# A note on simultaneous Diophantine approximation on planar curves


Victor Beresnevich[*] Sanju Velani[†]

MINSK YORK


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**Abstract**

Let $\mathcal{S}_n(\psi_1,\dots,\psi_n)$ denote the set of simultaneously $(\psi_1,\dots,\psi_n)$–approximable points in $\mathbb{R}^n$ and $\mathcal{S}_n^*(\psi)$ denote the set of multiplicatively $\psi$–approximable points in $\mathbb{R}^n$. Let $\mathcal{M}$ be a manifold in $\mathbb{R}^n$. The aim is to develop a metric theory for the sets $\mathcal{M}\cap\mathcal{S}_n(\psi_1,\dots,\psi_n)$ and $\mathcal{M}\cap\mathcal{S}_n^*(\psi)$ analogous to the classical theory in which $\mathcal{M}$ is simply $\mathbb{R}^n$. In this note, we mainly restrict our attention to the case that $\mathcal{M}$ is a planar curve $\mathcal{C}$. A complete Hausdorff dimension theory is established for the sets $\mathcal{C}\cap\mathcal{S}_2(\psi_1,\psi_2)$ and $\mathcal{C}\cap\mathcal{S}_2^*(\psi)$. A divergent Khintchine type result is obtained for $\mathcal{C}\cap\mathcal{S}_2(\psi_1,\psi_2)$; i.e. if a certain sum diverges then the one–dimensional Lebesgue measure on $\mathcal{C}$ of $\mathcal{C}\cap\mathcal{S}_2(\psi_1,\psi_2)$ is full. Furthermore, in the case that $\mathcal{C}$ is a rational quadric the convergent Khintchine type result is obtained for both types of approximation. Our results for $\mathcal{C}\cap\mathcal{S}_2(\psi_1,\psi_2)$ naturally generalize the dimension and Lebesgue measure statements of [2]. Within the multiplicative framework, our results for $\mathcal{C}\cap\mathcal{S}_2^*(\psi)$ constitute the first of their type.



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[†]Royal Society University Research Fellow

# 1 Introduction

## 1.1 Background: two types of simultaneous approximation

Throughout $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a real, positive decreasing function and will be referred to as an *approximating function.* Given approximating functions $\psi_1, \ldots, \psi_n$, a point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *simultaneously* $(\psi_1, \ldots, \psi_n)$*–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\|qy_i\| < \psi_i(q) \qquad\qquad 1 \leqslant i \leqslant n$$

where $\|x\| = \min\{|x-m| : m \in \mathbb{Z}\}$. In the case $\psi_i : h \to h^{-v_i}$ with $v_i > 0$, the point $\mathbf{y}$ is said to be *simultaneously* $(v_1, \ldots, v_n)$*–approximable.* The set of simultaneously $(\psi_1, \ldots, \psi_n)$–approximable points in $\mathbb{R}^n$ will be denoted by $\mathcal{S}_n(\psi_1, \ldots, \psi_n)$ and similarly $\mathcal{S}_n(v_1, \ldots, v_n)$ will denote the set of simultaneously $(v_1, \ldots, v_n)$–approximable points in $\mathbb{R}^n$. Geometrically, $\mathbf{y} \in \mathcal{S}_n(\psi_1, \ldots, \psi_n)$ if it lies in infinitely many $n$–dimensional 'rectangular' regions centred at rational points with 'size' determined by $\psi_1, \ldots, \psi_n$.

Next, given an approximating function $\psi$, a point $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called *multiplicatively* $\psi$*–approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\prod_{i=1}^{n} \|qy_i\| \;<\; \psi(q) \;.$$

In the case $\psi : h \to h^{-v}$ with $v > 0$ the point $\mathbf{y}$ is said to be *multiplicatively* $v$*–approximable.* The set of multiplicatively $\psi$–approximable points in $\mathbb{R}^n$ will be denoted by $\mathcal{S}_n^*(\psi)$ and similarly $\mathcal{S}_n^*(v)$ will denote the set of multiplicatively $v$–approximable points in $\mathbb{R}^n$. In this multiplicative setup, $\mathbf{y} \in \mathcal{S}_n^*(\psi)$ if it lies in infinitely many $n$–dimensional 'hyperbolic' regions centred at rational points with 'size' determined by $\psi$.

It is readily verified that

$$\mathcal{S}_n(\psi_1, \ldots, \psi_n) \subset \mathcal{S}_n^*(\psi) \quad \text{whenever} \quad \psi \geqslant \psi_1 \cdots \psi_n, \tag{1}$$

$$\mathcal{S}_n(v_1, \ldots, v_n) \subset \mathcal{S}_n^*(v) \quad \text{whenever} \quad v \leqslant v_1 + \cdots + v_n. \tag{2}$$

Also, in view of Minkowski's linear forms theorem which gives rise to a general $n$–dimensional version of Dirichlet's theorem,

$$\mathcal{S}_n(v_1, \ldots, v_n) = \mathbb{R}^n \qquad \text{if} \qquad v_1 + \cdots + v_n \leq 1 \;. \tag{3}$$

This together with (2) implies that

$$\mathcal{S}_n^*(v) = \mathbb{R}^n \qquad \text{if} \qquad v \leq 1 \;. \tag{4}$$

*The Lebesgue theory.* The following key results provide beautiful and simple criteria for the 'size' of the sets $\mathcal{S}_n(\psi_1, \ldots, \psi_n)$ and $\mathcal{S}_n^*(\psi)$ expressed in terms of $n$–dimensional Lebesgue measure $|\ |_{\mathbb{R}^n}$. The first is due to Khintchine [10] and the second is due to Gallagher [7].

