_2}{q|f''(x)|\delta_0 Q\psi(Q)} < \frac{c_2}{c_1\delta_0^2 Q^2\psi(Q)} = \frac{C_1}{Q^2\psi(Q)}\ , \tag{20}$$

where $C_1 := \frac{c_2}{c_1\delta_0^2}$. In view of (4) and the fact that $x \in \frac{3}{4}I$, we have that $p_1/q \in I$ for Q is sufficiently large. By Taylor's formula,

$$f\left(\tfrac{p_1}{q}\right) = f(x) + f'(x)\left(\tfrac{p_1}{q} - x\right) + \tfrac{1}{2}f''(\tilde{x})\left(\tfrac{p_1}{q} - x\right)^2$$

for some $\tilde{x}$ between x and p_1/q. Thus $\tilde{x} \in I$. Now the expression on the left hand side of the first inequality of (16) is equal to

$$\begin{aligned} &|q(xf'(x) - f(x)) - p_1 f'(x) + p_2| = |(qx - p_1)f'(x) + p_2 - qf(x)| \\ &= \left|(qx-p_1)f'(x) + p_2 - q\Big(f\left(\tfrac{p_1}{q}\right) - f'(x)\left(\tfrac{p_1}{q} - x\right) - \tfrac{1}{2}f''(\tilde{x})\left(\tfrac{p_1}{q} - x\right)^2\Big)\right| \\ &= \left|p_2 - qf\left(\tfrac{p_1}{q}\right) + \tfrac{q}{2}f''(\tilde{x})(x - \tfrac{p_1}{q})^2\right|\ . \end{aligned}$$

It follows from (4), (15), (16) and (20) that for Q sufficiently large

$$\begin{aligned} \left|qf\left(\tfrac{p_1}{q}\right) - p_2\right| &\leqslant \left|p_2 - qf\left(\frac{p_1}{q}\right) + \frac{q}{2}f''(\tilde{x})(x - \frac{p_1}{q})^2\right| + \left|\frac{q}{2}f''(\tilde{x})(x - \frac{p_1}{q})^2\right| \\ &\leqslant \delta_0\psi(Q) + \frac{Q}{2}c_2\Big(\frac{C_1}{Q^2\psi(Q)}\Big)^2 < 2\delta_0\psi(Q)\ . \end{aligned}$$

This inequality together with (19) implies that

$$\left|f\left(\tfrac{p_1}{q}\right) - \tfrac{p_2}{q}\right| < \tfrac{2\delta_0\psi(Q)}{q} < \tfrac{\psi(Q)}{Q}\ . \tag{21}$$

Thus, for any $x \in \frac{3}{4}I \setminus B(I,\delta,K,T)$ conditions (20) and (21) hold for some $(p_1,p_2)/q$ with $2\delta_0\, q < q \le Q$. Thus, $\mathbf{p}/q := (p_1,p_2)/q \in A_Q(I)$ and moreover, in view of (18) we have that

$$\left|\bigcup_{\mathbf{p}/q \in A_Q(I)}\{x \in I : \left|x - \tfrac{p_1}{q}\right| < \tfrac{C_1}{Q^2\psi(Q)}\}\right| \geqslant |I|/2\ ,$$

for all sufficiently large Q. The statement of the theorem now follows. ♠

5. Proof of Theorem 4

Throughout, ψ is an approximating function with $\lambda_\psi := \liminf_{t\to\infty} \frac{-\log\psi(t)}{\log t} \in (1/2, 1)$. It is readily verified that for any $\varepsilon > 0$

$$\psi(t) \leqslant t^{-\lambda_\psi+\varepsilon} \qquad \text{for all but finitely many } t \in \mathbb{N} \;\;, \tag{22}$$

and that there exists a strictly increasing sequence of natural numbers t_i such that

$$\psi(t_i) \geqslant t_i^{-\lambda_\psi-\varepsilon} \qquad \text{for all } i \in \mathbb{N} \;. \tag{23}$$

The dimension part of Theorem 4 is obtained by considering upper and lower bounds separately.

The upper bound. First notice that since f is continuously differentiable the map $x \mapsto (x, f(x))$ is locally bi-Lipshitz and thus preserves Hausdorff dimension [19, 25]. Hence, we will investigate $\dim \Omega_{f,\psi}$ instead of $\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi)$, where $\Omega_{f,\psi}$ is defined to be the set of $x \in I_0$ such that the system of inequalities

$$\begin{cases} \left|x - \frac{p_1}{q}\right| < \frac{\psi(q)}{q}, \\ \left|f(x) - \frac{p_2}{q}\right| < \frac{\psi(q)}{q} \end{cases} \tag{24}$$

is satisfied for infinitely many $\mathbf{p}/q \in \mathbb{Q}^2$. Furthermore, there is no loss of generality in assuming that $p_1/q \in I_0$ for solutions $\mathbf{p}/q$ of (24).

Next, without loss of generality, we can assume that I_0 is open in $\mathbb{R}$. Notice that the set $B := \{x \in I_0 : |f''(x)| = 0\}$ is closed in I_0. Thus the set $G := I_0 \setminus B := \{x \in I_0 : |f''(x)| \neq 0\}$ is open and a standard argument allows one to write G as a countable union of intervals I_i on which f satisfies (2) with I_0 replaced by I_i. Of course, the constants c_1 and c_2 appearing in (2) will depend on the particular interval I_i. The upper bound result will follow on showing that $\dim \Omega_{f,\psi} \cap I_i \leq d$, since by the conditions imposed on the theorem $\dim B \leq d$ and so

$$\dim \Omega_{f,\psi} \leq \dim \left(B \cup \bigcup_{i=1}^{\infty} \left(\Omega_{f,\psi} \cap I_i\right) \right) \;\leq\; d \;.$$

Without loss of generality, and for the sake of clarity we assume that f satisfies (2) on I_0.

For a point $\mathbf{p}/q \in \mathbb{Q}^2$, denote by $\sigma(\mathbf{p}/q)$ the set of $x \in I_0$ satisfying (24). Trivially, $|\sigma(\mathbf{p}/q)| \leqslant 2\psi(q)/q$. Assume that $\sigma(\mathbf{p}/q) \neq \emptyset$ and let $x \in \sigma(\mathbf{p}/q)$. By the mean value theorem, $f(x) = f(p_1/q) + f'(\tilde{x})(x - p_1/q)$ for some $\tilde{x} \in I_0$. We can assume that f' is bounded on I_0 since f'' is bounded and I_0 is a bounded interval. Suppose $2^t \leqslant q < 2^{t+1}$. By (24),

$$\left|f(\tfrac{p_1}{q}) - \tfrac{p_2}{q}\right| \;\leqslant\; \left|f(x) - \tfrac{p_2}{q}\right| \;+\; \left|f'(\tilde{x})\big(x - \tfrac{p_1}{q}\big)\right| \;\leqslant\; c_4\psi(q)/q \;\leqslant\; c_4\psi(2^t)/2^t$$

where $c_4 > 0$ is a constant. In view of (22), this implies that for any $\varepsilon \in (0,1)$ and t sufficiently large

$$\left|f(\tfrac{p_1}{q}) - \tfrac{p_2}{q}\right| \;\leqslant\; 4c_4 \; 2^{(t+1)(-\lambda_\psi+\varepsilon)}/2^{t+1} \;.$$

By (3), for t sufficiently large the number of $\mathbf{p}/q \in \mathbb{Q}^2$ with $2^t \leqslant q < 2^{t+1}$ and $\sigma(\mathbf{p}/q) \neq \emptyset$ is at most $2^{t(2-\lambda_\psi+3\varepsilon)}$. Therefore, with $\eta := (2 - \lambda_\psi + 4\varepsilon)/(\lambda_\psi + 1 - \varepsilon)$ it follows that

$$\begin{aligned} \sum_{\mathbf{p}/q\in\mathbb{Q}^2 \,:\, \sigma(\mathbf{p}/q)\neq\emptyset} |\sigma(\mathbf{p}/q)|^\eta \;&=\; \sum_{t=0}^{\infty} \; \sum_{\mathbf{p}/q\in\mathbb{Q}^2,\, \sigma(\mathbf{p}/q)\neq\emptyset,\, 2^t\leqslant q<2^{t+1}} |\sigma(\mathbf{p}/q)|^\eta \\ &\leqslant\; c' \sum_{t=0}^{\infty} 2^{t(-\lambda_\psi-1+\varepsilon)\eta} \cdot 2^{t(-\lambda_\psi+2+3\varepsilon)} \;=\; c' \sum_{t=0}^{\infty} 2^{-t\varepsilon} \;<\; \infty \;\;, \end{aligned}$$

where c' is a positive constant. By the Hausdorff–Cantelli Lemma [11, p. 68], $\dim \Omega_{f,\psi} \leqslant \eta$. As $\varepsilon > 0$ is arbitrary,

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi) \;=\; \dim \Omega_{f,\lambda_\psi} \;\leqslant\; d := \frac{2-\lambda_\psi}{\lambda_\psi+1} \quad . \tag{25}$$

The lower bound (modulo Theorem 3). This is a simple consequence of Theorem 3 and so all that is required is that the curve is non-degenerate at a single point.

Fix $\epsilon > 0$ such that $\lambda_\psi + \epsilon < 1$ and let

$$s \;:=\; \frac{2-\lambda_\psi-\epsilon}{1+\lambda_\psi+\epsilon} \;<\; d \quad .$$

Clearly, $s \in (1/2, 1)$. In view of (23) and the fact that ψ is decreasing, there exists a strictly increasing sequence m_i of natural numbers such that

$$\psi(2^{m_i}) \;\geq\; 2^{-(\lambda_\psi+\epsilon)}\; 2^{-m_i(\lambda_\psi+\epsilon)} \quad . \tag{26}$$

To see that this is the case, notice that for each t_i there exists a natural number m_i such that $2^{m_i} < t_i \leq 2^{m_i+1}$. It follows that $\psi(2^{m_i}) \geq \psi(t_i) \geq t_i^{-(\lambda_\psi+\epsilon)} \geq 2^{-(m_i+1)(\lambda_\psi+\epsilon)}$ and to ensure that $m_{i-1} < m_i$ simply choose a suitable subsequence. By (26) and the fact that ψ is decreasing, we obtain that

$$\begin{aligned}
\sum_{h=1}^{\infty} h^{1-s}\psi(h)^{s+1} \;&=\; \sum_{t=1}^{\infty}\; \sum_{2^{t-1}\leqslant h<2^t} h^{1-s}\,\psi(h)^{s+1} \;\gg\; \sum_{t=1}^{\infty} 2^{t(2-s)}\;\psi(2^t)^{s+1} \\
&\gg\; \sum_{i=1}^{\infty} 2^{m_i(2-s)}\; 2^{-m_i(\lambda_\psi+\epsilon)(s+1)} \;=\; \infty \quad .
\end{aligned}$$

Hence, Theorem 3 implies that $\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(\psi)) = \infty$ and so $\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi) \geq s$. As $\epsilon > 0$ can be made arbitrarily small, we obtain the required lower bound result.

The Hausdorff measure part of Theorem 4 is a direct consequence of Theorem 3. Simply note that if $\limsup_{h\to\infty} h^{2-d}\psi(h)^{d+1} > 0$ then $\sum h^{1-d}\psi(h)^{d+1} = \infty$ and also that if $\lambda_\psi \in (1/2, 1)$ then $d \in (1/2, 1)$. The latter is obvious. The former follows by first observing that if $\limsup_{h\to\infty} h^{2-d}\psi(h)^{d+1} > 0$, then there exists a strictly increasing sequence m_i of natural numbers such that $2^{m_i(2-d)}\;\psi(2^{m_i})^{d+1} \geq \eta > 0$. It follows that

$$\sum_{h=1}^{\infty} h^{1-d}\psi(h)^{d+1} \;\gg\; \sum_{t=1}^{\infty} 2^{t(2-d)}\;\psi(2^t)^{d+1} \;\gg\; \sum_{i=1}^{\infty}\; 2^{m_i(2-d)}\psi(2^{m_i})^{d+1} \;=\; \infty \quad ,$$

as required. ♠

Alternatively, the lower bound result for $\dim \mathcal{C}_f \cap \mathcal{S}_2(\psi)$ and the Hausdorff measure part of Theorem 4 can be deduced independently of Theorem 3 via Corollary 4. Note that the upper bound result is complete. It has been established without reference to any other result.

6. Proof of Theorem 1

As $\mathcal{C} := \mathcal{C}_f$ is non-degenerate almost everywhere, we can restrict our attention to a sufficiently small patch of $\mathcal{C}$, which can be written as $\{(x, f(x)) : x \in I\}$ where I is a sub-interval of I_0

and f satisfies (2) with I_0 replaced by I. Clearly, Theorem 4 is applicable to f restricted to I. However, without loss of generality and for clarity, we assume that f satisfies (2) on I_0.

