On a problem of K. Mahler: Diophantine approximation and Cantor sets

Jason Levesley
York

Cem Salp
York

Sanju Velani*
York

Dedicated to Maurice Dodson on his retirement – finally!

Abstract

Let K denote the middle third Cantor set and $\mathcal{A} := \{3^n : n = 0, 1, 2, \ldots\}$. Given a real, positive function ψ let $W_{\mathcal{A}}(\psi)$ denote the set of real numbers x in the unit interval for which there exist infinitely many $(p, q) \in \mathbb{Z} \times \mathcal{A}$ such that $|x - p/q| < \psi(q)$. The analogue of the Hausdorff measure version of the Duffin-Schaeffer conjecture is established for $W_{\mathcal{A}}(\psi) \cap K$. One of the consequences of this is that there exist very well approximable numbers, other than Liouville numbers, in K – an assertion attributed to K. Mahler. Explicit examples of irrational numbers satisfying Mahler's assertion are also given.

2000 *Mathematics Subject Classification*: Primary 11J83; Secondary 11J82, 11K55

Keywords and phrases: Diophantine approximation, Cantor sets, Hausdorff measure and dimension.

1 Introduction

1.1 A problem of K. Mahler

A real number x is said to be *very well approximable* if there exists some $\epsilon > 0$ such that

$$|x - p/q| \; < \; q^{-(2+\epsilon)} \qquad \text{for infinitely many} \qquad (p, q) \in \mathbb{Z} \times \mathbb{N} \, . \tag{1}$$

Note that in view of Dirichlet's theorem, if $\epsilon = 0$ then every real number satisfies the above inequality. The set $\mathcal{W}$ of very well approximable numbers is a set of Lebesgue measure zero but nevertheless is large in the sense that it has maximal Hausdorff dimension; i.e. $\dim \mathcal{W} = 1$. A real number x is said to be a Liouville number if (1) is satisfied for all $\epsilon > 0$. It is well known that the set $\mathcal{L}$ of Liouville numbers is uncountable and of zero Hausdorff dimension; i.e. $\dim \mathcal{L} = 0$.

*Royal Society University Research Fellow

Throughout, let K denote the standard middle third Cantor set. Thus, K is precisely the set of real numbers in the unit interval whose base three expansions are free of the digit one. It is well known that K is a set of Lebesgue measure zero and

$$\dim K \; = \; \gamma \; := \frac{\log 2}{\log 3} \; .$$

The following assertion is attributed to K. Mahler – see Problem 35 in [5, §10.2] and Remark (iii) below.

Mahler's Assertion. *There exists very well approximable numbers, other than Liouville numbers, in the middle third Cantor set; i.e.*

$$(\mathcal{W} \setminus \mathcal{L}) \cap K \; \neq \; \emptyset \; .$$

It is rather surprising that this claim remains unproved to this day. A simple consequence of our main result – Theorem 1 below, is that

$$\dim(\, (\mathcal{W} \setminus \mathcal{L}) \cap K) \; \geq \; \gamma/2 \; . \tag{2}$$

Clearly, this 'strongly' implies the assertion of Mahler. In fact, we shall prove a lot more than (2). We show that there exist real numbers in K with any prescribed exact order – see §3.

In §8.1 we give explicit examples of irrational numbers satisfying Mahlers assertion.

Remarks:

(i) It is easy to see that the real number,

$$\xi := 2 \sum_{n=1}^{\infty} 3^{-n!}$$

is a Liouville number and therefore very well approximable. Furthermore, ξ clearly lies in K since only the digits 0 and 2 appear in its base 3 expansion. Hence, $\mathcal{W} \cap K \neq \emptyset$ and it is therefore natural to exclude Liouville numbers in the above assertion.

(ii) A real number x is said to be badly approximable if there exists a constant $c(x) > 0$ such that $|x - p/q| \geq c(x)/q^2$ for all rational p/q. Let **Bad** denote the set of badly approximable numbers. It is well known that $\mathbf{Bad} \cap K$ is non-empty and moreover that $\dim(\mathbf{Bad} \cap K) = \gamma$ – see for example [10, 12].

(iii) We have not been able to find a direct source to Mahler in which the above form of the assertion is stated. In view of Remarks (i) & (ii), Mahler's assertion as stated above is in all likelihood a more precise reformulation of the following problem posed by Mahler in [13, §2]: *How close can irrational elements of Cantor's set be approximated by rational numbers?* In any case, the results obtained in §3 provide a satisfactory and precise solution to this rather vague problem. In short, the 'vague' answer is: *However close one wishes!*

2 The set $W_{\mathcal{A}}(\psi)$ and our approach

Throughout, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a real, positive function and, unless explicitly stated otherwise, $\mathcal{A} := \{3^n : n = 0, 1, 2, \ldots\}$. Also

$$W_{\mathcal{A}}(\psi) := \{x \in [0,1] : |x - p/q| < \psi(q) \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathcal{A}\} ,$$

where 'i.m' stands for 'infinitely many'. Thus, $W_{\mathcal{A}}(\psi)$ is simply a hybrid of the classical set $W(\psi)$ of ψ–well approximable numbers in which the denominator of the rational approximates are restricted to the set $\mathcal{A}$; in other words if we were to put $\mathcal{A} = \mathbb{N}$ then $W_{\mathcal{A}}(\psi) = W(\psi)$. In the case $\psi : r \to r^{-\tau}$ with $\tau > 0$, we write $W_{\mathcal{A}}(\tau)$ for $W_{\mathcal{A}}(\psi)$. Thus, when $\mathcal{A} = \mathbb{N}$ then $W_{\mathcal{A}}(\tau) = W(\tau)$ is the classical set of τ–well approximable numbers.

2.1 The general metric theory for $W_{\mathcal{A}}(\psi) \cap K$

Let f be a dimension function and let $\mathcal{H}^f$ denote the Hausdorff f–measure – see §4.1. In short, our aim is to provide a complete metric theory for the set $W_{\mathcal{A}}(\psi) \cap K$. The following result achieves this goal in that it provides a simple criteria for the 'size' of the set $W_{\mathcal{A}}(\psi) \cap K$ expressed in terms of the general measure $\mathcal{H}^f$.

Theorem 1 *Let f be a dimension function such that $r^{-\gamma} f(r)$ is monotonic. Then*

$$\mathcal{H}^f(W_{\mathcal{A}}(\psi) \cap K) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^{\gamma} < \infty \\ \\ \mathcal{H}^f(K) & \text{if} \quad \sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^{\gamma} = \infty \end{cases} .$$

The convergence part of the above theorem is relatively straightforward if not trivial – see §5. The main substance is the divergent part. It is worth stressing that we do not assume that the function ψ is monotonic. Thus, within the framework under consideration the above theorem establishes the analogue of the general form of the Duffin-Schaeffer conjecture as formulated in [4]. The fact that we have not imposed the condition that $(p,q) = 1$ in the definition of $W_{\mathcal{A}}(\psi)$ is irrelevant since $\phi(q) := \#\{1 \leq t \leq q : (t,q) = 1\} = \frac{2}{3} q$ for any $q \in \mathcal{A}$ and so

$$\sum_{n=1}^{\infty} f(\psi(3^n)) \times (3^n)^{\gamma} \asymp \sum_{n=1}^{\infty} f(\psi(3^n)) \times (\phi(3^n))^{\gamma} .$$

For details regarding the original statement of Duffin and Schaeffer see [5, 9].