**Theorem K (1926).** *Let $\psi$ be an approximating function. Then*

$$|\mathcal{S}_n(\psi_1,\ldots,\psi_n)|_{\mathbb{R}^n} = \begin{cases} \text{ZERO} & \text{if} \quad \sum \psi_1(h)\ldots\psi_n(h) \quad < \infty \\ \\ \text{FULL} & \text{if} \quad \sum \psi_1(h)\ldots\psi_n(h) \quad = \infty \end{cases} .$$

This theorem is a generalization of Khintchine's 1924 result which deals with the special case $\psi_1 = \psi_2 = \cdots = \psi_n$.

**Theorem G (1962).** *Let $\psi$ be an approximating function. Then*

$$|\mathcal{S}_n^*(\psi)|_{\mathbb{R}^n} = \begin{cases} \text{ZERO} & \text{if} \quad \sum \psi(h)^n \ (\log h)^{n-1} \ < \ \infty \\ \\ \text{FULL} & \text{if} \quad \sum \psi(h)^n \ (\log h)^{n-1} \ = \ \infty \end{cases} .$$

Here 'full' simply means that the complement of the set under consideration is of 'zero' measure. Thus the $n$–dimensional Lebesgue measure of the sets in question satisfy a 'zero-full' law. The divergence parts of the above statements constitute the main substance of the theorems. The convergence parts are a simple consequence of the Borel-Cantelli lemma from probability theory. Trivially, the convergence parts imply that

$$|\mathcal{S}_n(v_1,\ldots,v_n)|_{\mathbb{R}^n} = 0 \qquad \text{if} \qquad v_1+\cdots+v_n > 1$$

and

$$|\mathcal{S}_n^*(v)|_{\mathbb{R}^n} = 0 \qquad \text{if} \qquad v > 1 \ .$$

Note that the former statement is in fact a consequence of the latter and (2). In the case that the set in question is of Lebesgue measure zero, a more delicate attribute of the 'size' of the set is its Hausdorff measure and dimension. In this article we shall only be concerned with the dimension theory. The Hausdorff dimension of a set $X \in \mathbb{R}^n$ is defined as follows. For $\rho > 0$, a countable collection $\{B_i\}$ of Euclidean balls in $\mathbb{R}^n$ with diameter $\operatorname{diam}(B_i) \leq \rho$ for each $i$ such that $X \subset \bigcup_i B_i$ is called a $\rho$-cover for $X$. Let $s$ be a non-negative number and define $\mathcal{H}^s_\rho(X) \ = \ \inf\{\sum_i \operatorname{diam}(B_i)^s \ : \{B_i\}$ is a $\rho-$cover of $X\}$, where the infimum is taken over all possible $\rho$-covers of $X$. The *Hausdorff dimension* $\dim X$ of $X$ is defined by infimum over $s$ for which $\sup_{\rho>0} \mathcal{H}^s_\rho(X)$ is zero.

*The dimension theory.* The following relatively recent results provide exact formulae for the 'size' of the sets $\mathcal{S}_n(v_1,\ldots,v_n)$ and $\mathcal{S}_n^*(v)$ expressed in terms of Hausdorff dimension. The first is due to Rynne [11] and the second is due to Bovey & Dodson [6].

**Theorem R (1996).** *Let $v_1 \geq v_2 \geq \cdots \geq v_n$ and $v_1+v_2+\cdots+v_n \geq 1$. Then*

$$\dim \mathcal{S}_n(v_1,\ldots,v_n) \ = \ \min_{1\leq k\leq n} \left\{ \frac{n+1+\sum_{i=k}^n (v_k - v_i)}{1+v_k} \right\} .$$

In the case $v_1 = v_2 = \cdots = v_n$, the above statement reduces to the classical Jarník–Besicovitch theorem.

**Theorem BD (1978).** *Let $v \geq 1$. Then*

$$\dim \mathcal{S}_n^*(v) \ = \ n-1+\frac{2}{v+1} \quad .$$

### 1.2 Simultaneous approximation restricted to manifolds

Let $\mathcal{M}$ be a manifold in $\mathbb{R}^n$. In short, the aim is to develop a metric theory for the sets $\mathcal{M} \cap \mathcal{S}_n(\psi_1, \ldots, \psi_n)$ and $\mathcal{M} \cap \mathcal{S}_n^*(\psi)$ analogous to that described above in which $\mathcal{M}$ is simply $\mathbb{R}^n$. The fact that the points $\mathbf{y}$ of interest consist of dependent variables, reflecting the fact that $\mathbf{y} \in \mathcal{M}$ introduces major difficulties in attempting to describe the measure theoretic structure of either set. This is true even in the specific case that $\mathcal{M}$ is a planar curve – the main subject of this article.

In order to make any reasonable progress it is not unreasonable to assume that the manifolds $\mathcal{M}$ under consideration are **non-degenerate**. Essentially, these are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane. Formally, a manifold $\mathcal{M}$ of dimension $m$ embedded in $\mathbb{R}^n$ is said to be non-degenerate if it arises from a non–degenerate map $\mathbf{f} : U \to \mathbb{R}^n$ where $U$ is an open subset of $\mathbb{R}^m$ and $\mathcal{M} := \mathbf{f}(U)$. The map $\mathbf{f} : U \to \mathbb{R}^n : \mathbf{u} \mapsto \mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_n(\mathbf{u}))$ is said to be *non–degenerate at* $\mathbf{u} \in U$ if there exists some $l \in \mathbb{N}$ such that $\mathbf{f}$ is $l$ times continuously differentiable on some sufficiently small ball centred at $\mathbf{u}$ and the partial derivatives of $\mathbf{f}$ at $\mathbf{u}$ of orders up to $l$ span $\mathbb{R}^n$. The map $\mathbf{f}$ is *non–degenerate* if it is non–degenerate at almost every (in terms of $m$–dimensional Lebesgue measure) point in $U$; in turn the manifold $\mathcal{M} = \mathbf{f}(U)$ is also said to be non–degenerate. Any real, connected analytic manifold not contained in any hyperplane of $\mathbb{R}^n$ is non–degenerate.