Throughout this section, ψ will be an approximating function such that

$$\sum_{h=1}^{\infty} \psi^2(h) \;=\; \infty \;\; . \tag{27}$$

Step 1. We show that there is no loss of generality in assuming that

$$\psi(h) \leqslant h^{-1/2} \qquad \text{for all } h. \tag{28}$$

Define the auxiliary function $\tilde{\psi} : h \to \tilde{\psi}(h) := \min\{h^{-1/2}, \psi(h)\}$. Clearly $\tilde{\psi}$ is an approximating function. First we show that

$$\sum_{h=1}^{\infty} \tilde{\psi}^2(h) \;=\; \infty \; . \tag{29}$$

Assume that (29) is false. Then using the fact that $\tilde{\psi}$ is decreasing, we obtain

$$0 \underset{l\to\infty}{\longleftarrow} \sum_{[l/2]\leqslant h<l} \tilde{\psi}^2(h) \geqslant \sum_{[l/2]\leqslant h<l} \tilde{\psi}^2(l) \geqslant \tilde{\psi}^2(l) l/3 \; .$$

Thus, $\tilde{\psi}(l) l^{1/2} \to 0$ as $l \to \infty$. It follows that $\tilde{\psi}(l) = o(l^{-1/2})$ and so $\tilde{\psi}(l) = \psi(l)$ for all but finitely many l. This together with (27) implies (29), a contradiction.

By definition, $\mathcal{S}_2(\tilde{\psi}) \subseteq \mathcal{S}_2(\psi)$. Thus to complete the proof of Theorem 1 it suffices to prove the result with ψ replaced by $\tilde{\psi}$. Hence, without loss of generality, (28) can be assumed.

Step 2. We show that there is no loss of generality in assuming that

$$\psi(h) \geqslant h^{-2/3} \text{ for all } h. \tag{30}$$

To this end, define $\hat{\psi} : h \to \hat{\psi}(h) := \max\{\psi(h), h^{-2/3}\}$. It is readily verified that

$$\mathcal{S}_2(\hat{\psi}) = \mathcal{S}_2(\psi) \cup \mathcal{S}_2(h \mapsto h^{-2/3}).$$

By the upper bound result established in §5, we have that $\dim \mathcal{C}_f \cap \mathcal{S}_2(h \mapsto h^{-2/3}) \leq 4/5 < 1$. It follows from the definition of Hausdorff dimension that $\mathcal{H}^1(\mathcal{C}_f \cap \mathcal{S}_2(h \mapsto h^{-2/3})) = 0$; i.e. for almost all $x \in I_0$

$$(x, f(x)) \;\notin\; \mathcal{S}_2(h \mapsto h^{-2/3}) \;\; .$$

Thus,

$$\left|\{x \in I_0 : (x, f(x)) \in \mathcal{S}_2(\hat{\psi})\}\right| \;=\; \left|\{x \in I_0 : (x, f(x)) \in \mathcal{S}_2(\psi)\}\right|$$

and to complete the proof of Theorem 1 it suffices to prove that the set on the left has full measure. Hence, without loss of generality, (30) can be assumed.

Step 3. In view of Steps 1 and 2 above, the function ψ satisfies (4) and Corollary 5 is applicable to ψ. By (27) and the fact that ψ is decreasing, we obtain that

$$\infty \;=\; \sum_{t=0}^{\infty} \sum_{2^t\leqslant h<2^{t+1}} \psi(h)^2 \;\leqslant\; \sum_{t=0}^{\infty} \sum_{2^t\leqslant h<2^{t+1}} \psi(2^t)^2 \;=\; \sum_{t=0}^{\infty} 2^t \psi(2^t)^2 \;\; .$$

Hence

$$\sum_{t=0}^{\infty} 2^t \psi(2^t)^2 \; = \; \infty.$$

Next, define the increasing function $u : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$u(h) \; := \; \sum_{t=0}^{[h]} 2^t \; \psi(2^t)^2 \quad .$$

Trivially, $\lim_{t\to\infty} u(t) = \infty$. Let $a_t = 2^t\psi(2^t)^2$ and $u_t = u(t)$. Fix $k \in \mathbb{N}$. Then

$$\sum_{t=k}^{m} \frac{a_t}{u_t} \; \geqslant \; \sum_{t=k}^{m} \frac{a_t}{u_m} \; = \; \frac{u_m - u_{k-1}}{u_m} \to 1 \;\; \text{as} \;\; m \to \infty.$$

Hence

$$\sum_{t=k}^{\infty} \frac{a_t}{u_t} \; \geqslant \; 1 \;\; \text{for all } k.$$

This implies that the sum $\sum_{t=1}^{\infty} a_t/u_t$ diverges; i.e.

$$\sum_{t=0}^{\infty} \frac{2^t \; \psi(2^t)^2}{u(t)} \; = \; \infty \quad . \tag{31}$$

Now let $\Psi(t) = \Phi(t) := \psi(t)/t$ and $\rho(t) := u(\log_2 t)/(t^2\psi(t))$. By Corollary 5, $(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta)$ is locally ubiquitous relative to ρ, where the function β is given by (11). In view of (31),

$$\sum_{t=1}^{\infty} \frac{\Psi(2^t)}{\rho(2^t)} \; := \; \sum_{t=1}^{\infty} \frac{\frac{\psi(2^t)}{2^t}}{\frac{u(t)}{2^{2t}\psi(2^t)}} \; = \; \sum_{t=1}^{\infty} \frac{2^t \; \psi(2^t)^2}{u(t)} \; = \; \infty \quad .$$

Since ψ is decreasing,

$$\Psi(2^{t+1}) \; := \; \frac{\psi(2^{t+1})}{2^{t+1}} \; \leqslant \; \frac{1}{2} \cdot \frac{\psi(2^t)}{2^t} \; := \; \frac{1}{2} \; \Psi(2^t) \quad .$$

Thus the conditions of Lemma 3 are satisfied and it follows that the set $\Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta, \Psi)$ has full measure. By definition, the set $\Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta, \Psi)$ consists of points $x \in I_0$ such that the system of inequalities

$$\begin{cases} \left|x - \frac{p_1}{q}\right| < \Psi(q) = \frac{\psi(q)}{q} < \frac{2\psi(q)}{q} \\ \left|f(x) - \frac{p_2}{q}\right| < \Psi(q) + \Phi(q) = \frac{\psi(q)}{q} + \frac{\psi(q)}{q} \leqslant \frac{2\psi(q)}{q} \end{cases}$$

is satisfied for infinitely many $\mathbf{p}/q \in \mathbb{Q}^2$. Obviously, for $x \in \Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta, \Psi)$ the point $(x, f(x))$ is in $\mathcal{S}_2(2\psi)$. In order to complete the proof of Theorem 1, simply apply what has already been proved to the approximating function $\frac{1}{2}\psi$. ♠

7. Proof of Theorem 3

We are assuming that there exists at least one point on the curve $\mathcal{C}_f$ which is non-degenerate. Thus, there exists a sufficiently small patch of $\mathcal{C}_f$, which can be written as $\{(x, f(x)) : x \in I\}$ where I is a sub-interval of I_0 and f satisfies (2) with I_0 replaced by I. Clearly, Theorems 1 and 4 are applicable to f restricted to I. However, without loss of generality and for the sake of clarity, we assume that f satisfies (2) on I_0.

Throughout this section, $s \in (1/2, 1)$ and ψ will be an approximating function such that

$$\sum_{h=1}^{\infty} h^{1-s}\, \psi(h)^{s+1} \;=\; \infty \;\; . \tag{32}$$

Step 1. We show that there is no loss of generality in assuming that

$$\lim_{t\to\infty} \psi(t) \;=\; 0 \;\; . \tag{33}$$

Suppose on the contrary that $\limsup_{t\to\infty} \psi(t) > 0$. Then for any $s \leq 1$, we have that (32) holds. In particular, $\sum_{h=1}^{\infty} \psi^2(h) = \infty$ and so Theorem 1 implies that $\mathcal{H}^1(\mathcal{C}_f \cap \mathcal{S}_2(\psi)) > 0$. It follows that $\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(\psi)) = \infty$ for any $s < 1$. Hence, (33) can be assumed.

Step 2. Since $s > 1/2$, there exists $\eta > 0$ such that $s = \frac{1}{2} + \eta$. We show that there is no loss of generality in assuming that for all $h \in \mathbb{N}$,

$$\psi(h) \geqslant h^{-(1-\epsilon)} \qquad \text{where} \qquad 0 < \epsilon < 4\eta/(3+2\eta) \;\; . \tag{34}$$

To this end, define $\hat{\psi} : h \to \hat{\psi}(h) := \max\{\psi(h), h^{-(1-\epsilon)}\}$. It is readily verified that

$$\mathcal{S}_2(\hat{\psi}) = \mathcal{S}_2(\psi) \cup \mathcal{S}_2(h \mapsto h^{-(1-\epsilon)}).$$

By the upper bound result established in §5, $\dim \mathcal{C}_f \cap \mathcal{S}_2(h \mapsto h^{-(1-\epsilon)}) \leq (1+\epsilon)/(2-\epsilon) < s$ and so $\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(h \mapsto h^{-(1-\epsilon)})) = 0$. Thus,

$$\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(\hat{\psi})) = \mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(\psi))$$

and to complete the proof of Theorem 3 it suffices to prove that $\mathcal{H}^s(\mathcal{C}_f \cap \mathcal{S}_2(\hat{\psi})) = \infty$. Hence, without loss of generality, (34) can be assumed.

Step 3. In view of Steps 1 and 2 above, the function ψ satisfies (4) and Corollary 5 is applicable to ψ. In view of (32), we can find a strictly increasing sequence of positive integers $\{h_i\}_{i\in\mathbb{N}}$ such that

$$\sum_{h_{i-1}<h\leq\, h_i} h^{1-s}\, \psi(h)^{s+1} \;>\; 1 \qquad (h_0 := 0)\;.$$

Now simply define the increasing function u as follows:

$$u\,:\,h\;\to\;u(h)\;:=\;i \qquad \text{for} \qquad h_{i-1} < h \leq h_i \;\; .$$

Note that

$$\sum_{h=1}^{\infty} h^{1-s}\, \psi(h)^{s+1}\, u(h)^{-1} \;=\; \sum_{i=1}^{\infty} \;\sum_{h_{i-1}<h\leq\, h_i} h^{1-s}\, \psi(h)^{s+1}\, u(h)^{-1} \;>\; \sum_{i=1}^{\infty} i^{-1} \;=\; \infty\;.$$

In particular, since the function ψ^{s+1}/u is decreasing we have that

$$\infty \;=\; \sum_{t=0}^{\infty} \sum_{2^t \le h < 2^{t+1}} h^{1-s}\,\psi(h)^{s+1}\,u(h)^{-1} \;\le\; 2^{2-s} \sum_{t=0}^{\infty} 2^{t(2-s)}\;\psi(2^t)^{1+s}\;u(2^t)^{-1} \;\;.$$

Hence

$$\sum_{t=0}^{\infty} 2^{t(2-s)}\;\psi(2^t)^{1+s}\;u(2^t)^{-1} \;=\; \infty \;\;. \tag{35}$$

Now let $\Psi(t) = \Phi(t) := \psi(t)/t$ and $\rho(t) := u(t)/(t^2\psi(t))$. By Corollary 5, $(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta)$ is locally ubiquitous relative to ρ, where the function β is given by (11). In view of (35),

$$\sum_{t=1}^{\infty} \frac{\Psi(2^t)^s}{\rho(2^t)} \;:=\; \sum_{t=1}^{\infty} 2^{t(2-s)}\;\psi(2^t)^{1+s}\;u(2^t)^{-1} \;=\; \infty \;\;.$$

Since ψ is decreasing, $\Psi(2^{t+1}) \leqslant \frac{1}{2}\;\Psi(2^t)$. Thus the conditions of Lemma 4 are satisfied and it follows that the set $\Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta, \Psi)$ is of infinite s-dimensional Hausdorff measure. The statement of Theorem 3 now follows on repeating verbatim the argument given towards the end of the proof of Theorem 1. ♠

8. Various Generalizations

8.1 Theorem 3 for general Hausdorff measures

A *dimension function* $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, continuous function such that $h(r) \to 0$ as $r \to 0$. Let $\mathcal{H}^h$ denote the Hausdorff h–measure with respect to the dimension function h. With reference to §2.2, this measure is defined by replacing $l^s(C_i)$ in the definition of s–dimensional Hausdorff measure $\mathcal{H}^s$ by the quantity $h(l(C_i))$ – see [19, 25] for further details. In the case that $h : r \to r^s$ $(s \ge 0)$, the measure $\mathcal{H}^h$ is precisely $\mathcal{H}^s$. For Hausdorff h–measures, Jarník's Theorem can be stated as follows – see [6, §1.2 and §12.1].

Jarník's General Theorem (1931). *Let h be a dimension function such that $r^{-1}\,h(r) \to \infty$ as $r \to 0$ and $r^{-1}\,h(r)$ is decreasing. Let ψ be an approximating function. Then*

$$\mathcal{H}^h\left(\mathcal{S}_n(\psi)\right) = \begin{cases} 0 & \text{if} \quad \sum_{r=1}^{\infty} r^n\,h\left(\psi(r)/r\right) \;\; < \infty \\[2ex] \infty & \text{if} \quad \sum_{r=1}^{\infty} r^n\,h\left(\psi(r)/r\right) \;\; = \infty \end{cases} \;\;.$$

In the most simplest form, the following statement is the Hausdorff h–measure analogue of Theorem 3.