With $f : r \to r^s$ $(s \geq 0)$, an immediate consequence of Theorem 1 is the following corollary.

Corollary 1 *For $\tau \geq 1$, $\dim(W_{\mathcal{A}}(\tau) \cap K) = \gamma/\tau$. In particular, for $\tau > 1$*

$$\mathcal{H}^{\gamma/\tau}(W_{\mathcal{A}}(\tau) \cap K) = \infty .$$

Now if τ is strictly greater than two, then every point in $W_{\mathcal{A}}(\tau)$ is by definition very well approximable. Thus,

$$\dim(\mathcal{W} \cap K) \ \geq \ \gamma/2 \ .$$

This together with the fact that the set $\mathcal{L}$ of Liouville numbers is of dimension zero implies (2). In turn, this implies the assertion of Mahler.

3 Sets of exact order in K

Recall, that for $\tau > 0$ the set $W(\tau)$ of τ-well approximable numbers consists of real numbers such that

$$|x - p/q| \ < \ q^{-\tau} \qquad \text{for infinitely many} \qquad (p,q) \in \mathbb{Z} \times \mathbb{N} \ .$$

For a real number x, its *exact order* $\tau(x)$ is defined as follows:

$$\tau(x) := \sup\{\tau : x \in W(\tau)\} \ .$$

It follows from Dirichlet's theorem that $\tau(x) \geqslant 2$ for all $x \in \mathbb{R}$. For $\alpha \geq 2$, let $E(\alpha)$ denote the set of numbers with *exact order* α; that is

$$E(\alpha) := \{x \in \mathbb{R} : \tau(x) = \alpha\} \ .$$

Thus, $E(\alpha)$ consists of real numbers with 'order' of rational approximation sandwiched between $\alpha - \epsilon$ and $\alpha + \epsilon$ where $\epsilon > 0$ is arbitrarily small. The set $E(\alpha)$ is equivalent to the set of real numbers x for which Mahler's function $\theta_1(x)$ is equal to $\alpha - 1$; the general function $\theta_n(x)$ is central to Mahler's classification of transcendental numbers (see [1, 5, 8]). A simple consequence of Theorem 1 is the following result which trivially implies Mahler's assertion and provides a precise solution to the problem discussed in Remark (iii) of §1.1.

Corollary 2 *For $\alpha \geq 2$, $\dim(E(\alpha) \cap K) \ \geq \ \gamma/\alpha$. In particular, for $\alpha > 2$*

$$\mathcal{H}^{\gamma/\alpha}(E(\alpha) \cap K) \ = \ \infty \ .$$

Proof. When $\alpha = 2$, we have that $E(\alpha) = \mathbb{R}$ and so $\dim(E(\alpha) \cap K) = \dim K := \gamma$. Thus, without loss of generality assume that $\alpha > 2$. Let

$$\psi_1 : r \to r^{-\alpha} \qquad \text{and} \qquad \psi_2 : r \to r^{-\alpha}(\log r)^{-\frac{2\alpha}{\gamma}} \ .$$

It is easily verified that

$$W_{\mathcal{A}}(\psi_1) \backslash W_{\mathcal{A}}(\psi_2) \ \subset \ E(\alpha) \ . \tag{3}$$

Now with $f : r \to r^{\gamma/\alpha}$ we have that

$$\sum_{n=1}^{\infty} f(\psi_1(3^n)) \times (3^n)^{\gamma} \ = \ \sum_{n=1}^{\infty} 1 \ = \ \infty$$

and

$$\sum_{n=1}^{\infty} f(\psi_2(3^n)) \times (3^n)^\gamma \asymp \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .$$

By Theorem 1,

$$\mathcal{H}^{\gamma/\alpha}(W_{\mathcal{A}}(\psi_1) \cap K) = \infty \qquad \text{and} \qquad \mathcal{H}^{\gamma/\alpha}(W_{\mathcal{A}}(\psi_2) \cap K) = 0 .$$

Thus

$$\mathcal{H}^{\gamma/\alpha}(W_{\mathcal{A}}(\psi_1) \backslash W_{\mathcal{A}}(\psi_2) \cap K) = \infty ,$$

which together with (3) implies the desired measure and dimension statements. ♠

Explicit examples of irrational numbers in $E(\alpha) \cap K$ are given in §8.1.

Remark. It is evident from the above proof that the statement of Corollary 2 does not in anyway utilize the full power of Theorem 1. The argument outlined above can be modified to establish much stronger 'exact order' statements in the spirit of those in [2]. Essentially, the set $E(\alpha)$ in the statement of Corollary 2 can be replaced by sets $E(\psi, \phi)$ consisting of real numbers whose rational approximation properties are sandwiched between two functions ψ and ϕ with ϕ in some sense 'smaller' than ψ. In short, $E(\psi, \phi) := W_{\mathcal{A}}(\psi) \backslash W_{\mathcal{A}}(\phi)$.

4 Preliminaries

4.1 Hausdorff measures

In this section we give a brief account of Hausdorff measures. For further details see [7, 14]. A *dimension function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing function such that $f(r) \to 0$ as $r \to 0$. The Hausdorff f–measure with respect to the dimension function f will be denoted throughout by $\mathcal{H}^f$ and is defined as follows. Suppose F is a subset of $\mathbb{R}^k$. For $\rho > 0$, a countable collection $\{B_i\}$ of balls in $\mathbb{R}^k$ with radius $r(B_i) \leq \rho$ for each i such that $F \subset \bigcup_i B_i$ is called a *ρ-cover for F*. For a dimension function f define

$$\mathcal{H}^f_\rho(F) = \inf \sum_i f(r(B_i)),$$

where the infimum is taken over all ρ-covers of F. The *Hausdorff f–measure* $\mathcal{H}^f(F)$ of F with respect to the dimension function f is defined by

$$\mathcal{H}^f(F) := \lim_{\rho \to 0} \mathcal{H}^f_\rho(F) = \sup_{\rho > 0} \mathcal{H}^f_\rho(F) .$$

In the case that $f(r) = r^s$ $(s \geq 0)$, the measure $\mathcal{H}^f$ is the usual s–dimensional Hausdorff measure $\mathcal{H}^s$ and the Hausdorff dimension $\dim F$ of a set F is defined by

$$\dim F := \inf \{s : \mathcal{H}^s(F) = 0\} = \sup \{s : \mathcal{H}^s(F) = \infty\} .$$

In particular when s is an integer, $\mathcal{H}^s$ is comparable to the s–dimensional Lebesgue measure. Actually, $\mathcal{H}^s$ is a constant multiple of the s–dimensional Lebesgue measure.