Trivially, if the dimension $\dim \mathcal{M}$ of the manifold $\mathcal{M}$ in $\mathbb{R}^n$ is strictly less than $n$ then $|\mathcal{M}|_{\mathbb{R}^n} = 0$. Thus, in attempting to develop a Lebesgue theory for the sets $\mathcal{M} \cap \mathcal{S}_n(\psi_1, \ldots, \psi_n)$ and $\mathcal{M} \cap \mathcal{S}_n^*(\psi)$ it is natural to use the induced Lebesgue measure $| \, . \, |_{\mathcal{M}}$ on $\mathcal{M}$.

In 1998, D. Kleinbock & G. Margulis [9] proved the Baker-Sprindžuk conjecture:

**Theorem KM (1998)** *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$. Then*

$$|\mathcal{M} \cap \mathcal{S}_n^*(v)|_{\mathcal{M}} \; = \; 0 \qquad \text{if} \qquad v > 1 \ . \tag{5}$$

By inclusion (2), Theorem KM implies that for any non-degenerate manifold

$$|\mathcal{M} \cap \mathcal{S}_n(v_1, \ldots, v_n)|_{\mathcal{M}} \; = \; 0 \qquad \text{if} \qquad v_1 + \ldots + v_n > 1 \ . \quad {}^{1} \tag{6}$$

Also, note that in view of (3) and (4) both (5) and (6) are sharp. The first significant 'clear cut' statement was for planar curves. In 1964, Schmidt [12] established (6) in the case that $\mathcal{M}$ is a $C^{(3)}$ non-degenerate planar curve and $v_1 = v_2$.

The result of Kleinbock & Margulis gives some hope of developing a general metric theory for simultaneous approximation restricted to manifolds, analogous to that described in §1.1. As stepping stones, it is natural to consider the following explicit problems which ask for refinements of the measure zero statement of Kleinbock & Margulis.

**Problem S1 :** Given a non-degenerate manifold $\mathcal{M} \subset \mathbb{R}^n$ and $v > 1$ (respectively $v_1 + \ldots + v_n > 1$), what is the Hausdorff dimension of $\mathcal{M} \cap \mathcal{S}_n^*(v)$ (respectively $\mathcal{M} \cap \mathcal{S}_n(v_1, \ldots, v_n)$) ?

[1] When $v_1 = \ldots = v_n$ this statement verifies the conjecture of Sprindžuk which was stated for analytic manifolds only.

**Problem S2 :** Given a non-degenerate manifold $\mathcal{M} \subset \mathbb{R}^n$ and an approximating function $\psi$ (respectively $\psi_1, \ldots, \psi_n$), what is the weakest condition under which $\mathcal{M} \cap \mathcal{S}^*(\psi)$ (respectively $\mathcal{M} \cap \mathcal{S}_n(v_1, \ldots, v_n)$ ) is of Lebesgue measure zero?

Problem S1 motivates the dimension theory for simultaneous approximation restricted to manifolds whilst Problem S2 motivates the convergent aspects of the Lebesgue theory. A priori, convergent statements are usually easier to establish than their divergent counterparts.

Until recently, the existing metric theory for simultaneous approximation restricted to manifolds was rather ad-hoc – see [3] for an account. Even in the simplest geometric and arithmetic situation in which the manifold is a genuine curve in $\mathbb{R}^2$ the above problems seemed to have been impenetrable. However, in [3] we made significant progress towards developing a complete metric theory for the sets $\mathcal{M} \cap \mathcal{S}_2(\psi_1, \psi_2)$ with $\psi_1 = \psi_2$ and $\mathcal{M}$ a non-degenerate planar curve[2]. In this paper we study the general simultaneous settings given to us by the above problems. This therefore includes the multiplicative setup. As in [3], we will mainly direct our efforts towards the case that the manifold $\mathcal{M}$ is a planar curve $\mathcal{C}$. Thus, $\dim \mathcal{M} = 1$ and $n = 2$ in the above problems.

### 1.3 Statement of results

#### 1.3.1 The Lebesgue theory

**Theorem 1** *Let $\psi_1, \psi_2$ be approximating functions and let $\mathcal{C}$ be a $C^{(3)}$ non-degenerate planar curve. Then*

$$|\mathcal{C} \cap \mathcal{S}_2(\psi_1, \psi_2)|_{\mathcal{C}} \; = \; \text{FULL} \qquad \text{if} \qquad \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) = \infty \ .$$

The next theorem shows that the above result is best possible. We establish the complementary 'convergence result' for a class $\mathcal{Q}$ of non-degenerate rational quadrics. A planar curve $\mathcal{C}$ is in $\mathcal{Q}$ if it is the image of either the unit circle $\mathcal{C}_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, the parabola $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^2\}$ or the hyperbola $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = 1\}$ under a rational affine transformation of the plane.

**Theorem 2** *Let $\psi_1, \psi_2$ be approximating functions and $\mathcal{C} \in \mathcal{Q}$. Then*

$$\big|\mathcal{C} \cap \mathcal{S}_2(\psi_1, \psi_2)\big|_{\mathcal{C}} \; = \; \left\{ \begin{array}{lll} \text{ZERO} & \text{if} & \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) \; < \infty \ , \\ & & \\ \text{FULL} & \text{if} & \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) \; = \infty \ . \end{array} \right. \tag{7}$$

These theorems are a generalization of the results in [3] which deal with the situation $\psi_1 = \psi_2$. The next theorem is concerned with the multiplicative Lebesgue theory and is a refinement of Theorem KM for manifolds in $\mathcal{Q}$.