Theorem 8 *Let $f \in C^{(3)}(I_0)$, where I_0 is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Assume that there exists at least one point on the curve $\mathcal{C}_f$ which is non-degenerate. Let ψ be an approximation function and let h be a dimension function such that $r^{-1}\,h(r) \to \infty$ as $r \to 0$, $r^{-1}\,h(r)$ is decreasing and $r^{-(1/2+\epsilon)}\,h(r) \to 0$ as $r \to 0$ for $\epsilon > 0$ sufficiently small. Furthermore, suppose h satisfies the following growth condition: there exist constants $r_0, \lambda_1, \lambda_2 \in (0, 1)$ such that $h(\lambda_1 r) \le \lambda_2\,h(r)$ for $r \in (0, r_0)$. Then,*

$$\mathcal{H}^h((\mathcal{C}_f \cap \mathcal{S}_2(\psi))) \;=\; \infty \qquad \text{if} \qquad \sum_{r=1}^{\infty} r\;\psi(r)\;h(\psi(r)/r) \;=\; \infty \;\;.$$

Apart from the growth condition imposed on the dimension function, Theorem 8 is the precise analogue of the divergent part of Jarník's General Theorem for simultaneous Diophantine approximation on planar curves. The growth condition is not particularly restrictive and can be completely removed from the statement of the theorem in the case that $G := \limsup_{r\to\infty} h(\psi(r)/r)\,\psi(r)\,r^2 > 0$. Furthermore, when $G = 0$, if there exists a constant $\lambda \in (0,1)$ such that $\psi(2r) > \lambda\psi(r)$ for all sufficiently large r then the growth condition on h is again redundant.

Notice that if $h : r \to r^s$ ($s \geq 0$), then the growth condition is trivially satisfied and the above theorem reduces to Theorem 3.

Remark on the proof of Theorem 8. The first step is to obtain the analogue of Lemma 4 for general Hausdorff measures. This is easy, it follows directly from Corollary 3 of [6, §5] in the same way that Lemma 4 is deduced from Lemma 2. The proof of Theorem 8 then follows on modifying the argument used to prove Theorem 3 in §7. Note that Corollary 5, the important local ubiquity statement which gives the 'optimal' ubiquitous function ρ, is independent of any dimension function. The following useful fact concerning dimension functions is also required: if f and g are two dimension functions such that $f(r)/g(r) \to 0$ as $r \to 0$, then $\mathcal{H}^f(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$. We leave the details to the reader.

8.2 The Multiplicative Problems/Theory

Given an approximating function ψ, a point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *simultaneously multiplicatively ψ–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\prod_{1\leqslant i\leqslant n} \|qy_i\| \;<\; \psi(q)^n .$$

Thus, the maximum in the definition of simultaneously ψ–approximable is replaced by the product. Denote by $\mathcal{S}_n^{\mathrm{M}}(\psi)$ the set of simultaneously multiplicatively ψ–approximable points. Trivially, we have that

$$\mathcal{S}_n(\psi) \;\subset\; \mathcal{S}_n^{\mathrm{M}}(\psi) \ .$$

The two fundamental problems posed in the introduction can obviously be reinstated for the multiplicative setup. In a forthcoming paper [7], the first and third authors develop the simultaneous multiplicative theory for metric Diophantine approximation on planar curves. As an illustration of the type of results established in [7], we mention the following analogue of Theorem 4. With the same notation and hypotheses of Theorem 4,

$$\dim \mathcal{C}_f \cap \mathcal{S}_2^{\mathrm{M}}(\psi) \;=\; \frac{2-\lambda_\psi}{\lambda_\psi+1} \quad .$$

Appendix I: Proof of Ubiquity Lemmas

A. Ubiquity with respect to sequences

In this appendix we prove the ubiquity lemmas of §3.1 which are the key towards establishing the divergent results of this paper. It is both more convenient and no more difficult to consider a slightly more general setup in which the sequence $\{2^n\}$ is replaced by an arbitrary increasing sequence u. Apart from this the setup remains unchanged.

Let $u := \{u_n\}$ be a positive increasing sequence such that $\lim_{n\to\infty} u_n = \infty$ and let $J^u(n) := \{\alpha \in J : \beta_\alpha \leqslant u_n\}$. Given a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \rho(t) = 0$, let

$$\Delta^u(\rho, n) := \bigcup_{\alpha \in J^u(n)} B(R_\alpha, \rho(u_n)) \ .$$

Definition 4 *Suppose there exists a function ρ, a sequence u and an absolute constant $\kappa > 0$ such that for any interval $I \subseteq I_0$*

$$|\Delta^u(\rho, n) \cap I| \ \geqslant \ \kappa \ |I| \qquad \text{for } n \geq n_o(I). \tag{36}$$

Then the pair $(\mathcal{R}, \beta)$ is said to be locally ubiquitous in I_0 relative to (ρ, u).

Notice that any subsequence v of u will also do in the above definition; i.e. (36) is satisfied for $\Delta^v(\rho, n)$. In order to state the consequences of this slightly more general definition of ubiquity we introduce the following notion. Given a sequence u, a function h will be said to be **u-regular** if there exists a strictly positive constant $\lambda < 1$ such that for n sufficiently large

$$h(u_{n+1}) \leq \lambda \, h(u_n) \ . \tag{37}$$

The constant λ is independent of n but may depend on u. Clearly, if h is u-regular then it is v–regular for any subsequence v of u.

Theorem 9 *Suppose that $(\mathcal{R}, \beta)$ is locally ubiquitous in I_0 relative to (ρ, u) and let Ψ be an approximating function such that Ψ is u-regular. Then*

$$|\Lambda(\mathcal{R}, \beta, \Psi)| \ = \ \text{FULL} \ := \ |I_0| \qquad \text{if} \qquad \sum_{n=1}^{\infty} \frac{\Psi(u_n)}{\rho(u_n)} = \infty \ .$$

Theorem 10 *Suppose that $(\mathcal{R}, \beta)$ is locally ubiquitous in I_0 relative to (ρ, u) and let Ψ be an approximation function. Let $s \in (0, 1)$ and let*

$$G \ := \ \limsup_{t\to\infty} \ \frac{\Psi(u_n)^s}{\rho(u_n)} \ . \tag{38}$$

(i) *Suppose that $G = 0$ and that Ψ is u-regular. Then,*

$$\mathcal{H}^s(\Lambda(\mathcal{R}, \beta, \Psi)) \ = \ \infty \qquad \text{if} \qquad \sum_{n=1}^{\infty} \frac{\Psi(u_n)^s}{\rho(u_n)} \ = \ \infty \ .$$

(ii) *Suppose that $G > 0$. Then, $\mathcal{H}^s(\Lambda(\mathcal{R}, \beta, \Psi)) \ = \ \infty$.*

In the case $u = \{u_n\} := \{2^n\}$, these theorems clearly reduce to Lemmas 1 and 2 of §3.1.

A.1. Prerequisites

A.1.1. The Mass Distribution Principle and a covering lemma

A general and classical method for obtaining a lower bound for the s–dimensional Hausdorff measure of an arbitrary set F is the following mass distribution principle.

Lemma 7 (Mass Distribution Principle) *Let μ be a probability measure supported on a subset F of $\mathbb{R}$. Suppose there are positive constants c and r_o such that $\mu(B) \leqslant c\,r^s$ for any ball B with radius $r \le r_o$. Then, $\mathcal{H}^s(F) \ge 1/c$.*

The following covering result will be used at various stages during the proof of our theorems.

Lemma 8 (Covering lemma) *Let $\mathcal{B}$ be a finite collection of balls in $\mathbb{R}$ with common radius $r > 0$. Then there exists a disjoint sub-collection $\{B_i\}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \ \subset \ \bigcup_i \ 3B_i \ .$$

These lemmas are easily established and relatively standard, see [19, 25] and [6, §7].

A.1.2. Positive and full measure sets

Proposition 1 *Let $E \subset \mathbb{R}$ be a measurable set and let $I_0 \subset \mathbb{R}$ be an interval. Assume that there is a constant $c > 0$ such that for any finite interval $I \subset I_0$ we have that $|E \cap I| \geqslant c\,|I|$. Then E has full measure in I_0, i.e. $|I_0 \setminus E| = 0$.*

For the proof see [3, Lemma 2] and [6, §8].

Proposition 2 *Let $E_n \in \mathbb{R}$ be a sequence of measurable sets such that $\cup_{n=1}^{\infty} E_n$ is bounded and $\sum_{n=1}^{\infty} |E_n| = \infty$. Then*

$$|\limsup_{n\to\infty} E_n| \ \ge \ \limsup_{Q\to\infty} \frac{\left(\sum_{s=1}^{Q} |E_s|\right)^2}{\sum_{s,t=1}^{Q} |E_s \cap E_t|} \qquad .$$

This result is a generalization of the divergent part of the standard Borel–Cantelli lemma. For the proof see Lemma 5 in [29].

Proposition 3 *Let $E_n \in \mathbb{R}$ be a sequence of measurable sets and let $I \subset \mathbb{R}$ be a bounded interval. Suppose there exists a constant $c > 0$ such that* $\limsup_{n\to\infty} |I \cap E_n| \geqslant c\,|I|$. *Then,* $|I \cap \limsup_{n\to\infty} E_n| \ \geqslant \ c^2\,|I|$.

Proof of Proposition 3. For any $0 < \varepsilon < c$, there is a subsequence E_{n_i} with n_i strictly increasing such that $|I \cap E_{n_i}) \geqslant (c-\varepsilon)|I|$. Clearly $\big(\sum_{i=1}^{N} |I \cap E_{n_i}|\big)^2 \geqslant \big(\sum_{i=1}^{N} (c-\varepsilon)\,|I|\big)^2 = (c-\varepsilon)^2 N^2\,|I|^2$ and $\sum_{n,m=1}^{N} |I \cap E_n \cap E_m| \leqslant \sum_{m,n=1}^{N} |I| = |I|\,N^2$. Also notice that $\sum_{i=1}^{\infty} |I \cap E_{n_i}| \ge |I|\,\sum_{i=1}^{\infty} (c-\varepsilon) = \infty$. Thus on applying Proposition 2 and observing that $I \cap \limsup_{n\to\infty} E_n \supseteq I \cap \limsup_{i\to\infty} E_{n_i}$ we have that

$$\left| I \cap \limsup_{n\to\infty} E_n \right| \geqslant \limsup_{N\to\infty} \frac{(c-\varepsilon)^2 N^2 |I|^2}{|I| N^2} = (c-\varepsilon)^2\,|I| \ .$$

As $\varepsilon > 0$ is arbitrary, this completes the proof of the proposition. ♠

B. Proof of Theorem 9

Let B be an arbitrary ball in I_0 and let $r(B)$ denote its radius. In view of Proposition 1, the aim is to show that

$$|\Lambda(\mathcal{R},\beta,\Psi)\cap B| \ \geqslant\ |B|/C\ , \tag{39}$$

where $C>0$ is a constant independent of B.

B.1. The subset $A(\Psi,B)$ of $\Lambda(\mathcal{R},\beta,\Psi)\cap B$

Consider the collection of balls $\{B(R_\alpha,2\rho(u_n)) : \alpha\in J^u(n)\}$. By the covering lemma, there exists a disjoint sub-collection $\{B(R_\alpha,2\rho(u_n)) : \alpha\in G^u(n)\}$, where $G^u(n)$ is a subset of $J^u(n)$, such that

$$\overset{\circ}{\bigcup_{\alpha\in G^u(n)}} B(R_\alpha,\rho(u_n))\ \subset\ \Delta^u(\rho,n)\ \subset\ \bigcup_{\alpha\in G^u(n)} B(R_\alpha,6\rho(u_n))\ . \tag{40}$$

The left hand side follows from the fact that the balls $B(R_\alpha,2\rho(u_n))$ with $\alpha\in G^u(n)$ are disjoint. Choose n sufficiently large so that $24\rho(u_n)<r(B)$ (by definition, $\rho(u_n)\to 0$ as $n\to\infty$) and let

$$G^u_B(n)\ :=\ \left\{\alpha\in G^u(n)\ :\ R_\alpha\in \tfrac{1}{2}\,B\right\}\quad .$$

In view of (40),

$$\overset{\circ}{\bigcup_{\alpha\in G^u_B(n)}} B(R_\alpha,\rho(u_n))\ \subset\ \Delta^u(\rho,n)\ \cap\ B \tag{41}$$

and

$$\bigcup_{\alpha\in G^u_B(n)} B(R_\alpha,6\rho(u_n)))\ \supset\ \Delta^u(\rho,n)\ \cap\ \tfrac{1}{4}\,B\quad .$$

We now estimate the cardinality of $G^u_B(n)$. By (36), for n sufficiently large

$$\#G^u_B(n)\ \rho(u_n)\ \gg\ |{\textstyle\bigcup_{\alpha\in G^u_B(n)}} B(R_\alpha,6\rho(u_n))|\ \geq\ |\Delta^u(\rho,n)\cap\tfrac{1}{4}\,B|\ \ \gg\ \kappa\,|B|\ .$$

On the other hand, $|B|\geqslant\big|\overset{\circ}{\bigcup}_{\alpha\in G^u_B(n)}B(R_\alpha,\rho(u_n))\big|\gg \#G^u_B(n)\,\rho(u_n)$. The upshot is that

$$\#G^u_B(n)\ \asymp\ \frac{|B|}{\rho(u_n)}\quad . \tag{42}$$

Suppose for the moment that for some sufficiently large $n\in\mathbb{N}$ we have that $\Psi(u_n)\geq\rho(u_n)$. Then (36) implies that $|\Delta^u(\Psi,n)\cap B|\ \geq\ |\Delta^u(\rho,n)\cap B|\ \geq\ \kappa\,|B|$. Thus, if $\Psi(u_n)\geq\rho(u_n)$ for infinitely many $n\in\mathbb{N}$, Proposition 3 implies (39) and we are done. Hence, without loss of generality, we can assume that for n sufficiently large

$$\rho(u_n)>\ \Psi(u_n)\ . \tag{43}$$

Now let

$$A_n(\Psi,B)\ :=\ \overset{\circ}{\bigcup_{\alpha\in G^u_B(n)}} B(R_\alpha,\Psi(u_n))\quad .$$

The disjointness is a consequence (43). Indeed, for $\alpha \in G^u_B(n)$ the balls $B(R_\alpha, 2\rho(u_n))$ are disjoint. Therefore, $|A_n(\Psi, B)| \asymp \Psi(u_n)\ \#G^u_B(n)$ and in view of (42) we have that

$$|A_n(\Psi, B)| \ \asymp\ |B| \ \times\ \frac{\Psi(u_n)}{\rho(u_n)}\ . \tag{44}$$

Finally, let

$$A(\Psi, B) \ :=\ \limsup_{n\to\infty} A_n(\Psi, B) \ :=\ \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n(\Psi, B)\,.$$

By construction, we have $A_n(\Psi, B) \subset \Delta^u(\Psi, n) \cap B$ and it follows that $A(\Psi, B) \setminus \mathcal{R}$ is a subset of $\Lambda(\mathcal{R}, \beta, \Psi) \cap B$. Now in view of (39) and the fact that $\mathcal{R}$ is countable and therefore of measure zero, the proof of Theorem 9 will be completed on showing that

$$|A(\Psi, B)| = |A(\Psi, B) \cap B| \ \geqslant\ m(B)/C. \tag{45}$$

Notice that (44) together with the divergent sum hypothesis of the theorem implies that

$$\sum_{n=1}^{\infty} |A_n(\Psi, B)| = \infty\ . \tag{46}$$

In view of Proposition 2, this together with the following quasi-independence on average result implies (45) and thereby completes the proof of Theorem 9.