4.2 The Mass Transference Principle

Let X be a compact set in $\mathbb{R}^n$. Suppose there exist constants $\delta > 0$, $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that

$$c_1\, r^\delta \;\leqslant\; \mathcal{H}^\delta(B) \;\leqslant\; c_2\, r^\delta \;,$$

for any ball $B = B(x,r)$ in X with $x \in X$ and $r \leqslant r_0$. Next, given a dimension function f and a ball $B = B(x,r)$ we define

$$B^f := B(x, f(r)^{1/\delta})\,.$$

When $f(r) = r^s$ for some $s > 0$ we also adopt the notation B^s, *i.e.* $B^s := B^{(r\mapsto r^s)}$. Thus, by definition, $B^\delta(x,r) = B(x,r)$. Given a sequence of balls B_i in X, $i = 1,2,3,\ldots$, as usual its limsup set is

$$\limsup_{i\to\infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i\geqslant j} B_i \;.$$

By definition, $\limsup_{i\to\infty} B_i$ is precisely the set of points in X which lie in infinitely many balls B_i.

The following Mass Transference Principle allows us to transfer $\mathcal{H}^\delta$-measure theoretic statements for lim sup subsets of X to general $\mathcal{H}^f$-measure theoretic statements.

Theorem (Mass Transference Principle) *Let X be as above and let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in X with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $r^{-\delta} f(x)$ is monotonic and suppose that for any ball B in X*

$$\mathcal{H}^\delta\big(B \cap \limsup_{i\to\infty} B_i^f\,\big) = \mathcal{H}^\delta(B)\;.$$

Then, for any ball B in X

$$\mathcal{H}^f\big(B \cap \limsup_{i\to\infty} B_i^\delta\,\big) = \mathcal{H}^f(B)\;.$$

The theorem is essentially Theorem 3 in [4]. It is simplified for the particular application we have in mind.

4.3 Positive and full measure sets

Let X be a compact set in $\mathbb{R}^n$ and μ be a finite measure supported on X. The measure μ is said to be *doubling* if there exists a constant $\lambda > 1$ such that for $x \in X$

$$\mu(B(x,2r)) \;\leq\; \lambda\, \mu(B(x,r))\;. \tag{4}$$

In this section we state two measure theoretic results which will be required during the course of establishing the divergent part of Theorem 1.

Lemma 1 *Let X be a compact set in $\mathbb{R}^n$ and let μ be a finite doubling measure on X such that any open set is μ measurable. Let E be a Borel subset of X. Assume that there are constants $r_0, c > 0$ such that for any ball B with $r(B) < r_0$ and centre in X we have that*

$$\mu(E \cap B) \ \geqslant \ c\,\mu(B) \ .$$

Then, E has full measure in X, i.e. $\mu(X \setminus E) = 0$.

For the proof see [3, §8].

Lemma 2 *Let X be a compact set in $\mathbb{R}^n$ and let μ be a finite measure on X. Also, let E_n be a sequence of μ-measurable sets such that $\sum_{n=1}^{\infty} \mu(E_n) = \infty$. Then*

$$\mu(\limsup_{n\to\infty} E_n) \ \geq \ \limsup_{Q\to\infty} \frac{\left(\sum_{s=1}^{Q} \mu(E_s)\right)^2}{\sum_{s,t=1}^{Q} \mu(E_s \cap E_t)} \quad .$$

Lemma 2 is relatively well known and is a generalization of the divergent part of the standard Borel–Cantelli lemma, see Lemma 5 in [16].

5 Proof of Theorem 1: the convergence part

We are given that f is a dimension function such that

$$\sum_{n=1}^{\infty} f(\psi(3^n)) \times \ (3^n)^{\gamma} \ < \ \infty \ . \tag{5}$$

In view of this, $\psi(3^n) \to 0$ as $n \to \infty$. Thus given any $\rho > 0$, there exists an integer $n_0(\rho)$ such that

$$\psi(3^n) \leq \rho \quad \text{for all} \quad n \geq n_0(\rho) \ .$$

Furthermore and without loss of generality we can assume that $n_0(\rho) \to \infty$ as $\rho \to 0$. Now for $n \in \mathbb{N}$, let

$$\begin{aligned} A_n \quad &:= \bigcup_{0 \,\leq\, p \,\leq\, 3^n} B(\tfrac{p}{3^n}, \ \psi(3^n)) \ \cap \ K \\ &= \bigcup_{\substack{0 \,\leq\, p \,\leq\, 3^n: \\ \frac{p}{3^n} \in K}} B(\tfrac{p}{3^n}, \ \psi(3^n)) \ \cap \ K \end{aligned}$$

Then by definition, $W_{\mathcal{A}}(\psi) \cap K = \limsup_{n\to\infty} A_n$ and for each $m \in \mathbb{N}$ we have that

$$W_{\mathcal{A}}(\psi) \cap K \ \subset \ \bigcup_{n \geq m} A_n \ .$$

It now follows from the definition of $\mathcal{H}^f$ that for any $\rho > 0$,

$$\begin{aligned}\mathcal{H}^f_\rho(W_{\mathcal{A}}(\psi)\cap K) \quad &\ll \quad \sum_{n\geq n_0(\rho)} f(\psi(3^n)) \ \times \ \#\{0 \leq p \leq 3^n : \tfrac{p}{3^n} \in K\} \\ &\ll \quad \sum_{n\geq n_0(\rho)} f(\psi(3^n)) \times (3^n)^\gamma \ .\end{aligned}$$

In view of (5) and the fact that $n_0(\rho) \to \infty$ as $\rho \to 0$, we have that

$$\sum_{n\geq n_0(\rho)} f(\psi(3^n)) \times (3^n)^\gamma \quad \to \quad 0 \quad \text{as} \quad \rho \to 0 \ .$$

Thus, $\mathcal{H}^f(W_{\mathcal{A}}(\psi)\cap K) = 0$ as required. ♠

6 Proof of Theorem 1: the divergent part

The divergent part of Theorem 1 constitutes the main substance of theorem. The proof will be split into various key and natural steps.

6.1 A reduction to the measure μ

We begin by imposing a co-primeness condition. Let $W^*_{\mathcal{A}}(\psi)$ denote the set of $x \in [0,1]$ for which there exist infinitely many co-prime $(p,q) \in \mathbb{Z} \times \mathcal{A}$ such that

$$|x - p/q| < \psi(q) \ .$$

Trivially, $W^*_{\mathcal{A}}(\psi) \subset W_{\mathcal{A}}(\psi)$ and so

$$\mathcal{H}^f(W^*_{\mathcal{A}}(\psi)\cap K) = \mathcal{H}^f(K) \quad \Longrightarrow \quad \mathcal{H}^f(W_{\mathcal{A}}(\psi)\cap K) = \mathcal{H}^f(K) \ . \tag{6}$$

Recall, that $\gamma := \dim K$. Let μ denote the restriction of the γ–dimensional Hausdorff measure to K; that is

$$\mu \ := \ \mathcal{H}^\gamma|_K \ .$$

It is well known that μ is a finite measure supported on K and moreover there exist constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that

$$c_1 \, r^\gamma \ \leqslant \ \mathcal{H}^\gamma(B\cap K) \ := \ \mu(B) \ \leqslant \ c_2 \, r^\gamma \ , \tag{7}$$

for any ball $B = B(x,r)$ with $x \in X$ and $r \leqslant r_0$ – see for example [7, 14]. Note that (7) implies (4); i.e. the measure μ is doubling.