[2]Note that in the case the manifold is a planar curve $\mathcal{C}$, a point on $\mathcal{C}$ is non-degenerate if the curvature at that point is non-zero. Thus, $\mathcal{C}$ is a non-degenerate planar curve if the set of points on $\mathcal{C}$ at which the curvature vanishes is a set of one–dimensional Lebesgue measure zero. Moreover, it is not difficult to show that the set of points on a planar curve at which the curvature vanishes but the curve is non-degenerate is at most countable. In view of this, the curvature completely describes the non-degeneracy of planar curves.

**Theorem 3** *Let $\psi$ be an approximating function and $\mathcal{C} \in \mathcal{Q}$. Then*

$$|\mathcal{C} \cap \mathcal{S}_2^*(\psi)|_{\mathcal{C}} \; = \; 0 \qquad \text{if} \qquad \sum_{h=1}^{\infty} \psi(h) \log h < \infty \; . \tag{8}$$

### 1.3.2 The dimension theory

Regarding Problem S1, for planar curves we are able to give a complete description for either form of simultaneous approximation.

**Theorem 4** *Let $f \in C^{(3)}(I_0)$, where $I_0$ is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Let $v_1$ and $v_2$ be positive numbers such that $0 < \min(v_1, v_2) < 1$ and $v_1 + v_2 \geq 1$. Assume that*

$$\dim \big\{x \in I_0 : f''(x) = 0\big\} \leqslant \frac{2 - \min(v_1, v_2)}{1 + \max(v_1, v_2)} \; .$$

*Then*

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(v_1, v_2) \;\; = \;\; \frac{2 - \min(v_1, v_2)}{1 + \max(v_1, v_2)} \, . \tag{9}$$

In the case $v_1 = v_2$, this theorem generalizes the dimension results of [3]. Our next result is a general $n$–dimensional statement concerning Lipshitz manifolds; i.e. manifolds for which there exists an atlas of Lipshitz maps.

**Theorem 5** *Let $\mathcal{M}$ be an arbitrary Lipshitz manifold in $\mathbb{R}^n$ of dimension $\dim \mathcal{M}$. Then*

$$\dim \mathcal{M} \cap \mathcal{S}_n^*(v) \; \geqslant \; \dim \mathcal{M} \; - \; 1 \; + \; \frac{2}{1+v} \qquad \text{if} \qquad v \geq 1 \;\; . \tag{10}$$

In essence, the above theorem indicates that the lower bound for the Hausdorff dimension in the general multiplicative setup reduces to a one dimensional problem. For clarification of this remark, see (43) in §6.2.

We conjecture that for manifolds in $\mathbb{R}^n$ which are non-degenerate everywhere except possibly on a set of dimension at most $\dim \mathcal{M} - 1 + 2/(1+v)$, the lower bound given by Theorem 5 is in fact exact. The following result verifies the conjecture for planar curves.

**Theorem 6** *Let $f \in C^{(3)}(I_0)$, where $I_0$ is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Let $v > 1$ and assume that $\dim \{x \in I_0 : f''(x) = 0\} \leqslant 2/(1+v)$. Then*

$$\dim \mathcal{C}_f \cap \mathcal{S}_2^*(v) \;\; = \;\; \frac{2}{1+v} \quad . \tag{11}$$

*Remark.* Let $\psi$ be an approximating function for which the limit

$$\lambda(\psi) \; := \; \lim_{h \to \infty} \frac{-\log \psi(h)}{\log h}$$

exists and is positive. The quantity $\lambda(\psi)$ is usually referred to as the *order* of $1/\psi$ and indicates the limiting behavior of the function $1/\psi$ at infinity. On making use of the fact that for any $\varepsilon > 0$,

$$h^{-\lambda(\psi)-\varepsilon} \ \leq \ \psi(h) \ \leq \ h^{-\lambda(\psi)+\varepsilon} \tag{12}$$

for all sufficiently large $h$, the above dimension results (Theorems 4 – 6) can be easily generalized to approximating functions $\psi$ for which the order $\lambda(\psi)$ exists. For example, Theorem 6 becomes

**Theorem 6*** *Let $f \in C^{(3)}(I_0)$, where $I_0$ is an interval and $\mathcal{C}_f := \{(x, f(x)) : x \in I_0\}$. Let $\psi$ be an approximating function of order $\lambda(\psi) > 1$ and assume that* $\dim\{x \in I_0 : f''(x) = 0\} \leqslant 2/(1+\lambda(\psi))$. *Then*

$$\dim \mathcal{C}_f \cap \mathcal{S}_2^*(\psi) \ = \ \frac{2}{1+\lambda(\psi)} \quad .$$

*Proof.* Let $v := \lambda(\psi)$ and fix $\varepsilon > 0$ such that $v - \varepsilon > 1$. In view of (12), it follows that

$$\mathcal{C}_f \cap \mathcal{S}_2^*(v+\varepsilon) \ \subset \ \mathcal{C}_f \cap \mathcal{S}_2^*(\psi) \ \subset \ \mathcal{C}_f \cap \mathcal{S}_2^*(v-\varepsilon) \ .$$

Theorem 6* now follows from these inclusions, (10) and (11), by letting $\varepsilon \to 0$.

# 2 Preliminaries

First some useful notation. For any point $\mathbf{r} \in \mathbb{Q}^n$ there exists a smallest $q \in \mathbb{N}$ such that $q\mathbf{r} \in \mathbb{Z}^n$. Thus, every point $\mathbf{r} \in \mathbb{Q}^n$ has a unique representation in the form

$$\frac{\mathbf{p}}{q} = \frac{(p_1, \ldots, p_n)}{q} = \left(\frac{p_1}{q}, \ldots, \frac{p_n}{q}\right)$$

with $(p_1, \ldots, p_n) \in \mathbb{Z}^n$. Henceforth, we will only consider points of $\mathbb{Q}^n$ in this form. As usual, $C^{(n)}(I)$ will denote the set of $n$–times continuously differentiable functions defined on some interval $I$ of $\mathbb{R}$. Also, as usual the Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$, we write $a \asymp b$ and say that the quantities $a$ and $b$ are comparable.