Lemma 9 (Quasi–independence on average) *There exists a constant $C > 1$ such that for Q sufficiently large,*

$$\sum_{s,t=1}^{Q} |A_s(\Psi, B) \cap A_t(\Psi, B)| \ \leq\ \frac{C}{|B|} \left(\sum_{s=1}^{Q} |A_s(\Psi, B)| \right)^2 .$$

Proof of Lemma 9. Throughout, write $A_t(\Psi)$ for $A_t(\Psi, B)$. Also, let $s < t$ and note that

$$|A_s(\Psi) \cap A_t(\Psi)| \ \ =\ \sum_{\alpha \in G^u_{\mathrm{B}}(s)} |\, B(R_\alpha, \Psi(u_s)) \cap A_t(\Psi)\,|\ . \tag{47}$$

Let $B_s(\Psi)$ denote a generic ball $B(R_\alpha, \Psi(u_s))$ with $\alpha \in G^u_{\mathrm{B}}(s)$. We now obtain an upper bound for $|B_s(\Psi) \cap A_t(\Psi)|$. Trivially,

$$\begin{aligned} |B_s(\Psi) \cap A_t(\Psi)| \ \ &:=\ \ |B_s(\Psi)\ \cap\ \mathring{\bigcup}_{\alpha \in G^u_{\mathrm{B}}(t)}\ B(R_\alpha, \Psi(u_t))| \\ &=\ \sum_{\alpha \in G^u_{\mathrm{B}}(t)} |B_s(\Psi) \cap B(R_\alpha, \Psi(u_t))| \ \ll\ N(t, s)\ \Psi(u_t) \end{aligned} \tag{48}$$

where $N(t, s) := \#\{\alpha \in G^u_{\mathrm{B}}(t) : B_s(\Psi) \cap B(R_\alpha, \Psi(u_t)) \neq \emptyset\}$. We proceed by considering two cases.

Case (i): $t > s$ such that $\Psi(u_s) < \rho(u_t)$. On using the fact that the balls $B(R_\alpha, 2\rho(u_t))$ with $\alpha \in G^u_{\mathrm{B}}(t)$ are disjoint, it is easily verified that $N(t, s) \leq 1$. This together with (42), (47) and (48) implies that

$$|A_s(\Psi) \cap A_t(\Psi)| \ \ll\ \#G^u_{\mathrm{B}}(s)\ \Psi(u_t) \ \ll\ |B| \ \times\ \frac{\Psi(u_t)}{\rho(u_s)}\ .$$

Case (ii): $t > s$ such that $\Psi(u_s) \geqslant \rho(u_t)$. First note that if $B_s(\Psi) \cap B(R_\alpha, \rho(u_t)) \neq \emptyset$, then $B(R_\alpha, \Psi(u_t)) \subset 3B_s(\Psi)$. The balls $B(R_\alpha, \rho(u_t))$ with $\alpha \in G^u_{\rm B}(t)$ are disjoint and so $N(t,s) \ll \Psi(u_s)/\rho(u_t)$. It now follows, via (42), (44), (47) and (48), that

$$|A_s(\Psi) \cap A_t(\Psi)| \ \ll\ \frac{1}{|B|}\, |A_s(\Psi)|\; |A_t(\Psi)|\ .$$

The upshot of these two cases, is that for Q sufficiently large

$$\begin{aligned}
\sum_{s,t=1}^{Q} |A_s(\Psi) \cap A_t(\Psi)| = &\sum_{s=1}^{Q} |A_s(\Psi)| \ +\ 2\sum_{s=1}^{Q-1} \sum_{\substack{s+1\leq t\leq Q\\ \text{case(i)}}} |A_s(\Psi) \cap A_t(\Psi)| \\
&+ 2\sum_{s=1}^{Q-1} \sum_{\substack{s+1\leq t\leq Q\\ \text{case(ii)}}} |A_s(\Psi) \cap A_t(\Psi)| \\
\ll &\sum_{s=1}^{Q} |A_s(\Psi)| \quad + \quad \frac{1}{|B|}\Big(\textstyle\sum_{s=1}^{Q} |A_s(\Psi)|\Big)^2 \\
&+\ |B| \sum_{s=1}^{Q-1} \sum_{\substack{s+1\leq t\leq Q\\ \Psi(u_s)<\rho(u_t)}} \frac{\Psi(u_t)}{\rho(u_s)}\ .
\end{aligned}$$

We now make use of the fact that Ψ is u-regular. For $t > s$ with s sufficiently large, we have that $\Psi(u_t) \leq \lambda^{t-s}\Psi(u_s)$ for some $0 < \lambda < 1$. This together with (44), implies that for Q sufficiently large

$$|B| \sum_{s=1}^{Q-1} \sum_{\substack{s+1\leq t\leq Q\\ \Psi(u_s)<\rho(u_t)}} \frac{\Psi(u_t)}{\rho(u_s)} \ll |B| \sum_{s=1}^{Q-1} \frac{\Psi(u_s)}{\rho(u_s)} \sum_{s<t\leq Q} \lambda^{t-s} \ll \sum_{s=1}^{Q} |A_s(\Psi)|\ .$$

By (46), for Q sufficiently large $\sum_{s=1}^{Q} |A_s(\Psi)| \leqslant |B|^{-1}(\sum_{s=1}^{Q} |A_s(\Psi)|)^2$. The statement of Lemma 9 now readily follows. This completes the proof of Theorem 1. ♠

C. Proof of Theorem 10

To prove Theorem 10 we proceed as follows. For any fixed $\eta \gg 1$ we construct a Cantor subset $\mathbf{K}_\eta$ of $\Lambda(\mathcal{R}, \beta, \Psi)$ and a probability measure μ supported on $\mathbf{K}_\eta$ satisfying the condition that for an arbitrary ball A of sufficiently small radius $r(A)$

$$\mu(A) \ \ll\ \frac{r(A)^s}{\eta}\ , \tag{49}$$

where the implied constant is absolute. By the Mass Distribution Principle, the above inequality implies that $\mathcal{H}^s(\mathbf{K}_\eta) \gg \eta$. Since $\mathbf{K}_\eta \subset \Lambda(\mathcal{R}, \beta, \Psi)$, we obtain that $\mathcal{H}^s(\Lambda(\mathcal{R}, \beta, \Psi)) \gg \eta$. However, $\eta \gg 1$ is arbitrarily large whence $\mathcal{H}^s(\Lambda(\mathcal{R}, \beta, \Psi)) = \infty$ and this proves Theorem 10.

In view of the above outline, the whole strategy of our proof is centred around the construction of a 'right type' of Cantor set $\mathbf{K}_\eta$ which supports a measure μ with the desired property. The actual nature of the construction of $\mathbf{K}_\eta$ depends heavily on whether G defined by (38) is finite or infinite. We first deal with the case that $0 \leq G < \infty$. The case that $G = \infty$ is substantially easier.

C.1. Preliminaries

In this section we group together for clarity and convenience various concepts and results which will be required in constructing the Cantor set $\mathbf{K}_\eta$. Throughout, g will denote the function given by

$$g(r) := \Psi(r)^s \rho(r)^{-1} \qquad \text{and so} \qquad G := \limsup_{n\to\infty} g(u_n) \ .$$

C.1.1. The sets $G^u_B(n)$ again.

Let B be an arbitrary ball in I_0 with radius $r(B)$. Relabel the sets $G^u_B(n)$ constructed in §B by $\tilde{G}^u_B(n)$. By keeping track of constants, the estimate (42) for $\#\tilde{G}^u_B(n)$ is explicitly as follows:

$$\frac{\kappa}{24}\,\frac{r(B)}{\rho(u_n)} \ \leq\ \#\tilde{G}^u_B(n) \ \leq\ \frac{r(B)}{\rho(u_n)} \ ,$$

where κ is as in (36). Now let $0 < c_1 := \frac{\kappa}{24} < 1$ and define $G^u_B(n)$ to be any sub-collection of $\tilde{G}^u_B(n)$ such that

$$\#G^u_B(n) \ = \left[c_1\,\frac{r(B)}{\rho(u_n)}\right] \ ,$$

where $[x]$ denotes the integer part of a real number x. Thus, for n sufficiently large

$$\tfrac{1}{2}\ c_1\,\frac{r(B)}{\rho(u_n)} \ \leq\ \#G^u_B(n) \ \leq\ c_1\,\frac{r(B)}{\rho(u_n)} \ . \tag{50}$$

Remark. Recall, that by construction the balls $B(R_\alpha, 2\rho(u_n))$ with $\alpha \in G^u_{\mathrm{B}}(n)$ are disjoint. Also note, that we can assume that $\rho(u_n)^{-1}\,\Psi(u_n) \to 0$ as $n \to \infty$. If this was not the case then $\limsup \rho(u_n)^{-1}\,\Psi(u_n) > 0$ as $n \to \infty$ and Theorem 1 implies that $|\Lambda(\mathcal{R},\beta,\Psi)| = |I_0| > 0$. In turn, $\mathcal{H}^s(\Lambda(\mathcal{R},\beta,\Psi)) = \infty$ for any $s \in [0,1)$ and we are done. Hence, without loss of generality, we can assume that for n sufficiently large

$$2\,\Psi(u_n) \ <\ \rho(u_n) \qquad \text{and} \qquad \lim_{n\to\infty} \Psi(u_n) = 0 \ . \tag{51}$$

C.1.2. Working on a subsequence of u and the ubiquity function ρ.

The proof of Theorem 10 in the case that G is finite relies on the fact that the ubiquity function ρ can be taken to be u-regular with constant λ as small as we please. The fact that we have assumed that the approximating function Ψ is u-regular in the hypothesis of the theorem is purely for convenience with the application to planar curves in mind. To begin with recall the following simple facts: (i) if we have local ubiquity for a particular sequence u then we automatically have local ubiquity for any subsequence v and (ii) if a function h is u-regular then it is v-regular for any subsequence v. Also note that if G is finite, then $\limsup_{n\to\infty} g(v_n) < \infty$ for any subsequence v of u.

Suppose G is finite and fix some $\lambda \in (0,1)$. We now prove the existence of an appropriate subsequence v of u on which ρ is v-regular with constant λ and $\sum g(v_n) = \infty$. In the case $G = 0$ (part (i) of Theorem 10), we have that Ψ is u-regular and so there exists a constant $\lambda_* \in (0,1)$ such that $\Psi(u_{n+1}) \leq \lambda_*\,\Psi(u_n)$ for all n sufficiently large. It follows that for n sufficiently large

$$x_{n+1} := \Psi(u_{n+1})^s \leq \lambda_*^s \Psi(u_n)^s = \lambda_*^s\, x_n \ .$$

Next, fix some sufficiently large n_1 and for $k \geq 2$ let n_k be the least integer strictly greater than n_{k-1} such that $\rho(u_{n_k}) \leq \lambda\,\rho(u_{n_{k-1}})$. This is possible since $\rho(r) \to 0$ as $r \to \infty$. By construction, $\rho(u_m) \geq \lambda\,\rho(u_{n_{k-1}})$ for any integer $m \in [n_{k-1}, n_k - 1]$. It follows that

$$\begin{aligned}
\infty &= \sum_{n=n_1}^{\infty} g(u_n) = \sum_{n=n_1}^{\infty} x_n\,\rho(u_n)^{-1} = \sum_{k=2}^{\infty} \sum_{n_{k-1}\leq m<n_k} x_m\,\rho(u_m)^{-1} \\
&\leq \sum_{k=2}^{\infty} \sum_{n_{k-1}\leq m<n_k} x_m\,\rho(u_{n_{k-1}})^{-1}\,\lambda^{-1} = \lambda^{-1}\sum_{k=2}^{\infty}\rho(u_{n_{k-1}})^{-1} \sum_{n_{k-1}\leq m<n_k} x_m \\
&\ll \sum_{k=2}^{\infty}\rho(u_{n_{k-1}})^{-1}x_{n_{k-1}} \sum_{i=0}^{\infty}(\lambda_*^s)^i \ll \sum_{k=1}^{\infty}\rho(u_{n_k})^{-1}x_{n_k} := \sum_{k=1}^{\infty} g(u_{n_k}).
\end{aligned}$$

Now set $v := \{u_{n_k}\}$. By construction, ρ is v-regular with constant λ and $\sum g(v_n) = \infty$. Next notice that if $0 < G < \infty$, then there exists a strictly increasing sequence $\{n_i\}$ such that $g(u_{n_i}) \geq G/2 > 0$. Since $\lim_{r\to\infty}\rho(r) = 0$, it follows that for any $\lambda < 1$ there exists a subsequence v of $\{u_{n_i}\}$ such that $\rho(v_{t+1}) < \lambda\,\rho(v_t)$ and $\sum g(v_t) = \infty$. The upshot is that in establishing Theorem 10 for the case that $0 \leq G < \infty$ we can assume that ρ is u regular with constant λ as small as we please.