As we shall soon see, the following theorem is an important restatement of the divergent part of Theorem 1 in terms of the set $W^*_{\mathcal{A}}(\psi)$ and the measure μ.

Theorem 2

$$\mu(W^*_{\mathcal{A}}(\psi)) = \mu(K) \qquad \text{if} \qquad \sum_{n=1}^{\infty} (\psi(3^n) \times \ 3^n)^{\gamma} \ = \ \infty \ \ .$$

Note that by definition we have that

$$\mu(W^*_{\mathcal{A}}(\psi)) \ = \ \mathcal{H}^{\gamma}(W^*_{\mathcal{A}}(\psi) \cap K) \qquad \text{and} \qquad \mu(K) \ = \ \mathcal{H}^{\gamma}(K) \ .$$

The statement of Theorem 2 is the precise analogue of the standard Duffin-Schaeffer conjecture for the set $W^*_{\mathcal{A}}(\psi) \cap K$. The following result enables us to reduce the proof of the divergent part of Theorem 1 to that of establishing Theorem 2.

Theorem 3

Theorem 2 $\implies$ *Theorem 1 (divergent part)*

This theorem is a simple consequence of the Mass Transference Principle.

Proof of Theorem 3. Without loss of generality assume that $\psi(3^n) \to 0$ as $n \to \infty$. Otherwise, $W^*_{\mathcal{A}}(\psi) = [0,1]$ and the statement is obvious. We are given that $r^{-\gamma} f(r)$ is monotonic and that

$$\sum_{n=1}^{\infty} f(\psi(3^n)) \times \ (3^n)^{\gamma} \ = \ \infty \ \ .$$

Let $\theta : r \to \theta(r) := f(\psi(r))^{\frac{1}{\gamma}}$. Then,

$$\sum_{n=1}^{\infty} (\theta(3^n) \times \ 3^n)^{\gamma} \ = \ \infty \ \ .$$

Thus, Theorem 2 implies that $\mathcal{H}^{\gamma}\big(B \cap (W^*_{\mathcal{A}}(\theta) \cap K)\big) = \mathcal{H}^{\gamma}(B)$ for any ball B in K. It now follows via the Mass Transference Principle that $\mathcal{H}^{f}\big(B \cap (W^*_{\mathcal{A}}(\psi) \cap K)\big) = \mathcal{H}^{f}(B)$ for any ball B in K. In particular, this implies that $\mathcal{H}^{f}\big(W^*_{\mathcal{A}}(\psi) \cap K) = \mathcal{H}^{f}(K)$ which together with (6) completes the proof of Theorem 3. ♠

The upshot is that divergent part of Theorem 1 is a consequence of Theorem 2.

Remark. For any ball B in the unit interval $[0,1]$ and $n \in N$, we have that

$$\mu\Big(B \ \cap \bigcup_{0 \le p \le 3^n} B(\tfrac{p}{3^n}, \ 3^{-n})\Big) \ = \ \mu(B) \qquad (\mu := \mathcal{H}^{\gamma}|_K) \ . \tag{8}$$

This simply makes use of the fact that the distance between consecutive rationals with fixed denominator q is $1/q$. In view of (8), if we assume that ψ is monotonic then the divergent part of Theorem 1 is easily seen to be a straightforward consequence of the local m-ubiquity results established in [3]. Note that Theorem 1 under the assumption that ψ is monotonic is enough to determine Corollaries 1 and 2. Recall, that Mahler's assertion trivially follows from the dimension part of Corollary 1. In fact, (8) together with the m-ubiquity result established in [6] some ten years ago is already enough to yield the dimension part of Corollary 1.

6.2 Proof of Theorem 2

We are given that

$$\sum_{n=1}^{\infty} (\psi(3^n) \times \; 3^n)^{\gamma} \;=\; \infty \quad . \tag{9}$$

Step 1. We notice that there is no loss of generality in assuming that

$$\psi(3^n) \leq c\, 3^{-n} \qquad \text{for all } n \in \mathbb{N} \text{ and } c > 0 \ .$$

Suppose for the moment that this was not case and define

$$\Psi : r \rightarrow \Psi(r) := \min\{c/r, \psi(r)\} \ .$$

In view of (9), it is easily verified that

$$\textstyle\sum_{n=1}^{\infty} (\Psi(3^n) \times \; 3^n)^{\gamma} \;=\; \infty \quad .$$

Furthermore, $W^*_{\mathcal{A}}(\Psi) \subset W^*_{\mathcal{A}}(\psi)$ and so it suffices to establish Theorem 2 for Ψ. In particular, there is no loss of generality in assuming that

$$\psi(3^n) \;<\; \tfrac{1}{2}\, 3^{-n} \qquad \text{for all} \qquad n \in \mathbb{N} \quad . \tag{10}$$

Step 2. Let B be an arbitrary ball centered at a point in K such that $\mu(2B)$ satisfies (7). Trivially, $\mu(B) \asymp \mu(2B)$. The aim is to show that

$$\mu(W^*_{\mathcal{A}}(\psi) \cap B) \;\geq\; \mu(B)/C \ , \tag{11}$$

where $C > 0$ is a constant independent of B. Theorem 2 is then a consequence of Lemma 1.

Let $r(B)$ denote the radius of B. Throughout, $t_0 := t_0(B)$ is a sufficiently large integer so that

$$3^{-t_0} \;<\; r(B) \quad . \tag{12}$$

For $n \in N$, let

$$A^*_n(B) \;:=\; \bigcup^{\circ}_{\substack{0 \,\leq\, p \,\leq\, 3^n: \\ (p,3^n)=1}} B(\tfrac{p}{3^n},\; \psi(3^n)) \;\cap\; B$$

The fact that the above union is disjoint is a consequence of (10). Furthermore, let $B^*_n(r)$ denote a generic ball centered at a reduced rational $p/3^n$ $(0 \leq p \leq 3^n)$ and radius $2r \leq 3^{-n}$. By considering the n'th level L_n of the Cantor set construction in which there are 2^n intervals I_n of common length 3^{-n}, it is easily verified that if $B^*_n(r) \cap K \neq \emptyset$ then its center $p/3^n \in K$; i.e. the rational $p/3^n$ is an end point of some interval I_n in L_n. Now the measure μ is supported on K and satisfies (7). Hence, for $n > t_0$

$$\mu\big(B^*_n(r)\big) \asymp r^{\gamma} \qquad \text{if} \qquad B^*_n(r) \cap K \neq \emptyset \ .$$

Also, it is easily verified that for $n > t_0$

$$\begin{aligned} \#\{B_n^*(r) \subset B : B_n^*(r) \cap K \neq \emptyset\} &\asymp \#\{B_n^*(\tfrac{1}{2}3^{-n}) \subset B : B_n^*(\tfrac{1}{2}3^{-n}) \cap K \neq \emptyset\} \\ &\asymp \frac{\mu(B)}{(3^{-n})^\gamma} \ . \end{aligned} \tag{13}$$

In view of the above discussion regarding $B_n^*(r)$, a straight forward geometric argument yields that for $n > t_0$

$$\mu(A_n^*(B)) \asymp \frac{\mu(B)}{(3^{-n})^\gamma} \times (\psi(3^n))^\gamma = \mu(B) \ (\psi(3^n) \times \ 3^n)^\gamma \quad . \tag{14}$$

This together with (9) implies that

$$\sum_{n=1}^{\infty} \mu(A_n^*(B)) = \infty \ . \tag{15}$$

Finally, note that

$$W_{\mathcal{A}}^*(\psi) \cap B = \limsup_{n \to \infty} A_n^*(B) \ .$$

Step 3. The key to establishing Theorem 2 is the following 'local' pairwise quasi–independence result.