## 2.1 Rational points close to a curve

The following estimates on the number of rational points close to a reasonably defined curve will be crucial towards establishing our convergence and (upper bound) dimension results.

Let $I_0$ denote a finite, open interval of $\mathbb{R}$ and let $f$ be a function in $C^{(3)}(I_0)$ such that

$$c_1 < |f''(x)| < c_2 \qquad \text{for all} \ \ x \in I_0 \ . \tag{13}$$

Here $c_1$ and $c_2$ are positive constants. Given an approximating function $\psi$ and $Q \in \mathbb{R}^+$ consider the counting function $N_f(Q, \psi, I_0)$ given by

$$N_f(Q, \psi, I_0) \ := \ \#\{\mathbf{p}/q \in \mathbb{Q}^2 \ : \ q \leqslant Q, \, p_1/q \in I_0, \, |f(p_1/q) - p_2/q| < \psi(Q)/Q\}.$$

In short, the function $N_f(Q,\psi,I_0)$ counts the number of rational points with bounded denominator lying within a specified neighbourhood of the curve $\mathcal{C}_f := \{(x,f(x)) : x \in I_0\}$ parameterized by $f$. Now let

$$\lim_{t\to+\infty} \psi(t) = \lim_{t\to+\infty} \frac{1}{t\psi(t)} = 0. \tag{14}$$

In [8], Huxley obtains the following upper bound: *For $\varepsilon > 0$ and $Q$ sufficiently large*

$$N_f(Q,\psi,I_0) \ \leqslant\ \psi(Q)\, Q^{2+\varepsilon}. \tag{15}$$

For this exact form of Huxley's estimate we refer the reader to [3, §1.4]. In the case that the curve is the unit circle the above estimate can be sharpened. For $n \in \mathbb{N}$, let $r(n)$ denote the number of representations of $n$ as the sum of two squares. A simple consequence of Theorem A in [3, §A.1] is the following statement.

*There is a constant $C > 0$ such that for any choice of real numbers $Q$ and $\Psi$ satisfying*

$$Q^{-1}\,(\log Q)^{260} \ \leqslant\ \Psi \ <\ 1 \qquad \text{and} \qquad Q > 1\ , \tag{16}$$

*one has that*

$$\sum_{Q<q\leqslant 2Q} \quad \sum_{\substack{n:\\ |q-\sqrt{n}|<\Psi}} r(n) \ \leqslant\ C\ \Psi\ Q^2\ . \tag{17}$$

Notice that if $n$ is the sum of two square, say $n = p_1^2 + p_2^2$ then the inequality $|q-\sqrt{n}| < \Psi$ appearing in (17) implies that the rational point $(p_1/q, p_2/q)$ lies within a constant times $\Psi/Q$ neighbourhood of the unit circle. Thus, we obtain the following sharpening of Huxley's estimate.

*If $\mathcal{C}_f$ is the unit circle and $\psi(q) \geqslant q^{-1}(\log q)^{260}$ for all sufficiently large $q$, then*

$$N_f(Q,\psi,I_0) \ll \psi(Q)Q^2\ .$$

In fact, on adapting the arguments of [3, §2] it is relatively straightforward to extend the statement to any planar curve $\mathcal{C}_f$ in $\mathcal{Q}$; i.e to any non-degenerate rational quadric. However, we shall not make use of this stronger fact.

### 2.2 Ubiquitous systems

The divergence and (lower bound) dimension results stated in this paper will be established via a general technique developed in [3]. The 'general technique' is based on the notion of 'ubiquity' as introduced in [2].

Let $I_0$ be an interval in $\mathbb{R}$ and $\mathcal{R} := (R_\alpha)_{\alpha\in\mathcal{J}}$ be a family of *resonant points* $R_\alpha$ of $I_0$ indexed by an infinite set $\mathcal{J}$. Next let $\beta : \mathcal{J} \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on $\mathcal{J}$. Thus, the function $\beta$ attaches a 'weight' $\beta_\alpha$ to the resonant point $R_\alpha$. Also, for $t \in \mathbb{N}$ let $J_t := \{\alpha \in \mathcal{J} : \beta_\alpha \leqslant 2^t\}$ and assume that $\#J_t$ is always finite.

Throughout, $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a function satisfy $\lim_{t\to\infty} \rho(t) = 0$ and is usually referred to as the *ubiquitous function*. Also $B(x,r)$ will denote the ball (or rather the interval) centred at $x$ of radius $r$.

**Definition 1 (Ubiquitous systems on the real line)** *Suppose there exists a ubiquitous function $\rho$ and an absolute constant $\kappa > 0$ such that for any interval $I \subseteq I_0$*

$$\liminf_{t\to\infty} \ \left|\bigcup_{\alpha\in J_t}\left(B(R_\alpha,\rho(2^t))\cap I\right)\right| \ \geqslant \ \kappa\,|I| \ .$$

*Then the system $(\mathcal{R};\beta)$ is called* locally ubiquitous in $I_0$ with respect to $\rho$.

In [3] the theory of ubiquity is developed to incorporate the situation in which the resonant points of interest lie within some specified neighborhood of a given curve in $\mathbb{R}^n$.