C.2. Proof of Theorem 10: $0 \leq G < \infty$

C.2.1. The Cantor set $\mathbf{K}_\eta$

Let $G^* := \max\{2, \sup_{n\in\mathbb{N}} g(u_n)\}$ and fix a real number $\eta > G^*$. Thus

$$g(u_n) \;<\; G^* \;<\; \eta \qquad \text{for all } n \;\;.$$

To avoid cumbersome expressions, let ϖ denote the following repeatedly occurring constant

$$\varpi \;:=\; \tfrac{c_1}{96} \;<\; 1 \;\;. \tag{52}$$

In view of the discussion of §C.1.2, we can assume that for n sufficiently large

$$\rho(u_{n+1}) \;\leq\; \lambda\,\rho(u_n) \qquad \text{with} \qquad 0 < \lambda < \tfrac{1}{9} \;. \tag{53}$$

Constructing the first level $\mathbf{K}(1)$.

Choose t_1 large enough so that

$$g(u_{t_1}) \;<\; G^* \;< \frac{\eta}{24\varpi} \;, \tag{54}$$

$$\psi(u_{t_1})^{s-1} > \frac{\eta}{\varpi} \;, \tag{55}$$

and so that the counting estimate (50) is valid for the set $G^u_{I_0}(t_1)$; i.e. with $B = I_0$. Note that the first of these inequalities is possible since $g(u_n) < G^* < \eta$. The latter inequality is possible in view of (51) and since $s < 1$. Let $k_1 \geq 1$ be the unique integer such that

$$\frac{6\varpi}{\eta}\sum_{i=0}^{k_1-1} g(u_{t_1+i}) \;\leqslant\; \frac{1}{4} \;<\; \frac{6\varpi}{\eta}\sum_{i=0}^{k_1} g(u_{t_1+i}) \;. \tag{56}$$

Note, the fact that $k_1 \geq 1$ is a consequence of (54). The first level $\mathbf{K}(1)$ of the Cantor set $\mathbf{K}_\eta$ will consist of sub-levels $K(t_1+i)$ where $0 \leq i \leq k_1$.

- **The sub-level** $K(t_1)$ **:** This consists of balls of common radius $\psi(u_{t_1})$ defined as follows:-

$$K(t_1) := \bigcup_{\alpha \in V^u_{I_0}(t_1)}^{\circ} B(R_\alpha, \psi(u_{t_1})) \quad \text{where} \quad V^u_{I_0}(t_1) := G^u_{I_0}(t_1) .$$

- **The sub-levels** $K(t_1+i)$ **for** $1 \leq i \leq k_1$ **:** These are constructed inductively. The key to the whole procedure is the existence of 'special' subsets $V^u_{I_0}(t_1+i)$ of $G^u_{I_0}(t_1+i)$. Suppose for $0 \leq j \leq i-1$ we have constructed the sub-levels

$$K(t_1+j) = \bigcup_{\alpha \in V^u_{I_0}(t_1+j)}^{\circ} B(R_\alpha, \psi(u_{t_1+j})) .$$

We proceed to construct $K(t_1+i)$ – equivalently $V^u_{I_0}(t_1+i)$. Let

$$h(u_{t_1+j}) := \frac{\varpi}{\eta} \Psi(u_{t_1+j})^s .$$

Note that in view of (54) and (55) we have that $\psi(u_{t_1+j}) < h(u_{t_1+j}) < \rho(u_{t_1+j})$. Define

$$T(t_1+j) := \left\{ B(R_\alpha, h(u_{t_1+j})) : \alpha \in V^u_{I_0}(t_1+j) \right\} .$$

Now for each $\alpha \in G^u_{I_0}(t_1+i)$ construct the ball $B(R_\alpha, \rho(u_{t_1+i}))$. Clearly, the balls in this collection are also disjoint and we proceed by disregarding any of those which lie too close to balls from any of the previous sub-levels $K(t_1+j)$. To make this precise, we introduce the sets

$$\begin{aligned} U^u_{I_0}(t_1+i) &:= \{\alpha \in G^u_{I_0}(t_1+i) : B(R_\alpha, \rho(u_{t_1+i})) \cap \bigcup_{j=0}^{i-1} T(t_1+j) \neq \emptyset\} \\ V^u_{I_0}(t_1+i) &:= G^u_{I_0}(t_1+i) \setminus U^u_{I_0}(t_1+i) . \end{aligned}$$

By construction, $V^u_{I_0}(t_1+j) \subseteq G^u_{I_0}(t_1+j)$ for $0 \leqslant j \leqslant i$. In particular, the balls in $T(t_1+j)$ are disjoint. Thus, $\#T(t_1+j) = \#V^u_{I_0}(t_1+j) \leq \#G^u_{I_0}(t_1+j)$. We claim that $\#V^u_{I_0}(t_1+i) \geq \frac{1}{2}\, G^u_{I_0}(t_1+i)$. There are two cases to consider.

Case (i): $0 \leq j \leq i-1$ such that $\rho(u_{t_1+i}) < h(u_{t_1+j})$. The number of disjoint balls of radius $\rho(u_{t_1+i})$ that can possibly intersect a ball in $T(t_1+j)$ is $\leqslant 3h(u_{t_1+j})/\rho(u_{t_1+i})$.

Case (ii): $0 \leq j \leq i-1$ such that $\rho(u_{t_1+i}) \geq h(t_1+j)$. In this case, each ball in $T(t_1+j)$ can intersect at most one ball $B(R_\alpha, \rho(u_{t_1+i}))$ with $\alpha \in G^u_{I_0}(t_1+i)$. This makes use of the fact that the corresponding enlarged balls $B(R_\alpha, 2\rho(u_{t_1+i}))$ are disjoint.

It follows that

$$\#U^u_{I_0}(t_1+i) \leqslant \sum_{\text{case (i)}} \frac{3\varpi}{\eta} \, \frac{\psi(u_{t_1+j})^s}{\rho(u_{t_1+i})} \, \#T(t_1+j) + \sum_{\text{case (ii)}} \#T(t_1+j).$$

Recall that $\#T(t_1+j) \leq \#G^u_{I_0}(t_1+j)$. Thus, the contribution from the sum over case (i) is:

$$\leq \sum_{j=0}^{k_1-1} \frac{6\varpi}{\eta} \, g(u_{t_1+j}) \, \#G^u_{I_0}(t_1+i) \leqslant \frac{1}{4} \, \#G^u_{I_0}(t_1+i) ,$$

by (50) and the choice of k_1 – see (56). The contribution from the sum over case (ii) is:

$$\begin{aligned}
&\leq \sum_{\text{case (ii)}} \#G^u_{I_0}(t_1+j) \ \leq\ 2\sum_{j=0}^{i-1} \frac{\rho(u_{t_1+i})}{\rho(u_{t_1+j})} \#G^u_{I_0}(t_1+i) \\
&\leq\ 2\ \#G^u_{I_0}(t_1+i)\sum_{j=0}^{i-1}\lambda^{i-j}\ <\ 2\,\#G^u_{I_0}(t_1+i)\sum_{s=1}^{\infty}\lambda^s\ <\ \frac{1}{4}\ \#G^u_{I_0}(t_1+i)\ ,
\end{aligned}$$

by (50) and the choice of λ – see (53). Hence, $\#U^u_{I_0}(t_1+i) < \frac{1}{2}\ \#G^u_{I_0}(t_1+i)$ so that

$$\#V^u_{I_0}(t_1+i)\ \geq\ \tfrac{1}{2}\ \#G^u_{I_0}(t_1+i). \tag{57}$$

The sub-level $K(t_1+i)$ is defined to be:

$$K(t_1+i)\ :=\ \bigcup_{R_\alpha\in V^u_{I_0}(t_1+i)} B(R_\alpha,\psi(u_{t_1+i}))\ .$$

Also, note that by construction $K(t_1+i)\cap K(t_1+j)=\emptyset$ for $0\leq i\neq j\leq k_1$. The first level $\mathbf{K}(1)$ of the Cantor set is defined to be

$$\mathbf{K}(1)\ :=\ \bigcup_{i=0}^{k_1} K(t_1+i)\ .$$

Higher levels $\mathbf{K}(n)$ and the Cantor set $\mathbf{K}_\eta$.

For any integer $n\geq 2$, the n'th level $\mathbf{K}(n)$ will be defined recursively in terms of local levels $\mathrm{K}(n,B)$ associated with balls B from the previous level $\mathbf{K}(n-1)$:

$$\mathbf{K}(n)\ :=\ \overset{\circ}{\bigcup_{B\in\mathbf{K}(n-1)}}\ \mathrm{K}(n,B)\ ,$$

where

$$\mathrm{K}(n,B)\ :=\ \bigcup_{i=0}^{k_n(B)} K(t_n+i,B)\ .$$

To start with, choose $t_n>t_{n-1}$ sufficiently large so that for any ball $B\in\mathbf{K}(n-1)$ the counting estimate (50) is valid and so that

$$\psi(u_{t_n})^{s-1} > \frac{r(B)^{s-1}}{\varpi}. \tag{58}$$

In view of (51), (55), the fact that $g(u_n)<G^*$ for all n and that $s<1$, we have that

$$g(u_{t_n})\ <\ G^*\ <\frac{r(B)^{s-1}}{24\varpi}\qquad\forall\quad B\in\mathbf{K}(n-1). \tag{59}$$

Fix a ball B in $\mathbf{K}(n-1)$ and let $k_n(B)\geq 1$ be the unique integer such that

$$\frac{6\varpi}{r(B)^{s-1}}\sum_{i=0}^{k_n(B)-1} g(u_{t_n+i})\ \leqslant\ \frac{1}{4}\ <\ \frac{6\varpi}{r(B)^{s-1}}\sum_{i=0}^{k_n(B)} g(u_{t_n+i})\ . \tag{60}$$

The fact that $k_n(B) \geq 1$ is a consequence of (59). We now construct the local level $\mathrm{K}(n, B)$.

- **The local sub-level** $K(t_n, B)$ **:** Let

$$K(t_n, B) \ := \bigcup_{\alpha \in V_B^u(t_n)} B(R_\alpha, \psi(u_{t_n})) \quad \text{where} \quad V_B^u(t_n) := G_B^u(t_n) \ .$$

By construction, $K(t_n, B) \subset B$ – see (41).

- **The local sub-levels** $K(t_n + i, B)$ **for** $1 \leq i \leq k_n(B)$ **:** Suppose for $0 \leq j \leq i-1$ we have constructed the local sub-levels

$$K(t_n + j, B) \ = \overset{\circ}{\bigcup_{\alpha \in V_B^u(t_n+j)}} B(R_\alpha, \psi(u_{t_n+j})) \ .$$

Let

$$h_B(u_{t_n+j}) := \frac{\varpi\, \psi(u_{t_n+j})^s}{r(B)^{s-1}} \ .$$

In view of (58) and (59) we have that

$$\psi(u_{t_n+j}) \ < \ h_B(u_{t_n+j}) < \rho(u_{t_n+j}). \tag{61}$$

Define

$$T(t_n + j, B) := \{B(R_\alpha, h_B(u_{t_n+j})) : \alpha \in V_B^u(t_n + j)\} \ .$$

Next, introduce the sets

$$\begin{aligned} U_B^u(t_n + i) \ &:= \ \{\alpha \in G_B^u(t_n + i) : B(R_\alpha, \rho(u_{t_n+i})) \cap \bigcup_{j=0}^{i-1} T(t_n + j, B) \neq \emptyset\} \\ V_B^u(t_n + i) \ &:= \ G_B^u(t_n + i) \setminus U_B^u(t_n + i) \quad . \end{aligned}$$

By construction, $V_B^u(t_n + j) \subseteq G_B^u(t_n + j)$ for $0 \leqslant j \leqslant i$ and so the balls in $T(t_n + j)$ are disjoint. By adapting the argument used in establishing (57), it is easily verified that

$$\# V_B^u(t_n + i) \ \geq \ \tfrac{1}{2}\, \# G_B^u(t_n + i). \tag{62}$$

Now let

$$K(t_n + i, B) \ := \bigcup_{\alpha \in V_B^u(t_n+i)} B(R_\alpha, \psi(u_{t_n+i})) \ .$$

This completes the inductive step and the construction of the local level $\mathrm{K}(n, B)$ associated with $B \in \mathbf{K}(n-1)$. Clearly, for $0 \leq i \neq j \leq k_n(B)$ we have that

$$K(t_n + i, B) \ \cap \ K(t_n + j, B) \ = \ \emptyset \ .$$

Furthermore, by construction $\mathrm{K}(n, B)$ is contained in B. Therefore, $\mathbf{K}(n) \subset \mathbf{K}(n-1)$. The Cantor set $\mathbf{K}_\eta$ is defined as

$$\mathbf{K}_\eta \ := \ \bigcap_{n=1}^{\infty} \mathbf{K}(n) \ .$$

Strictly speaking, $\mathbf{K}_\eta \setminus \mathcal{R} \subset \Lambda(\mathcal{R}, \beta, \Psi)$ and not $\mathbf{K}_\eta \subset \Lambda(\mathcal{R}, \beta, \Psi)$. However, this is irrelevant since $\mathcal{R}$ is countable and so $\mathcal{H}^s(\mathbf{K}_\eta \setminus \mathcal{R}) = \mathcal{H}^s(\mathbf{K}_\eta)$. Before constructing a measure on $\mathbf{K}_\eta$, we state an important lemma. The proof is a simple consequence of (50), (57) and (62).