Lemma 3 (Local pairwise quasi–independence) *There exists a constant $C > 0$ such that for all $n > m > t_0$,*

$$\mu(A_m^*(B) \cap A_n^*(B)) \ \leq \ \frac{C}{\mu(B)} \ \mu(A_m^*(B)) \ \mu(A_n^*(B)) \quad . \tag{16}$$

Hence, (15) and Lemma 3 together with Lemma 2 implies (11). This completes the proof of Theorem 2 assuming of course the local pairwise quasi–independence result – this we now prove.

6.3 Proof of Lemma 3: Local pairwise quasi–independence

Recall, that B is some fixed ball centered at a point in K such that $\mu(2B)$ satisfies (7) and $t_0 := t_0(B)$ is chosen sufficiently large so that (12) is satisfied.

Fix a pair n and m with $n > m > t_0$. We proceed by consider two cases depending on the size of $\psi(3^n)$ compared to $\frac{1}{2}\, 3^{-n}$.

Case (i): $n > m > t_0$ such that $3^{-n} \geq 2\,\psi(3^m)$. Fix some ball $B(p/3^m, \psi(3^m))$ where $0 \leq p \leq 3^m$. It is easily verified on assuming (10), that

$$B\big(\tfrac{p}{3^m}, \psi(3^m)\big) \ \cap \ B\big(\tfrac{t}{3^n}, \psi(3^n)\big) \ = \ \emptyset \qquad \text{for any} \ \ 0 \leq t \leq 3^n \ \ \text{with} \ \ (t, 3^n) = 1 \ .$$

In view of this,

$$A_m^*(B) \cap A_n^*(B) \ = \ \emptyset \ .$$

Thus, $\mu(A_m^*(B) \cap A_n^*(B)) = 0$ and (16) is trivially satisfied for any constant $C \geq 0$.

Case (ii): $n > m > t_0$ such that $3^{-n} < 2\,\psi(3^m)$. For the sake of clarity, let us write $B_n^*(\psi)$ for the generic ball $B_n^*(\psi(3^n))$ and simply B_n^* for the generic ball $B_n^*(\frac{1}{2}\,3^{-n})$. Recall, that by definition a generic ball $B_n^*(r)$ is centered at a reduced rational $p/3^n$ $(0 \leq p \leq 3^n)$ and if it has non-empty intersection with K then $p/3^n \in K$ – see Step 2 in §6.2.

It is easily verified that

$$\begin{aligned}
\mu(A_m^*(B) \cap A_n^*(B)) \ &:= \ \mu\Big(\Big(\bigcup^{\circ}_{\substack{0 \,\leq\, p \,\leq\, 3^m: \\ (p,3^m)=1}} B(\tfrac{p}{3^m}, \ \psi(3^m)) \ \cap \ B \Big) \ \cap \ A_n^*(B) \Big) \\
&\leq \ \sum_{\substack{0 \,\leq\, p \,\leq\, 3^m \,:\, (p,3^m)=1, \\ B(\frac{p}{3^m},\psi(3^m))\,\cap\, B \,\cap\, K \neq \emptyset}} \mu\big(B(\tfrac{p}{3^m}, \ \psi(3^m)) \ \cap \ A_n^*(B)\big) \\
&\leq \ \mathcal{N}(m,B) \ \times \ \mu\big(B_m^*(\psi) \ \cap \ A_n^*(B)\big) \ , \qquad (17)
\end{aligned}$$

where

$$\mathcal{N}(m,B) \ := \ \#\{B_m^*(\psi) \ : \ B_m^*(\psi) \ \cap \ B \ \cap \ K \ \neq \ \emptyset\} \ .$$

In view of the fact that $m > t_0$ and t_0 satisfies (12), we have that

$$\begin{aligned}
\mathcal{N}(m,B) \ &\overset{(10)}{\leq} \ \#\{B_m^*(\psi) \subset 2B \ : \ B_m^*(\psi) \ \cap \ K \ \neq \ \emptyset\} \\
&\overset{(13)}{\leq} \ c_3 \ \mu(B) \ \times \ (3^m)^\gamma \ \ , \qquad (18)
\end{aligned}$$

where $c_3 > 0$ is a constant. We now obtain an upper bound for $\mu\big(B_m^*(\psi) \cap A_n^*(B)\big)$. Without loss of generality, we assume that $B_m^*(\psi) \cap B \cap K \neq \emptyset$ since otherwise $\mu\big(B_m^*(\psi) \cap A_n^*(B)\big) = 0$ and there is nothing to prove. For a fixed generic ball $B_m^*(\psi)$, a relatively simple geometric argument yields that

$$\begin{aligned}
\#\{B_n^*(\psi) \ : \ B_n^*(\psi) \ \cap \ B_m^*(\psi)\, K \ \neq \ \emptyset\} \ \ &\leq \ \ \#\{B_n^* \ : \ B_n^* \ \cap \ B_m^*(\psi) \ \cap \ K \ \neq \ \emptyset\} \\
&\overset{(7)}{\leq} \ \tfrac{c_2}{c_1} \ \big(\psi(3^m) \ 3^{-n}\big)^\gamma \ + \ 2 \ .
\end{aligned}$$

The 'plus 2' term above simply accounts for 'edge effects'. Hence,

$$\begin{aligned}
\mu\big(B_m^*(\psi) \ \cap \ A_n^*(B)\big) \ \ &\leq \ \ \mu\big(B_n^*(\psi)\big) \ \times \ \big(\tfrac{c_2}{c_1} \ \big(\psi(3^m) \ 3^{-n}\big)^\gamma \ + \ 2 \big) \\
&\overset{(7)}{\leq} \ \tfrac{c_2^2}{c_1} \ \big(\psi(3^n) \ \psi(3^m) \ 3^{-n}\big)^\gamma \ + \ 2\,c_2\big(\psi(3^n)\big)^\gamma \ \ . \qquad (19)
\end{aligned}$$

On combining (17), (18) and (19), we obtain that

$$\begin{aligned}
\mu\big(A^*_m(B) \cap A^*_n(B)\big) \quad &\leq \quad c_2\, c_3\; \mu(B)\; \big(\, 3^n\, \psi(3^n)\; 3^m\, \psi(3^m)\,\big)^{\gamma} \times \big(\tfrac{c_2}{c_1} + 2\,\big(3^n\psi(3^m)\big)^{-\gamma}\big) \\
&\overset{case\,(ii)}{\leq} \quad c_4\; \mu(B)\; \big(\, 3^n\, \psi(3^n)\; 3^m\, \psi(3^m)\,\big)^{\gamma} \\
&\overset{(14)}{\leq} \quad \frac{C}{\mu(B)}\, \mu\big(A^*_m(B)\big)\; \mu\big(A^*_n(B)\big)\ ,
\end{aligned}$$

where $c_4 > 0$ and $C > 0$ are absolute constants. Thus, (16) is satisfied.