With $n \geq 2$, let $\mathcal{R} := (R_\alpha)_{\alpha\in\mathcal{J}}$ be a family of *resonant points* $R_\alpha$ of $\mathbb{R}^n$ indexed by an infinite set $\mathcal{J}$. As before, $\beta : \mathcal{J} \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ is a positive function on $\mathcal{J}$. For a point $R_\alpha$ in $\mathcal{R}$, let $R_{\alpha,k}$ represent the $k$th coordinate of $R_\alpha$. Thus, $R_\alpha := (R_{\alpha,1}, R_{\alpha,2}, \ldots, R_{\alpha,n})$. Throughout this section and the remainder of the paper we will use the notation $\mathcal{R}_{\mathcal{C}}(\Phi)$ to denote the sub-family of resonant points $R_\alpha$ in $\mathcal{R}$ which are "$\Phi$–close" to the curve $\mathcal{C} = \mathcal{C}_{\mathbf{f}} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$ where $\Phi$ is an approximating function, $\mathbf{f} = (f_1, \ldots, f_n) : I_0 \to \mathbb{R}^n$ is a continuous map with $f_1(x) = x$ and $I_0$ is an interval in $\mathbb{R}$. Formally, and more precisely

$$\mathcal{R}_{\mathcal{C}}(\Phi) := (R_\alpha)_{\alpha\in\mathcal{J}_{\mathcal{C}}(\Phi)} \qquad \text{where} \qquad \mathcal{J}_{\mathcal{C}}(\Phi) := \{\alpha\in\mathcal{J} : \max_{1\leqslant k\leqslant n} |f_k(R_{\alpha,1}) - R_{\alpha,k}| < \Phi(\beta_\alpha)\} \ .$$

Finally, we will denote by $\mathcal{R}_1$ the family of first co-ordinates of the points in $\mathcal{R}_{\mathcal{C}}(\Phi)$; that is

$$\mathcal{R}_1 \ := \ (R_{\alpha,1})_{\alpha\in\mathcal{J}_{\mathcal{C}}(\Phi)} \ .$$

By definition, $\mathcal{R}_1$ is a subset of the interval $I_0$ and can therefore be regarded as a set of resonant points for the theory of ubiquitous systems in $\mathbb{R}$. This leads us naturally to the following definition in which the ubiquity function $\rho$ is as above.

**Definition 2 (Ubiquitous systems near curves)** *The system $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is called locally ubiquitous with respect to $\rho$ if the system $(\mathcal{R}_1, \beta)$ is locally ubiquitous in $I_0$ with respect to $\rho$.*

Next, given an approximating function $\Psi$ let $\Lambda(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)$ denote the set $x \in I_0$ for which the system of inequalities

$$\begin{cases} \quad |x - R_{\alpha,1}| \ < \ \Psi(\beta_\alpha) \\ \max\limits_{2\leqslant k\leqslant n} |f_k(x) - R_{\alpha,k}| \ < \ \Psi(\beta_\alpha) + \Phi(\beta_\alpha) \end{cases}$$

is satisfied for infinitely many $\alpha \in \mathcal{J}$. The following lemmas are stated and proved in [3, §3].

**Lemma 1** *Consider the curve $\mathcal{C} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$, where $f_2, \ldots, f_n$ are locally Lipshitz in a finite interval $I_0$. Suppose that $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is a locally ubiquitous system with respect to $\rho$. Let $\Psi$ be an approximating function such that $\Psi(2^{t+1}) \leqslant \frac{1}{2}\Psi(2^t)$ for $t$ sufficiently large. Then*

$$|\,\Lambda\,(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi)\,| \ = \ |I_0|$$

*whenever*

$$\sum_{t=1}^{\infty} \frac{\Psi(2^t)}{\rho(2^t)} = \infty$$

**Lemma 2** *Consider the curve $\mathcal{C} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$, where $f_2, \ldots, f_n$ are locally Lipshitz in a finite interval $I_0$. Suppose that $(\mathcal{R}_{\mathcal{C}}(\Phi), \beta)$ is a locally ubiquitous system with respect to $\rho$ and let $\Psi$ be an approximating function. Then*

$$\dim \Lambda\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right) \ \geqslant \ d \ := \ \min\left\{1, \left|\limsup_{t\to\infty} \frac{\log \rho(2^t)}{\log \Psi(2^t)}\right|\right\}.$$

**Lemma 3** *Let $I_0$ denote a finite, open interval of $\mathbb{R}$ and let $f$ be a function in $C^{(3)}(I_0)$ satisfying* (13). *Let $\psi$ be an approximating function satisfying* (14). *Let $\mathcal{C} := \{(x, f(x)) : x \in I_0\}$. With reference to the ubiquitous framework above, set*

$$\beta : \ \mathcal{J} := \mathbb{Z}^2 \times \mathbb{N} \to \mathbb{N} : (\mathbf{p}, q) \to q \ , \qquad \Phi : t \to t^{-1}\psi(t) \qquad \textit{and} \qquad \rho : t \to u(t)/(t^2\psi(t)) \quad (18)$$

*where $u : \mathbb{R}^+ \to \mathbb{R}^+$ is any function such that $\lim_{t\to\infty} u(t) = \infty$. Then the system $(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta)$ is locally ubiquitous with respect to $\rho$.*

*Remark.* In Lemma 3, the curve $\mathcal{C}$ is obviously a planar curve. Also, given $\alpha = (\mathbf{p}, q) \in \mathcal{J}$ the associated resonant point $R_\alpha$ in the ubiquitous system is simply the rational point $\mathbf{p}/q$ in the plane. Furthermore, $\mathcal{R} := \mathbb{Q}^2$.

## 3 Proof of Theorem 1

As $\mathcal{C} := \mathcal{C}_f$ is non-degenerate almost everywhere, we can restrict our attention to a sufficiently small patch of $\mathcal{C}$, which can be written as $\{(x, f(x)) : x \in I\}$ where $I$ is a sub-interval of $I_0$ and $f$ satisfies (13) with $I_0$ replaced by $I$. However, without loss of generality and for clarity, we assume that $f$ satisfies (13) on $I_0$.