Lemma 10 (i) *For* $0 \leqslant i \leqslant k_1$,

$$\#V^u_{I_0}(t_1+i)\ \psi(u_{t_1+i})^s \ \geqslant\ \frac{c_1|I_0|}{8}\ g(u_{t_1+i}) \quad .$$

(ii) *For* $n \geqslant 2$, *let* B *be a ball in* $\mathbf{K}(n-1)$. *Then, for* $0 \leqslant i \leqslant k_n(B)$

$$\#V_B(t_n+i)\ \psi(u_{t_n+i})^s \ \geqslant\ \frac{c_1|B|}{8}\ g(u_{t_n+i}) \quad .$$

C.2.2. A measure on $\mathbf{K}_\eta$

In this section, we construct a probability measure μ supported on $\mathbf{K}_\eta$ satisfying (49). Suppose $n \geq 2$ and $B \in \mathbf{K}(n)$. For $1 \leq m < n$, let B_m denote the unique ball in $\mathbf{K}(m)$ containing the ball B. With this notation in mind we now define μ. For any $B \in \mathbf{K}(n)$, we attach a weight $\mu(B)$ defined recursively as follows: For $n = 1$,

$$\mu(B) \ :=\ \frac{r(B)^s}{\sum_{B' \in \mathbf{K}(1)} r(B')^s}$$

and for $n \geqslant 2$,

$$\mu(B) \ :=\ \frac{r(B)^s}{\sum_{B' \in K(n,B_{n-1})} r(B')^s} \ \times\ \mu(B_{n-1}) \ .$$

This procedure defines inductively a mass on any ball appearing in the construction of $\mathbf{K}_\eta$. In fact a lot more is true: The probability measure μ constructed above is supported on $\mathbf{K}_\eta$ and for any Borel subset F of I_0

$$\mu(F) := \mu(F \cap \mathbf{K}_\eta) \ =\ \inf \sum_{B \in \mathcal{B}} \mu(B) \ ,$$

where the infimum is taken over all coverings $\mathcal{B}$ of $F \cap \mathbf{K}_\eta$ by balls $B \in \{\mathbf{K}(n) : n \in \mathbb{N}\}$.

For further details see [19, Prop. 1.7] . It remains to establish (49) for μ.

Measure of a ball in the Cantor construction. If $B \in \mathbf{K}(n)$ for some $n \in \mathbb{N}$, then

$$\begin{aligned}\mu(B) \ &:=\ \frac{r(B)^s}{\sum_{B' \in K(n,B_{n-1})} r(B')^s} \ \times\ \mu\left(B_{n-1}\right) \\ &=\ \frac{r(B)^s}{\sum_{B' \in \mathbf{K}(1)} r(B')^s} \prod_{m=1}^{n-1} \frac{r(B_m)^s}{\sum_{B' \in K(m+1,B_m)} r(B')^s} \ . \end{aligned} \tag{63}$$

The product term is taken to be one when $n = 1$. To proceed we require the following lemma which gives us a lower bound on the terms in the denominator of the above expression.

Lemma 11

$$\sum_{B \in \mathbf{K}(1)} r(B)^s \ \geqslant\ \tfrac{\eta}{2}\,|I_0| \qquad \text{and} \qquad \sum_{B \in K(n,B_{n-1})} r(B)^s \ \geqslant\ r(B_{n-1})^s \qquad (n \geqslant 2).$$

Proof of Lemma 11. By Lemma 10, the choice of k_1 (56) and ϖ (52) it follows that

$$\sum_{B\in\mathbf{K}(1)} r(B)^s \;=\; \sum_{i=0}^{k_1} \;\#V^u_{I_0}(t_1+i)\;\psi(u_{t_1+i})^s \;\geqslant\; \frac{c_1|I_0|}{8}\;\sum_{i=0}^{k_1} g(u_{t_1+i}) \;>\; \frac{c_1|I_0|}{192}\frac{\eta}{\varpi} \;\geqslant\; \tfrac{\eta}{2}\,|I_0|\;.$$

The statement for $n\geq 2$ follows in a similar fashion – use (60) rather than (56). ♠

In view of the above lemma, it now follows from (63) that for any ball $B\in\mathbf{K}(n)$

$$\mu(B) \;\leqslant\; \frac{2\,r(B)^s}{|I_o|\;\eta} \;\ll\; \frac{r(B)^s}{\eta}\;. \tag{64}$$

Measure of an arbitrary ball. The aim is to show that $\mu(A)\ll r(A)^s/\eta$ for an arbitrary ball A with radius $r(A)\leq r_o$. The measure μ is supported on $\mathbf{K}_\eta$. Thus, without loss of generality we can assume that $A\cap\mathbf{K}_\eta\neq\emptyset$; otherwise $\mu(A)=0$ and there is nothing to prove. We can also assume that for every n large enough A intersects at least two balls in $\mathbf{K}(n)$; since if B is the only ball in $\mathbf{K}(n)$ which has non–empty intersection with A, then in view of (64)

$$\mu(A)\;\leq\;\mu(B)\;\ll\; r(B)^s/\eta\;\to\;0 \qquad \text{as} \qquad n\to\infty$$

($r(B)\to 0$ as $n\to\infty$) and again there is nothing to prove. Thus we may assume that there exists an integer $n\geq 2$ such that A intersects only one ball $\widetilde{B}$ in $\mathbf{K}(n-1)$ and at least two balls from $\mathbf{K}(n)$. The case that A intersects two or more balls from the first level can be excluded by choosing $r(A)$ sufficiently small. This follows from the fact that by construction balls in any one level are disjoint. Furthermore, we can assume that

$$r(A)\;<\;r(\widetilde{B})\;.$$

Otherwise, $\mu(A)\;\leq\;\mu(\widetilde{B})\;\leq\;r(\widetilde{B})^s/\eta\;\leq\;r(A)^s/\eta$ and we are done. Given that A only intersects the ball $\widetilde{B}$ in $\mathbf{K}(n-1)$, the balls from level $\mathbf{K}(n)$ which intersect A must be contained in the local level

$$\mathrm{K}(n,\widetilde{B})\;:=\;\bigcup_{i=0}^{k_n(\widetilde{B})} K(t_n+i,\widetilde{B})\;.$$

By construction, any ball $B(R_\alpha,\psi(u_{t_n+i}))$ in $\mathrm{K}(n,\widetilde{B})$ is contained in the ball $B(R_\alpha,h_{\widetilde{B}}(u_{t_n+i}))$. Thus A intersects at least one ball in $T(t_n+i,\widetilde{B})$ for some $0\leq i\leq k_n(\widetilde{B})$.

Let $K(t_n+i',\widetilde{B})$ be the first local sub-level associated with $\widetilde{B}$ such that

$$K(t_n+i',\widetilde{B})\;\cap\;A\;\neq\;\emptyset\;.$$

Thus, A intersects at least one ball $B(R_\alpha,\psi(u_{t_n+i'}))$ from $K(t_n+i',\widetilde{B})$ and such balls are indeed the largest balls from the n'th level $\mathbf{K}(n)$ that intersect A. Clearly, A intersects at least one ball B_* in $T(t_n+i',\widetilde{B})$. We now prove a trivial but crucial geometric lemma.

Lemma 12 *For $i\geq i'$, if A intersects $B(R_\alpha,\psi(u_{t_n+i}))\subset B(R_\alpha,h_{\widetilde{B}}(u_{t_n+i}))\neq B_*$ then*

$$r(A)>\tfrac{1}{2}\,\rho(u_{t_n+i})\;.$$

Proof of Lemma 12. If $i = i'$ then as balls of radius $\rho(u_{t_n+i'})$ are disjoint we have that $r(A) > \rho(u_{t_n+i'})$. Assume that $i > i'$, then by construction $B(R_\alpha, \rho(u_{t_n+i})) \cap B_* = \emptyset$. Hence, $r(A) > \rho(u_{t_n+i}) - \psi(u_{t_n+i}) > \frac{1}{2}\rho(u_{t_n+i})$ – see (51). ♠

In view of the definition of i' and (64), we have that

$$\begin{aligned} \mu(A) &\leq \sum_{i=i'}^{k_n(\tilde{B})} \sum_{\substack{\alpha \in V^u_{\tilde{B}}(t_n+i): \\ B(R_\alpha, \psi(u_{t_n+i})) \cap A \neq \emptyset}} \mu(B(R_\alpha, \psi(u_{t_n+i}))) \\ &\leq \frac{2}{\eta\,|I_0|} \sum_{i=i'}^{k_n(\tilde{B})} \psi(u_{t_n+i})^s \sum_{\substack{\alpha \in V^u_{\tilde{B}}(t_n+i): \\ B(R_\alpha, \psi(u_{t_n+i})) \cap A \neq \emptyset}} 1 \ . \end{aligned} \tag{65}$$

In view of Lemma 12, if A intersects some ball $B(R_\alpha, h_{\widetilde{B}}(t_n+i))$ in $T(t_n+i, \widetilde{B})$ then the ball $B(R_\alpha, \rho(u_{t_n+i}))$ which contains it is itself contained in the ball $5A$. Let N_i denote the number of balls $B(R_\alpha, \rho(u_{t_n+i}))$ with $\alpha \in V^u_{\widetilde{B}}(t_n+i)$ that can possibly intersect A. By construction these balls are disjoint. Thus, $2\rho(u_{t_n+i}) \times N_i \ \leq \ |5A| = 10\ r(A)$. This implies, via (65) that

$$\mu(A) \ \leq \ \frac{2}{\eta\,|I_0|} \sum_{i=i'}^{k_n(\tilde{B})} \psi(u_{t_n+i})^s \ \ N_i \ \leq \ \frac{10}{\eta\,|I_0|}\ r(A) \sum_{i=0}^{k_n(\tilde{B})} g(u_{t_n+i}) \ .$$

By (60),

$$\sum_{i=0}^{k_n(B)-1} g(u_{t_n+i}) \ \leq \ \frac{r(\widetilde{B})^{s-1}}{24\varpi} \ ,$$

and by (59) together with the fact that $g(u_n) < G^*$ for all n

$$g(u_{t_n+k_n(\tilde{B})}) \ < \ \frac{r(\widetilde{B})^{s-1}}{24\varpi} \ .$$

Hence

$$\mu(A) \ \ll \ \frac{1}{\eta}\ r(A)\ r(\widetilde{B})^{s-1} \tag{66}$$

However, $r(A) < r(\widetilde{B})$ and $s < 1$. The desired inequality, namely (49) now readily follows. This completes the proof of Theorem 10 in the case that G is finite. ♠

C.3. Proof of Theorem 10: $G = \infty$

The proof of Theorem 10 in the case that G is infinite follows the same strategy as the proof when G is finite. However, to execute the strategy is far simpler than in the finite case.

C.3.1. The Cantor set K and the measure μ

We start by defining a Cantor set **K** which is dependent on a certain, strictly increasing sequence of natural numbers $\{t_i : i \in \mathbb{N}\}$. The main difference between this case and the previous case is that we do not need to consider sublevels.

The Cantor set $\mathbf{K}$. Choose t_1 sufficiently large so that the counting estimate (50) is valid for the set $G^u_{I_0}(t_1)$ and define the first level $\mathbf{K}(1)$ of the Cantor set $\mathbf{K}$ as follows:

$$\mathbf{K}(1) \ := \bigcup_{\alpha \in G^u_{I_0}(t_1)} B(R_\alpha, \psi(u_{t_1})) \ .$$

For $n \geq 2$ we define the n'th level $\mathbf{K}(n)$ recursively as follows:

$$\mathbf{K}(n) \ := \overset{\circ}{\bigcup_{B \in \mathbf{K}(n-1)}} \mathrm{K}(n, B) \ ,$$

where

$$\mathrm{K}(n, B) \ := \bigcup_{\alpha \in G^u_B(t_n)} B(R_\alpha, \psi(u_{t_n}))$$

is the n'th local level associated with the ball $B := B(R_\alpha, \psi(u_{t_{n-1}})) \in \mathbf{K}(n-1)$. Here $t_n > t_{n-1}$ is chosen sufficiently large so that (50) is valid for any ball B in $\mathbf{K}(n-1)$. By construction $\mathrm{K}(n, B) \subset B$ and so $\mathbf{K}(n) \subset \mathbf{K}(n-1)$. The Cantor set $\mathbf{K}$ is simply given by

$$\mathbf{K} \ := \bigcap_{n=1}^{\infty} \mathbf{K}(n) \ .$$

By construction, $\mathbf{K} \setminus \mathcal{R} \subset \Lambda(\mathcal{R}, \beta, \Psi)$ and since $\mathcal{R}$ is countable $\mathcal{H}^s(\Lambda(\mathcal{R}, \beta, \Psi)) \geq \mathcal{H}^s(\mathbf{K})$.