On combining the above two cases concludes the proof of the local pairwise quasi–independence statement – Lemma 3. ♠

7 General 'missing digit' sets $K_{J(b)}$

Let $b \geq 3$ be an integer and let $J(b)$ be a proper subset of $S := \{0, 1, \ldots, b-1\}$ with $\#J(b) \geq 2$. Furthermore, let $K_{J(b)}$ denote the set of real numbers in the unit interval $[0, 1]$ whose base b expansions consist exclusively of digits within $J(b)$. Equivalently, $K_{J(b)}$ is the set of real numbers in the unit interval whose base b expansions are free of the digits in $S \setminus J(b)$. Thus, $K_{J(b)}$ is a natural generalization of the middle third Cantor set K – simply put $b = 3$ and let $J(b) = \{0, 2\}$. Naturally, one can ask whether Mahler's assertion remains valid with the middle third Cantor set K replaced by the general Cantor set $K_{J(b)}$.

It is easily versified that

$$\dim K_{J(b)} \ = \gamma^* \ := \ \frac{\log \#J(b)}{\log b}\ .$$

Moreover, there exists a finite measure μ supported on $K_{J(b)}$ such that

$$\mathcal{H}^{\gamma^*}(B \cap K_{J(b)}) \ := \ \mu(B) \ \asymp \ r^{\gamma^*}\ ,$$

for any ball $B = B(x.r)$ with $x \in K_{J(b)}$ and $r \leq r_0$. Both the dimension and measure statements above can be deduced from standard results in fractal geometry – see for example [7, 14].

Now let $\mathcal{A}(b) := \{b^n : n = 0, 1, 2, \ldots\}$ and consider the set

$$W_{\mathcal{A}(b)}(\psi) := \{x \in [0, 1] : |x - p/q| < \psi(q) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathcal{A}(b)\}\ .$$

The arguments involved in establishing Theorem 1 can be modified in the obvious manner to yield the following generalization of Theorem 1.

Theorem 4 *Let f be a dimension function such that $r^{-\gamma^*}f(r)$ is monotonic. Then*

$$\mathcal{H}^f(W_{\mathcal{A}(b)}(\psi)\cap K_{J(b)}) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} f(\psi(b^n)) \times \ (b^n)^{\gamma^*} \ < \ \infty \\ \mathcal{H}^f(K_{J(b)}) & \text{if} \quad \sum_{n=1}^{\infty} f(\psi(b^n)) \times \ (b^n)^{\gamma^*} \ = \ \infty \end{cases} .$$

Simple consequences of the theorem are the following corollaries which clearly imply the existence of very well approximable numbers, other than Liouville numbers, in the Cantor set $K_{J(b)}$. In fact, it follows that

$$\dim(\,(\mathcal{W}\setminus\mathcal{L})\cap K_{J(b)}) \ \geq \ \gamma^*/2 \ .$$

Corollary 3 *For $\tau \geq 1$, $\dim(W_{\mathcal{A}(b)}(\tau)\cap K_{J(b)}) \ = \ \gamma^*/\tau$. In particular, for $\tau > 1$*

$$\mathcal{H}^{\gamma^*/\tau}(W_{\mathcal{A}(b)}(\tau)\cap K_{J(b)}) \ = \ \infty \ .$$

Corollary 4 *For $\alpha \geq 2$, $\dim(E(\alpha)\cap K_{J(b)}) \ \geq \ \gamma^*/\alpha$. In particular, for $\alpha > 2$*

$$\mathcal{H}^{\gamma^*/\alpha}(E(\alpha)\cap K_{J(b)}) \ = \ \infty \ .$$

8 Concluding Remarks

8.1 Mahler's assertion – explicit examples

For any real number $\tau > 2$, consider the irrational number

$$\xi \ = \ \xi(\tau) \ := \ 2\sum_{n=1}^{\infty} 3^{-\tau_n} \qquad \text{where} \qquad \tau_n := [\tau^n] \ .$$

Clearly ξ is irrational since its base 3 expansion is not periodic. Clearly ξ lies in K since only the digits 0 and 2 appear in its base 3 expansion. The following result implies that

$$\xi \ \in \ (\mathcal{W}\setminus\mathcal{L})\cap K \ .$$

Thus, for each $\tau > 2$ the explicit irrational number $\xi = \xi(\tau)$ satisfies Mahler's assertion.

Lemma 4 *(i) If $\tau \geq \frac{1}{2}(\sqrt{5}+3)$, then*

$$\xi \ \in \ E(\tau) \ \cap \ K \ .$$

(ii) If $2 < \tau < \frac{1}{2}(\sqrt{5}+3)$, then for any $\epsilon > 0$

$$\xi \ \in \ \big(W(\tau-\epsilon)\setminus W\big(\tfrac{2\tau-1}{\tau-1}+\epsilon\big)\big) \ \cap \ K \ .$$

Remark. Note that we are only able to conclude the stronger exact order statement (part (i)) under the assumption that $\tau \geq \frac{1}{2}(\sqrt{5}+3)$. However, statements of this type are reminiscent of numerous results in transcendence theory; see, for example [5, Theorems 7.7 & 8.8]. By the definition of the exact order set $E(\tau)$, we trivially have that $\xi \in \mathcal{W} \setminus \mathcal{L}$. Regarding part (ii) of the above lemma, by choosing $\epsilon > 0$ sufficiently small so that $\tau - \epsilon > 2$, we also have that $\xi \in \mathcal{W} \setminus \mathcal{L}$.

Proof of Lemma 4. As already mentioned above, the fact that $\xi \in K$ is trivial. For $s \in \mathbb{N}$, let

$$q_s \ := \ 3^{\tau_s} \quad \text{and} \quad p_s \ := \ q_s \ \times \ 2 \sum_{n=1}^{s} 3^{-\tau_n} \ .$$

It is easily verified that for all $s \in \mathbb{N}$, $(p_s, q_s) = 1$ and that

$$\tfrac{1}{3}\, {q_s}^{\tau} \ < \ q_{s+1} \ < \ 3^{\tau}\, {q_s}^{\tau} \ . \tag{20}$$

Furthermore,

$$\frac{2}{3^{\tau}} \, \frac{1}{{q_s}^{\tau}} \ \overset{(20)}{<} \ \frac{2}{q_{s+1}} \ < \ \left| \xi - \frac{p_s}{q_s} \right| \ < \ \frac{3}{q_{s+1}} \ \overset{(20)}{<} \ \frac{9}{{q_s}^{\tau}} \ . \tag{21}$$

Fix some $\epsilon > 0$. Since $9/q_s^{\tau} < 1/q_s^{\tau-\epsilon}$ for all sufficiently large s, we have that

$$\xi \in W(\tau - \epsilon) \ .$$

By assumption, $\tau > 2$. Therefore, there exists an integer $s_{\rm o}$, such that for all $s \geq s_{\rm o}$

$$\left| \xi - \frac{p_s}{q_s} \right| \ \overset{(21)}{<} \ \frac{9}{{q_s}^{\tau}} \ < \ \frac{1}{2\, {q_s}^2} \ .$$