We are given that $\psi_1$ and $\psi_2$ are approximating functions such that

$$\sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) = \infty \ . \qquad (19)$$

Thus, at least one of the following two sums diverges:

$$\sum_{h\in\mathbb{N},\, \psi_1(h)\geqslant\psi_2(h)} \psi_1(h)\psi_2(h) \qquad\qquad \sum_{h\in\mathbb{N},\, \psi_1(h)\leqslant\psi_2(h)} \psi_1(h)\psi_2(h) \quad .$$

Throughout, let us assume that the sum on the right is divergent. The argument below can easily be modified to deal with the case that only the sum on the left is divergent.

**Step 1.** We show that there is no loss of generality in assuming that

$$\psi_2(h) \geqslant \psi_1(h) \qquad \text{for all} \quad h \in \mathbb{N} \ . \qquad (20)$$

Define the auxiliary function $\psi_1^* : h \to \psi_1^*(h) := \min\{\psi_1(h), \psi_2(h)\}$. Then the sum

$$\sum_{h=1}^{\infty} \psi_1^*(h)\psi_2(h)$$

diverges since by assumption it contains a divergent sub-sum. It is readily verified that $\psi_1^*$ is an approximating function and that $\mathcal{S}_2(\psi_1^*,\psi_2)\subset\mathcal{S}_2(\psi_1,\psi_2)$. Thus to complete the proof of Theorem 1 it suffices to prove the result with $\psi_1$ replaced by $\psi_1^*$. Hence, without loss of generality, (20) can be assumed.

**Step 2.** We show that there is no loss of generality in assuming that

$$\psi_i(h)\to 0 \qquad \text{as} \qquad h\to\infty \qquad (i=1,2)\ . \tag{21}$$

Define the increasing function $v:\mathbb{R}^+\to\mathbb{R}^+$ as follows

$$v(h):=\sum_{t=1}^{[h]}\psi_1(t)\psi_2(t)\ .$$

In view of (19), $\lim_{t\to\infty}v(t)=\infty$. Fix $k\in\mathbb{N}$. Then

$$\sum_{t=k}^{m}\frac{\psi_1(t)\psi_2(t)}{v(t)}\ \geqslant\ \sum_{t=k}^{m}\frac{\psi_1(t)\psi_2(t)}{v(m)}\ =\ \frac{v(m)-v(k-1)}{v(m)}\ \to\ 1\quad \text{as}\quad m\to\infty.$$

Hence

$$\sum_{t=k}^{\infty}\frac{\psi_1(t)\psi_2(t)}{v(t)}\ \geqslant\ 1\quad \text{for all } k.$$

This implies that the sum $\sum_{t=1}^{\infty}\psi_1(t)\psi_2(t)/v(t)$ diverges. Next, for $i=1,2$ consider the functions

$$\psi_i^*:h\to\psi_i^*(h):=\psi_i(h)/\sqrt{v(h)}\ .$$

Then both $\psi_1^*(h)$ and $\psi_2^*(h)$ are decreasing, tend to 0 as $h\to\infty$ and $\sum_{q=1}^{\infty}\psi_1^*(q)\psi_2^*(q)=\infty$. Furthermore $\mathcal{S}_2(\psi_1^*,\psi_2^*)\subset\mathcal{S}_2(\psi_1,\psi_2)$. Therefore, it suffices to establish Theorem 1 for $\psi_1^*,\psi_2^*$.

**Step 3.** We show that there is no loss of generality in assuming that

$$\psi_2(h)\geqslant h^{-2/3}\ \text{ for all } h. \tag{22}$$

To this end, define $\hat\psi_2(h)=\max\{\psi_2(h),h^{-2/3}\}$. In view of (20), it is readily verified that

$$\mathcal{S}_2(\psi_1,\hat\psi_2)\ \subseteq\ \mathcal{S}_2(\psi_1,\psi_2)\cup\mathcal{S}_2(h\mapsto h^{-2/3},h\mapsto h^{-2/3})\ .$$

By Schmidt's theorem [12], for almost all $x\in I_0$ we have that

$$(x,f(x))\notin\mathcal{S}_2(h\mapsto h^{-2/3},h\mapsto h^{-2/3})\ .$$

Hence

$$\big|\{x\in I_0:(x,f(x))\in\mathcal{S}_2(\hat\psi_1,\psi_2)\}\big|\ \leq\ \big|\{x\in I_0:(x,f(x))\in\mathcal{S}_2(\psi_1,\psi_2)\}\big|\ ,$$

and to complete the proof of Theorem 1 it suffices to prove that the set on the left has full measure. In turn, this justifies (22).

**Step 4.** In view of Steps 2 and 3 above, the function $\psi_2$ satisfies (14) and Lemma 3 is applicable with $\psi=\psi_2$. By (19) and the fact that $\psi_1$ and $\psi_2$ are decreasing we obtain that

$$\infty=\sum_{t=0}^{\infty}\sum_{2^t\leqslant h<2^{t+1}}\psi_1(h)\psi_2(h)\ \leqslant\ \sum_{t=0}^{\infty}\sum_{2^t\leqslant h<2^{t+1}}\psi_1(2^t)\psi_2(2^t)\ =\ \sum_{t=0}^{\infty}2^t\,\psi_1(2^t)\psi_2(2^t)\ .$$

Hence

$$\sum_{t=0}^{\infty} 2^t\,\psi_1(2^t)\psi_2(2^t) = \infty\ . \tag{23}$$

Next, define the increasing function $u : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$u(h) = \sum_{t=0}^{[h]} 2^t\,\psi_1(2^t)\psi_2(2^t)\ .$$

Trivially, $\lim_{t\to\infty} u(t) = \infty$. On using the same argument as in Step 2 above we verify that

$$\sum_{t=0}^{\infty} \frac{2^t\,\psi_1(2^t)\psi_2(2^t)}{u(t)} \ = \ \infty\ . \tag{24}$$