The measure μ. Suppose $n \geq 2$ and $B \in \mathbf{K}(n)$. For $1 \leq m < n$, let B_m denote the unique ball in $\mathbf{K}(m)$ containing B. For any $B \in \mathbf{K}(n)$, we attach a weight $\mu(B)$ defined recursively as follows:

For $n = 1$,

$$\mu(B) \ := \ \frac{1}{\#G^u_{I_0}(t_1)}$$

and for $n \geqslant 2$,

$$\mu(B) \ := \ \frac{1}{\#G^u_{B_{n-1}}(t_n)} \times \ \mu(B_{n-1}) \ .$$

By the definition of μ and the counting estimate (50), it follows that

$$\mu(B) \ \leq \ \frac{2}{|I_0|} \ c_1^{-n} \ \rho(u_{t_n}) \ \times \prod_{m=1}^{n-1} \frac{\rho(u_{t_m})}{\psi(u_{t_m})} \ . \tag{67}$$

The product term is taken to be one when $n = 1$.

C.3.2. Completion of the proof

Fix $\eta \geq 1$. Since $G = \infty$, the sequence $\{t_i\}$ associated with the construction of the Cantor set $\mathbf{K}$ can clearly be chosen so that

$$\eta \ \times \ \frac{2}{|I_0|} \ c_1^{-i} \ \times \prod_{j=1}^{i-1} \frac{\rho(u_{t_j})}{\psi(u_{t_j})} \leq \ g(u_{t_i}) \ . \tag{68}$$

The product term is one when $i = 1$. It now immediately follows from (67) that for any $B \in \mathbf{K}(n)$,

$$\mu(B) \ \leq \ r(B)^s \, \frac{2}{|I_0|} \ \ c_1^{-n} \, \frac{1}{g(u_{t_n})} \ \times \ \prod_{m=1}^{n-1} \frac{\rho(u_{t_m})}{\psi(u_{t_m})} \ \leq \ r(B)^s/\eta \ .$$

We now show that $\mu(A) \ll r(A)^s/\eta$ where A is an arbitrary ball. The same reasoning as before enables us to assume that $A \cap \mathbf{K} \neq \emptyset$ and that there exists an integer $n \geq 2$ such that A intersects only one ball $\widetilde{B}$ in $\mathbf{K}(n-1)$ and at least two balls from $\mathrm{K}(n, \widetilde{B}) \subset \mathbf{K}(n)$. Thus,

$$\rho(u_{t_n}) \ \leq \ r(A) \ \leq \ r(\widetilde{B}) := \Psi(u_{t_{n-1}}) \ . \tag{69}$$

The left hand side of (69) makes use of the fact that $B(R_\alpha, \psi(u_{t_n})) \subset B(R_\alpha, \rho(u_{t_n}))$ and that the balls $B(R_\alpha, 2\rho(u_{t_n}))$ with $\alpha \in G^u_{\widetilde{B}}(t_n)$ are disjoint. A simple geometric argument yields that $N := \#\{\alpha \in G^u_{\widetilde{B}}(t_n) : B(R_\alpha, \rho(u_{t_n})) \cap A \neq \emptyset\} \ \leq \ 3\, r(A)/\rho(u_{t_n})$. In view of (67), (68), (69) and the fact that $s < 1$, we obtain

$$\begin{aligned}
\mu(A) \ \ \leq \ \ & N \ \mu(B(R_\alpha, \psi(u_{t_n}))) \ \leqslant \ r(A) \, \frac{6}{|I_0|} \ \ c_1^{-n} \prod_{m=1}^{n-1} \frac{\rho(u_{t_m})}{\psi(u_{t_m})} \\
\leqslant \ \ & r(A)^s \ \psi(u_{t_{n-1}})^{1-s} \, \frac{6}{|I_0|} \ \ c_1^{-n} \prod_{m=1}^{n-1} \frac{\rho(u_{t_m})}{\psi(u_{t_m})} \\
\leqslant \ \ & r(A)^s \ \frac{6}{|I_0|} \ \ c_1^{-n} \, \frac{1}{g(u_{t_{n-1}})} \prod_{m=1}^{n-2} \frac{\rho(u_{t_m})}{\psi(u_{t_m})} \ \ \leq \ 3\, c^{-1} \, \frac{r(A)^s}{\eta} \ .
\end{aligned}$$

The upshot is that (49) is satisfied and thereby completes the proof. ♠

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Appendix II: Sums of Two Squares Near Perfect Squares

R.C. Vaughan

A.1 The theorem

Let $r(n)$ denote the number of representations of a number n as the sum of two squares of integers and let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative decreasing function. We prove the following theorem.

Theorem A *Let Q^* denote the smallest integer with $Q^* > Q$. Then for each real number Q and natural number N with $N \leqslant Q^3$,*

$$\sum_{Q<q\leqslant 2Q} \sum_n{}' r(n) = \sum_{Q<q\leqslant 2Q} 4\pi q\psi(q)$$
$$+ \quad O\Big(Q\log Q + Q^{\frac{3}{2}}\psi(Q^*)^{\frac{1}{2}}(\log Q)^{64} + Q^2\psi(Q^*)^{\frac{1}{2}}(\log Q)^{64}N^{-\frac{1}{4}}$$
$$+ \quad N^{\frac{3}{4}}(\log N)^3 Q^{\frac{1}{2}}\psi(Q^*) + N^{\frac{1}{4}}(\log N)Q^{\frac{1}{2}} \sum_{Q<q\leqslant 2Q} \psi(q) + Q^2(\log Q)^3 N^{-\frac{1}{2}}\Big)$$

where $\sum'$ indicates that the sum is over n with $|q-\sqrt{n}| \leqslant \psi(q)$ and that any terms with $|q-\sqrt{n}| = \psi(q)$ are counted with weight $\frac{1}{2}$.

When $\psi(Q^*)$ has the same order of magnitude as $Q^{-1}\sum_{Q<q\leqslant 2Q}\psi(q)$ and the sum $\sum_{Q<q\leqslant 2Q}\psi(q)$ is large, a good choice for N is

$$Q^2\left(\sum_{Q<q\leqslant 2Q}\psi(q)\right)^{-1}.$$

This leads to the error estimate

$$\ll Q\left(\sum_{Q<q\leqslant 2Q}\psi(q)\right)^{\frac{3}{4}}(\log Q)^{64}.$$

Then the main term will dominate provided that $\sum_{Q<q\leqslant 2Q}\psi(q)$ is large compared with $(\log Q)^{256}$. A concomitant remark pertains if the theorem is averaged over Q with, say $R < Q \leqslant 2R$. It may well be possible to replace the $(\log Q)^{64}$ in the error term by a smaller power of $\log Q$, but that some power of a logarithm has to be present follows from either of the observations that

$$\sum_{q\leqslant Q} r(q^2) \sim \tfrac{4}{\pi}Q\log Q$$

(see (75) below) and

$$\sum_{q\leqslant Q} r(q^2+1) \sim \tfrac{12}{\pi}Q\log Q.$$

A.2 Proof of Theorem

Let $R(x) = \sum_{1 \leqslant n \leqslant x} r(n)$, $\Delta(x) = R(x) - \pi x$, and $\Delta_0(x) = \Delta(x)$ when $x \notin \mathbb{N}$ and $\Delta_0(x) = \Delta(x) - \frac{1}{2} r(x)$ when $x \in \mathbb{N}$. Then our motivation is the formula of Hardy [1, pg 265] which, for real $x \geqslant x_0$, we restate as

$$\Delta_0(x) = -1 + \sqrt{x} \sum_{n=1}^{\infty} r(n) n^{-1/2} \; J_1\left(2\pi\sqrt{nx}\right)$$

where J_1 denotes the usual Bessel function. However the convergence is only conditional and we require a form of this in which the tail of the infinite series is more readily accessible.

By Theorem 1 and Lemma 2 of [2] for any δ with $0 < \delta < 1$ and provided that $x \geqslant x_0(\delta)$ and $N > N_0(\delta)$ we have

$$\begin{aligned} \Delta_0(x) \; = \; -1 \; &+ \; \sqrt{x} \sum_{1 \leqslant n \leqslant N} \frac{r(n)}{n^{\frac{1}{2}}} \; J_1\left(2\pi\sqrt{nx}\right) \\ &- \; x^{\frac{5}{4}} \sum_{x(1-\delta) < n < x(1+\delta)} \frac{r(n)}{\pi n^{\frac{5}{4}}} \operatorname{sgn}\left(\sqrt{\frac{n}{x}} - 1\right) \int_{2\pi|\sqrt{n}-\sqrt{x}|\sqrt{N}}^{\infty} \frac{\sin\alpha}{\alpha} \, d\alpha \\ &+ \; O\left((xN)^{-\frac{1}{4}} + x^{\frac{1}{4}} N^{-\frac{5}{12}}\right) \end{aligned}$$

where we have used $\Delta(x) \ll x^{1/3}$ of [3] and $\operatorname{sgn}(u)$ is -1, 0 or 1 according as $u < 0$, $u = 0$ or $u > 0$. A standard estimate for J_1 [4, pg 199] gives for $\alpha > \alpha_0$

$$J_1(2\pi\alpha) \; = \; -\frac{1}{\pi} \, \alpha^{-1/2} \cos\left(2\pi\alpha + \frac{\pi}{4}\right) + O\left(\alpha^{-3/2}\right) \; .$$

For convenience we put

$$S(x) \; = \; x^{1/4} \sum_{1 \leqslant n \leqslant N} r(n) n^{-3/4} \cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)$$

and

$$E(x) \; = \; x^{\frac{5}{4}} \sum_{x(1-\delta) < n < x(1+\delta)} \frac{r(n)}{n^{\frac{5}{4}}} \; \operatorname{sgn}\left(\sqrt{\frac{n}{x}} - 1\right) \int_{2\pi|\sqrt{n}-\sqrt{x}|\sqrt{N}}^{\infty} \frac{\sin\alpha}{\alpha} \, d\alpha$$

so that

$$\begin{aligned} \Delta_0(x) \; &= \; -1 - \pi^{-1}\big(S(x) + E(x)\big) + O\left(x^{-\frac{1}{4}} + x^{\frac{1}{4}} N^{-\frac{5}{12}}\right) \\ &= \; -1 - \pi^{-1}\big(S(x) + E(x)\big) + O\left((x/N)^{\frac{1}{2}}\right) \end{aligned}$$

since $x^{-\frac{1}{4}} + x^{\frac{1}{4}} N^{-\frac{5}{12}} \ll (x/N)^{\frac{1}{2}}$ whenever $N \ll x^{3/2}$. For $x_0 \leqslant x \leqslant y$ we have

$$S(y) - S(x) = \sum_{1 \leqslant n \leqslant N} \frac{r(n)}{n^{3/4}} \int_x^y \Re\left(\left(\frac{1}{4} u^{-3/4} + \pi i u^{-1/4} n^{1/2}\right) e\left(\sqrt{nu} + \frac{1}{8}\right)\right) du.$$

The contribution to $S(y) - S(x)$ from the

$$\frac{1}{4} u^{-3/4} e\left(\sqrt{nu} + \frac{1}{8}\right)$$

part of the integrand is $\ll x^{-3/4}N^{1/4}(y-x)$. Here we have used partial summation and the fact that $r(n)$ is on average π. We shall do this several times hereafter without comment.