It follows, via a standard result in the theory of continued fractions (Legendre's theorem), that for each $s \geq s_{\rm o}$ the rational p_s/q_s is a convergent to ξ. Now, with a slight abuse of notation, it is well known that succussive convergents p_n/q_n and p_{n+1}/q_{n+1} to any irrational number x lie on either side of x and that

$$\frac{1}{q_n(q_n + q_{n+1})} \ < \ \left| x - \frac{p_n}{q_n} \right| \ < \ \frac{1}{q_n q_{n+1}} \ . \tag{22}$$

Also, the denominators $\{q_n : n \geq 2\}$ of the convergents form a strictly increasing sequence. For these standard statements from the theory of continued fractions the reader is referred to [5, §1.2]. By construction or rather by definition, we have that

$$\frac{p_s}{q_s} \ < \ \frac{p_{s+1}}{q_{s+1}} \ < \ \xi \ .$$

Thus, for $s \geq s_{\rm o}$ the convergents p_s/q_s and p_{s+1}/q_{s+1} are not succussive convergents to ξ and so there exists at least one other convergent with denominator between q_s and q_{s+1}. With this in mind, let p_*/q_* be the next convergent to ξ after p_s/q_s. Thus, $\frac{p_*}{q_*} \neq \frac{p_{s+1}}{q_{s+1}}$ and $q_s < q_* < q_{s+1}$. In view of (22), we have that

$$\frac{1}{q_s(q_s + q_*)} \ < \ \left| \xi - \frac{p_s}{q_s} \right| \ < \ \frac{1}{q_s q_*} \ .$$

This together with (21) and the fact that $\tau > 2$, implies that there exists an integer $s_1 \geq s_\text{o}$ such that for $s \geq s_1$

$$\tfrac{1}{10}\, q_s^{\tau-1} \ < \ q_* \ < \ \tfrac{3^\tau}{2}\, q_s^{\tau-1} \quad . \tag{23}$$

Now suppose there exists a rational p/q and $s \geq s_1$ such that $q_s < q < q_{s+1}$ and

$$\left|\xi - \frac{p}{q}\right| \ < \ \frac{1}{q^{\tau+\epsilon}} \quad . \tag{24}$$

Thus, p/q is a convergent to ξ with

$$q \ \geq \ q_* \quad . \tag{25}$$

Now let $\tilde{p}/\tilde{q}$ be the next convergent to ξ after p/q. Then,

$$\tilde{q} \ \leq \ q_{s+1} \quad . \tag{26}$$

In view of (22), we have that

$$\frac{1}{q(q+\tilde{q})} \ < \ \left|\xi - \frac{p}{q}\right| \ < \ \frac{1}{q\,\tilde{q}} \quad .$$

This together with (24) implies that

$$\begin{aligned} \tilde{q} \ > \ \tfrac{1}{2}\, q^{\tau+\epsilon-1} \ & \overset{(25)}{\geq} \ \tfrac{1}{2}\, q_*^{\tau+\epsilon-1} \\ & \overset{(23)}{>} \ \tfrac{1}{2}\left(\tfrac{1}{10}\right)^{\tau+\epsilon-1} \left(q_s^{\tau-1}\right)^{\tau+\epsilon-1} . \end{aligned} \tag{27}$$

Suppose for the moment that $\tau \geq \frac{1}{2}(\sqrt{5}+3)$. Then,

$$(\tau-1)(\tau+\epsilon-1) \ > \ \tau \quad . \tag{28}$$

It follows that there exists an integer $s_2 \geq s_1$ such that for $s \geq s_2$

$$\tfrac{1}{2}\left(\tfrac{1}{10}\right)^{\tau+\epsilon-1} \left(q_s^{\tau-1}\right)^{\tau+\epsilon-1} \ > \ 3^\tau q_s^\tau \ \overset{(20)}{>} \ q_{s+1} \quad .$$

This together with (27) implies that $\tilde{q} > q_{s+1}$ which in view of (26) is a contradiction. The upshot is that there are at most finitely many rationals p/q satisfying inequality (24). Hence,

$$\xi \notin W(\tau+\epsilon)$$

and this completes the proof of part (i) of the lemma. Part (ii) of the lemma follows from the observation that if

$$\epsilon \ > \ \frac{\tau}{\tau-1} - \tau + 1 \ > \ 0 \quad ,$$

then (28) is satisfied and we are still able to force the contradiction that $\tilde{q} > q_{s+1}$.

♠

Remark. Fix some $\tau \geq \frac{1}{2}(\sqrt{5}+3)$. Let $\lambda > 0$ be a real number. On adapting the above argument in the obvious manner, it is readily verified that

$$\xi = \xi(\tau,\lambda) \ := \ 2\sum_{n=1}^{\infty} 3^{-[\lambda\tau^n]} \quad \in \quad E(\tau) \cap K \quad .$$

This clearly yields uncountably many explicit irrational numbers in $E(\tau) \cap K$.

8.2 Mahler's assertion – what do we really expect?

We suspect that lower bound estimate (2) for $\dim((\mathcal{W} \setminus \mathcal{L}) \cap K)$ is far from the truth. It is highly likely that:

$$\dim((\mathcal{W} \setminus \mathcal{L}) \cap K) \ = \ \gamma \ := \ \dim K \ \ . \tag{29}$$

Recall, estimate (2) is obtained by considering rationals with denominators restricted to the set $\mathcal{A} := \{3^n : n = 0, 1, 2, \ldots\}$ and showing that $\dim(W_{\mathcal{A}}(\tau) \cap K) = \gamma/\tau$ for $\tau \geq 1$ – Corollary 1. The restriction of the denominators to the set $\mathcal{A}$ is absolutely paramount to most of the arguments employed in this paper – it forces the rationals of interest to lie in K. For a brief moment, let us forget about intersecting well approximable sets with K. It is well known that $\dim W_{\mathcal{A}}(\tau) = 1/\tau$ $(\tau \geq 1)$ where as $\dim W(\tau) = 2/\tau$ $(\tau \geq 2)$ – see [3, §12.5] and [9, Chp.10]. Thus, by restricting the denominators to the set $\mathcal{A}$, the dimension is reduced or rather re-scaled by a factor of 1/2. It is therefore reasonable to speculate that this same scaling is present when considering the dimensions of the sets $W_{\mathcal{A}}(\tau) \cap K$ and $W(\tau) \cap K$. This leads us to the following statement which would imply (29) in the same way that Corollary 1 is used to establish (2).

Statement 1. For $\tau \geq 2$, $\dim(W(\tau) \cap K) \ = \ 2\,\gamma/\tau$.

Remark. It is reasonably easy to obtain the upper bound estimate; namely $\dim(W(\tau) \cap K) \leq 2\,\gamma/\tau$. It also follows directly from Corollary 2 of [15]. Thus the problem lies in establishing the complementary lower bound estimate.

Even if the above statement turns out to be false, we have every reason to believe the following weaker statement which would still imply (29).

Statement 2. Let $\tau > 2$. Then, $\dim(W(\tau) \cap K) \ \to \ \gamma$ as $\tau \to 2$.

Ideally, one would like to obtain a complete metric theory for the set $W(\psi) \cap K$. The above statements would then be simple corollaries of such a theory. Equivalently, one would like to obtain the analogue of Theorem 1 for the set $W(\psi) \cap K$. Without, imposing the condition that ψ is monotonic the problem seems harder than establishing the Duffin-Schaeffer conjecture. It is quite likely that the necessary ideas and techniques required in proving the analogue of Theorem 1 for the set $W(\psi) \cap K$ would also lead to a proof of the Duffin-Schaeffer conjecture. In view of this, suppose we impose the condition that ψ is monotonic. It is then relatively straightforward to obtain the following convergent result:

$$\mathcal{H}^f(W(\psi) \cap K) \ = \ 0 \qquad \text{if} \qquad \sum_{n=1}^{\infty} f(\psi(n)) \times \ n^{2\gamma - 1} \ < \ \infty \ .$$

When $f : r \to r^s$, the above statement follows directly from Theorem 2 of [15]. The ideas in [15] can be easily modified to deal with general dimension functions. As far as we are aware, there has been absolutely no progress towards establishing a divergent theory even in the case that $\mathcal{H}^f$ is the Cantor measure $\mu := \mathcal{H}^\gamma|_K$.

It is worth mentioning that progress has recently been made towards establishing a convergent theory for general fractal sets supporting so called 'friendly' measures. The Cantor set K together with the Cantor measure μ fall into this general theory – see [11, 15, 17].

8.3 Another problem of Mahler concerning Cantor sets

It is believed by most experts that the base b expansion of any irrational algebraic number x is normal. Restricting to the case $b = 3$, this would mean that each of the digits $0, 1$ and 2 occurs the expected number of times in the base 3 expansion of x. In particular, for N sufficiently large we expect the digit 1 to occur around $N/3$ times in the first N terms of the expansion of x in base 3. However, it is not even know that the digit 1 occurs at least once. Equivalently,

Mahler's Problem [13]. *Are irrational elements of Cantor's set necessarily transcendental? Thus does Cantor's set contain no irrational algebraic elements?*

Remark. Roth's theorem on rational approximation to algebraic numbers states that irrational algebraic numbers are not very well approximable. Thus, irrational algebraic numbers are clearly excluded from Mahler's Assertion of §1.1.

To our knowledge, it is not even known whether or not quadratic irrationals avoid the middle third Cantor set K. It is easy to see that the golden ratio $(\sqrt{5} - 1)/2$ is not in K. Let $[a_1, a_2, a_3, \ldots]$ be the standard continued fraction expansion of a real number x in [0,1]. If $a_1 = a_2 = 1$, then it is easily verified that x lies in the interval $[\frac{1}{2}, \frac{2}{3})$. However, this interval clearly misses K. Now, $(\sqrt{5} - 1)/2 = [\bar{1}] := [1, 1, 1, \ldots]$ and so is not in K.

Refining the problem even further, it is not at all obvious (to us at any rate) that all 'simple' quadratic irrationals avoid K; i.e. if $x = [\bar{n}]$ then $x \notin K$ for any $n \in \mathbb{N}$. It may well be the case that Mahler's problem is no easier for quadratic irrationals. The fact that quadratic irrationals have periodic continued fraction expansions may simply be a red herring!

Acknowledgments: SV would like to thank the laughing girls Mr Tiger Scuba Diver Wiver Tiger (Iona) and Dorothy Princess Angel Flower Star (Ayesha) for their continuous gibberish – all of which is of the highest quality !! SV would also like to thank Yann Bugeaud for his detailed comments/suggestions during the 'writing up' period of the paper. JL would like to thank SV for his persistence and for his near legendary opening gambit "Alright me dear? 'ere, I've been thinking!"

References

[1] A. Baker and W. M. Schmidt. Diophantine approximation and Hausdorff dimension. *Proc. Lond. Math. Soc.*, (3) 21 (1970) 1-11.

[2] V. Beresnevich, H. Dickinson and S. L. Velani : Sets of exact 'logarithmic' order in the theory of Diophantine approxiamtion. *Math. Ann.* 321 (2001) 253–273.

[3] V. Beresnevich, H. Dickinson and S. L. Velani : Measure Theoretic Laws for limsup sets, Pre-print (97pp): arkiv:math.NT/0401118. To appear: *Memoirs of the AMS.*

[4] V. Beresnevich and S. L. Velani : A Mass Transference Principle and the Duffin–Schaeffer conjecture for Hausdorff measures, Pre-print (22pp): arkiv:math.NT/0401118. To appear: *Annals of Math.*.

[5] Y. Bugeaud : *Approximation by algebraic numbers.* Cambridge Tracts in Mathematics 160, C.U.P., (2004).

[6] M. M. Dodson, M. V. Meli*á*n, D. Pestana and S. L. Velani : Patterson measure and Ubiquity. *Ann. Acad. Sci. Fenn.*, 20:1 (1995) 37–60.

[7] K. Falconer : *Fractal Geometry : Mathematical Foundations and Applications.* John Wiley & Sons, (1990).

[8] R. Güting : On Mahler's function θ_1. *Michigan Math. J.*, 10 (1963), 161-179.

[9] G. Harman : *Metric Number Theory.* LMS Monographs 18, Clarendon Press, Oxford, (1998).

[10] D. Kleinbock and B. Weiss : Badly approximable vectors on fractals. Pre-print (31pp): http://people.brandeis.edu/ kleinboc/Pub/bad.pdf. To appear: *Israel J. Math.*

[11] D. Kleinbock, E. Lindenstrauss and B. Weiss : On fractal measures and Diophantine approximation. Pre-print (41pp): http://people.brandeis.edu/ kleinboc/Pub/friendly.pdf. To appear: *Selecta Mathematica.*

[12] S. Kristensen, R. Thorn and S. L. Velani : Diophantine approximation and badly approximable sets. Pre-print (40pp): arkiv:math.NT/0405433. To appear: *Advances in Math.*

[13] K. Mahler : Some suggestions for further research. *Bull. Austral. Math. Soc.*, 29 (1984), 101-108.

[14] P. Mattila : *Geometry of sets and measures in Euclidean space*, CUP, Cambridge studies in advance mathematics **44** (1995)

[15] A. D. Pollington and S. L. Velani : Metric Diophantine approximation and 'absolutely friendly' measures, Pre-print (11pp): http://www.arxiv.org/abs/math.NT/0401149. To appear: *Selecta Mathematica.*

[16] V. G. Sprindžuk : *Metric theory of Diophantine approximation* (translated by R. A. Silverman). V. H. Winston & Sons, Washington D.C. (1979).

[17] B. Weiss : Almost no points on a Cantor set are very well approximable. *Proc. R. Soc. London.* 457 (2001), 949–952.

Jason Levesley: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.

e-mail: jl107@york.ac.uk

Cem Salp: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.

e-mail: cs502@york.ac.uk

Sanju L. Velani: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.

e-mail: slv3@york.ac.uk