To prove the theorem we may suppose that $Q > Q_0$. In particular Q_0 can be chosen so that $q-\psi(q) > 2$ whenever $q > Q$. Thus, when $Q < q \leqslant 2Q$,

$$\Delta_0\big((q+\psi(q))^2\big) \;-\; \Delta_0\big((q-\psi(q))^2\big) \;=\; -\frac{1}{\pi}\,T(q,N) \\ +\,O\left(N^{\frac14}q^{-\frac12}\psi(q)+E_+(q,N)+E_-(q,N)+qN^{-\frac12}\right) \tag{70}$$

where

$$T(q,N) \;=\; \sum_{1\leqslant n\leqslant N}\frac{r(n)}{n^{1/4}}\int_{(q-\psi(q))^2}^{(q+\psi(q))^2}\Re\left(\pi i u^{-1/4}e\left(\sqrt{nu}+\frac18\right)\right)du$$

and

$$E_\pm(q,N) \;=\; \sum_{\frac12 Q^2<n<8Q^2} r(n)\min\left(1,\frac{1}{|\sqrt{n}-(q\pm\psi(q)|\sqrt{N}}\right)\;.$$

In the integral in $T(q,N)$ we make the change of variables, $u=(q+t)^2$, so that

$$T(q,N)=\sum_{1\leqslant n\leqslant N}\frac{r(n)}{n^{1/4}}\int_{-\psi(q)}^{\psi(q)}\Re\left(2\pi i(q+t)^{1/2}e\left((q+t)\sqrt{n}+\frac18\right)\right)dt\;.$$

The factor $(q+t)^{1/2}$ in the integrand is $q^{1/2}+O(|t|q^{-1/2})$ and so

$$T(q,N)=U(q,N)+O\left(q^{-\frac12}\psi(q)^2N^{3/4}\right)\;, \tag{71}$$

where

$$U(q,N)=\sum_{1\leqslant n\leqslant N}\frac{r(n)}{n^{1/4}}\int_{-\psi(q)}^{\psi(q)}\Re\left(2\pi i q^{1/2}e\left((q+t)\sqrt{n}+\frac18\right)\right)dt\;.$$

Collecting together the estimates (70) and (71) we have

$$\Delta_0\left((q+\psi(q))^2\right) \;-\; \Delta_0\big((q-\psi(q))^2\big) \;=\; -\frac{1}{\pi}\,U(q,N) \\ +\,O\left(N^{\frac34}q^{-\frac12}\psi(q)^2+N^{\frac14}q^{-\frac12}\psi(q)+E_-(q,N)+E_+(q,N)+qN^{-\frac12}\right)\;. \tag{72}$$

Let Q^* denote the smallest integer q with $q>Q$. Then

$$\sum_{Q<q\leqslant 2Q}U(q,n)=\sum_{1\leqslant n\leqslant N}\frac{r(n)}{n^{1/4}}\int_{-\psi(Q^*)}^{\psi(Q^*)}\Re\left(2\pi i e\left(t\sqrt{n}+\frac18\right)\sum_{Q<q\leqslant 2Q:\psi(q)\geqslant|t|}q^{\frac12}e(q\sqrt{n})\right)dt\;.$$

We also have

$$\sum_{Q<q\leqslant 2Q}\quad\sum_{\frac12Q^2<n<8Q^2}r(n)\;\min\left(1,\frac{1}{|\sqrt{n}-(q\pm\psi(q)|\sqrt{N}}\right) \\ \ll\quad\sum_{Q<q\leqslant 2Q}\;\sum_{\frac12Q^2<n<8Q^2}r(n)\;\min\left(1,\frac{Q}{|n-(q\pm\psi(q))^2|\sqrt{N}}\right) \\ \ll\quad\sum_{Q<q\leqslant 2Q}\;\sum_{-q^2<h\leqslant 8Q^2}r(q^2+h)\;\min\left(1,\frac{Q}{|h\mp 2q\psi(q)-\psi(q)|\sqrt{N}}\right).$$

and

$$\sum_{Q<q\leqslant 2Q}\ \sum_{\substack{4q\psi(q)+2\psi(q)^2<|h|\leqslant 8Q^2\\ h>-q^2}} r(q^2+h)\ \min\left(1,\frac{Q}{|h\mp 2q\psi(q)-\psi(q)|\sqrt{N}}\right)$$
$$\ll \sum_{Q<q\leqslant 2Q}\ \sum_{\substack{4q\psi(q)+2\psi(q)^2<|h|\leqslant 8Q^2\\ h>-q^2}} d(q^2+h)\ \min\left(1,\frac{Q}{|h|\sqrt{N}}\right).$$

Here we observe that

$$r(n)\leqslant 4d(n)\leqslant 8\sum_{\substack{l|n\\ l\leqslant\sqrt{n}}}1\ .$$

Below we state a bound for the number of solutions of a quadratic congruence which we use several times over and which is readily established using elementary facts about such congruences.

Lemma 13 *Suppose that $m\in\mathbb{N}$, $h\in\mathbb{Z}$ and define d_1 and d_2 uniquely by taking $(m,h)=d_1d_2^2$ where d_1 is square free. Further let $\rho(m;h)$ denote the number of solutions of the congruence $y^2+h\equiv 0 \pmod m$ in y modulo m. Then*

$$\rho(m;h)\leqslant 2d_2 d\left(\frac{m}{d_1d_2^2}\right).$$

By Lemma 13

$$\sum_{Q<q\leqslant 2Q}\ \sum_{\substack{4q\psi(q)+2\psi(q)^2<|h|\leqslant 8Q^2\\ h>-q^2}} d(q^2+h)\ \min\left(1,\frac{Q}{|h|\sqrt{N}}\right)$$
$$\ll \sum_{Q<q\leqslant 2Q}\ \sum_{4q\psi(q)+2\psi(q)^2<|h|\leqslant 8Q^2}\ \sum_{m|q^2+h,m\ll q}\min\left(1,\frac{Q}{|h|\sqrt{N}}\right)$$
$$\ll \sum_{0<d_1d_2^2|j|\ll Q}\ \sum_{l\ll Q/d_1d_2^2}\frac{Q^2d(l)}{d_1^2d_2^3l|j|\sqrt{N}} \ \ll\ Q^2N^{-\frac{1}{2}}(\log Q)^3.$$

Hence, by (72),

$$\begin{aligned}\sum_{Q<q\leqslant 2Q}\ \big(\Delta_0\big((q+\psi(q))^2\big)-\Delta_0\big((q-\psi(q))^2\big)\big)\ &=\ V(Q,N)\\ +\ O\Bigg(N^{\frac{3}{4}}\sum_{Q<q\leqslant 2Q}q^{-\frac{1}{2}}\psi(q)^2+N^{\frac{1}{4}}&\sum_{Q<q\leqslant 2Q}q^{-\frac{1}{2}}\psi(q)\\ +\ Q^2N^{-\frac{1}{2}}(\log Q)^3+F_-(Q,N)+F_+(Q,N)\Bigg)&\qquad(73)\end{aligned}$$

where

$$V(Q,N) = \sum_{1\leqslant n\leqslant N} \frac{r(n)}{n^{1/4}} \Im \int_{-\psi(Q^*)}^{\psi(Q^*)} 2e\left(t\sqrt{n}+\frac{1}{8}\right) \sum_{\substack{Q<q\leqslant 2Q\\ \psi(q)\geqslant |t|}} q^{1/2} e\left(q\sqrt{n}\right) dt$$

and

$$F_{\pm}(Q,N) = \sum_{Q<q\leqslant 2Q} \sum_{\substack{|h|\leqslant 4q\psi(q)+2\psi(q)^2\\ h>-q^2}} r(q^2+h) \min\left(1, \frac{Q}{|h\mp 2q\psi(q)-\psi(q)^2|\sqrt{N}}\right). \tag{74}$$

When $q>Q$, let

$$W(q) = \sum_{r=Q^*}^{q} e(r\sqrt{n})$$

and suppose $Q^* \leqslant m \leqslant 2Q$. Then

$$\begin{aligned}\sum_{q=Q^*}^{m} q^{\frac12} e\left(q\sqrt{n}\right) &= \sum_{q=Q^*}^{m} q^{\frac12}(W(q)-W(q-1)) = -\sum_{q=Q^*}^{m-1} \left((q+1)^{\frac12}-q^{\frac12}\right) W(q) + m^{\frac12} W(m)\\ &\ll Q^{1/2}\min\left(m-Q^*+1, \frac{1}{\|\sqrt{n}\|}\right) = Q^{1/2}\min\left(\sum_{q=Q^*}^{m} 1, \frac{1}{\|\sqrt{n}\|}\right).\end{aligned}$$

We have

$$\int_{-\psi(Q^*)}^{\psi(Q^*)} \min\left(\sum_{\substack{Q<q\leqslant 2Q\\ \psi(q)\geqslant|t|}} 1, \frac{1}{\|\sqrt{n}\|}\right) dt \ll \min\left(\int_{-\psi(Q^*)}^{\psi(Q^*)} \sum_{\substack{Q<q\leqslant 2Q\\ \psi(q)\geqslant|t|}} dt, \frac{\psi(Q^*)}{\|\sqrt{n}\|}\right).$$

Therefore,

$$V(Q,N) \ll Q^{1/2} \sum_{1\leqslant n\leqslant N} \frac{r(n)}{n^{1/4}} \min\left(\sum_{Q<q\leqslant 2Q} \psi(q), \frac{\psi(Q^*)}{\|\sqrt{n}\|}\right).$$

Suppose that $1 \leqslant m \leqslant \sqrt{N}+\frac12$, and consider those n with $\left(m-\frac12\right)^2 < n \leqslant \left(m+\frac12\right)^2$. Then $\|\sqrt{n}\| = |\sqrt{n}-m| = \frac{|n-m^2|}{\sqrt{n}+m} \gg \frac{|n-m^2|}{m}$. Hence, when $m>1$,

$$\begin{aligned}\sum_{(m-1/2)^2<n\leqslant(m+1/2)^2} \frac{r(n)}{n^{1/4}} \quad \min\left(\sum_{Q<q\leqslant 2Q}\psi(q), \frac{\psi(Q^*)}{\|\sqrt{n}\|}\right)&\\ \ll \frac{r(m^2)}{m^{\frac12}} \sum_{Q<q\leqslant 2Q} \psi(q) + m^{\frac12} \sum_{0<|h|\leqslant m} \frac{r(m^2+h)}{|h|}\psi(Q^*)&\ .\end{aligned}$$

The Dirichlet series generating function for $r(m^2)$ is

$$4(1+2^{-s})^{-1}\zeta(s)^2 L(s)\zeta(2s)^{-1}\ ,$$

where $L(s)$ is the Dirichlet L-function formed from the non-trivial character modulo 4. Thus

$$\sum_{m\leqslant M} r(m^2) \sim \frac{4}{\pi} M\log M \tag{75}$$

and hence

$$\sum_{m\leqslant M} \frac{r(m^2)}{m^{\frac12}} \ll M^{\frac12}\log M\ .$$

As in the analysis of $E_\pm$ above we have

$$\sum_{2\leqslant m\leqslant M}\ \sum_{0<|h|\leqslant m} \frac{r(m^2+h)}{|h|} \ \ll\ M(\log M)^3\ .$$

Hence

$$V(Q,N)\ \ll\ N^{\frac14}(\log N)Q^{\frac12}\sum_{Q<q\leqslant 2Q}\psi(q) + N^{3/4}(\log N)^3 Q^{\frac12}\psi(Q^*)\ .$$

Hence, assuming $N\leqslant Q^3$, by (73),

$$\begin{aligned}\sum_{Q<q\leqslant 2Q} &\big(\Delta_0\big((q+\psi(q))^2\big)-\Delta_0\big((q-\psi(q))^2\big)\big)\\ &\ll\ N^{\frac14}(\log N)Q^{\frac12}\sum_{Q<q\leqslant 2Q}\psi(q) + N^{\frac34}(\log N)^3 Q^{\frac12}\psi(Q^*)\\ &\quad+\ \ Q^2N^{-\frac12}(\log Q)^3 + F_-(Q,N)+F_+(Q,N)\ .\end{aligned}\tag{76}$$

We now turn our attention to $F_\pm$. Were the factor $r(q^2+h)$ not to be present this would be a routine matter. The natural way to remove it is to consider an application of the Cauchy-Schwarz inequality. However one is then dependent on being able to bound $r(n)^2$, or $d(n)^2$ in terms of the divisors of n of order of magnitude at most $\sqrt{n}$. This is readily effected by an application of a combinatorial lemma.

Lemma 14 *Let $n\in\mathbb{N}$. Then there is a divisor m of n such that $m\leqslant\sqrt{n}$ and $d(n)\leqslant \max(2,d(m)^3)$.*

Proof. The conclusion follows at once when n has a prime factor p with $p>\sqrt{n}$. Otherwise choose a sequence $\{m_j\}$ as follows. Let m_1 be the largest divisor of n not exceeding $\sqrt{n}$. Then given $m_1,m_2,\dots,m_j$ with $m_1\dots m_j|n$ and no m_k exceeding $\sqrt{n}$ choose m_{j+1} to be the largest divisor of $n/(m_1\dots m_j)$ not exceeding $\sqrt{n}$. It follows that $m_4=1$ since otherwise we would have $m_1m_2>\sqrt{n}$ and $m_3m_4>\sqrt{n}$. Hence $n=m_1m_2m_3$ and $d(n)\leqslant d(m_j)^3$ for some j. ♠

By Lemma 14,

$$\begin{aligned}\sum_{Q<q\leqslant 2Q}\ &\sum_{\substack{0<|h|\leqslant 4q\psi(q)+2\psi(q)^2\\ h>-q^2}} d(q^2+h)^2\\ &\ll \sum_{0<|h|\leqslant 4Q\psi(Q^*)+2\psi(Q^*)^2}\ \sum_{Q<q\leqslant 2Q}\ \sum_{m|q^2+h,\ m\ll Q} d(m)^6\end{aligned}$$

and, by Lemma 13, this is

$$\begin{aligned} &\ll \sum_{0<d_1 d_2^2|j|\leqslant 4Q\psi(Q^*)+2\psi(Q^*)^2} \quad \sum_{l\ll Q/(d_1 d_2^2)} d(l d_1 d_2^2)^6 \, \frac{Q}{l d_1 d_2} \, d(l) \\ &\ll Q^2\,\psi(Q^*)\,(\log Q)^{128}\,. \end{aligned}$$

We also have

$$\sum_{Q<q\leqslant 2Q} \quad \sum_{|h|\leqslant 4q\psi(q)+2\psi(q)^2} \min\left(1, \frac{Q^2}{|h\mp 2q\psi(q)-\psi(q)^2|^2 N}\right) \ll Q + Q^2 N^{-\frac{1}{2}}\ .$$

Hence, by (75) and the Cauchy-Schwarz inequality,

$$F_{\pm}(Q,N) \ \ll\ Q\log Q \ +\ Q^{\frac{3}{2}}\psi(Q^*)^{\frac{1}{2}}(\log Q)^{64} \ +\ Q^2\psi(Q^*)^{\frac{1}{2}}(\log Q)^{64} N^{-\frac{1}{4}}\ ,$$

and the theorem follows from (76). ♠

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