Now let $\Phi(t) := \psi_2(t)/t$ and $\rho(t) := u(\log_2 t)/(t^2\psi_2(t))$. By Lemma 3, $(\mathbb{Q}^2_{\mathcal{C}}(\Phi),\beta)$ is locally ubiquitous relative to $\rho$, where $\beta$ is given by (18). Let $\Psi(t) = \psi_1(t)/t$. In view of (24),

$$\sum_{t=1}^{\infty} \frac{\Psi(2^t)}{\rho(2^t)} \ = \ \sum_{t=1}^{\infty} \frac{\dfrac{\psi_2(2^t)}{2^t}}{\dfrac{u(t)}{2^{2t}\psi_1(2^t)}} \ = \ \sum_{t=1}^{\infty} \frac{2^t\psi_1(2^t)\psi_2(2^t)}{u(t)} \ = \ \infty\ .$$

Since $\psi_1$ is decreasing,

$$\Psi(2^{t+1}) \ := \ \frac{\psi_1(2^{t+1})}{2^{t+1}} \ \leqslant\ \frac{1}{2}\cdot\frac{\psi_1(2^t)}{2^t} \ :=\ \frac{1}{2}\Psi(2^t)\ .$$

Thus the conditions of Lemma 1 are satisfied and it follows that $\Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi),\beta,\Psi)$ is of full measure. By definition and (20), the set $\Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi),\beta,\Psi)$ consists of points $x \in I_0$ such that the system

$$\left\{\begin{array}{l} \left|x - \frac{p_1}{q}\right| < \Psi(q) = \frac{\psi_1(q)}{q} < \frac{2\psi_1(q)}{q}, \\ \left|f(x) - \frac{p_2}{q}\right| < \Psi(q) + \Phi(q) = \frac{\psi_1(q)}{q} + \frac{\psi_2(q)}{q} \leqslant \frac{2\psi_2(q)}{q} \end{array}\right.$$

has infinitely many solutions $\mathbf{p}/q \in \mathbb{Q}^2$. Obviously for $x \in \Lambda(\mathbb{Q}^2_{\mathcal{C}}(\Phi),\beta,\Psi)$ the point $(x, f(x))$ is in $\mathcal{S}_2(2\psi_1, 2\psi_2)$. To complete the proof of Theorem 1 we simply apply what has already been proved to the approximating functions $\frac{1}{2}\psi_1$ and $\frac{1}{2}\psi_2$.

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# 4 Proof of Theorem 3

By definition any rational quadric $\mathcal{C} \in \mathcal{Q}$ is the image of either the unit circle $\mathcal{C}_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, the parabola $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^2\}$ or the hyperbola $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = 1\}$ under a rational affine transformation of the plane. It is easily verified that the measure 'zero' statement of Theorem 3 is invariant under rational affine transformations of the plane. In view of this, it suffices to establish the statement of the theorem for the unit circle, the parabola and the hyperbola. Below, we only consider the case of the unit circle $\mathcal{C}_1$

and leave the hyperbola and parabola to the reader. The required modifications are relatively straightforward once the reader is armed with the arguments appearing in [3, §2.1].

From this point onwards $\mathcal{C} = \mathcal{C}_1$ – the unit circle. First, notice that it suffices to prove the theorem for every arc of $\mathcal{C}_1$ given by $\mathcal{C}_1^\varepsilon = \{(x,y) \in \mathcal{C}_1 : \varepsilon < x, y < 1-\varepsilon\}$ with $\varepsilon > 0$. Next, we are given that

$$\sum_{q=1}^{\infty} \psi(q) \log q < \infty \ . \tag{25}$$

Therefore, without loss of generality we can assume that

$$q^{-1}(\log q)^{-3} \ < \ \psi(q) \ < \ q^{-1}(\log q)^{-1} \quad \text{for sufficiently large } q \ . \tag{26}$$

To see this, note that since $\psi$ is decreasing

$$\sum_{h/2<q<h} \psi(q) \log q \ \geqslant \ \sum_{h/2<q<h} \psi(h) \log(h/2) \ \asymp \ h\,\psi(h) \ \log h$$

for every natural number $h$. In view of (25), we have that the left hand side of the above inequality tends to zero as $h \to \infty$. It follows that

$$h\,\psi(h) \ \log h \to 0 \quad \text{as} \quad h \to \infty \quad .$$

This establishes the right hand side inequality of (26). Next, if the left hand side inequality of (26) is not satisfied then we replace $\psi$ with the auxiliary function

$$\widetilde{\psi} : q \to \widetilde{\psi}(q) := \max\{\psi(q), \ q^{-1}(\log q)^{-3}\} \ .$$

Then $\widetilde{\psi}$ is clearly an approximating function for which both (25) and the left hand side inequality of (26) are satisfied with $\psi$ replaced by $\widetilde{\psi}$. Furthermore,

$$\mathcal{S}_2^*(\widetilde{\psi}) \ \supset \ \mathcal{S}_2^*(\psi) \ .$$

Thus it suffices to prove the theorem with $\psi$ replaced by $\widetilde{\psi}$. Hence, without loss of generality, (26) can be assumed.

The lim sup set $\mathcal{C}_1^\varepsilon \cap \mathcal{S}_2^*(\psi)$ has the following natural representation:

$$\mathcal{C}_1^\varepsilon \cap \mathcal{S}_2^*(\psi) = \bigcap_{n=1}^{\infty} \bigcup_{q=n}^{\infty} \bigcup_{(p_1,p_2)\in \mathbb{Z}^2} \Big\{(x,y) \in \mathcal{C}_1^\varepsilon : \Big|x - \frac{p_1}{q}\Big| \cdot \Big|y - \frac{p_2}{q}\Big| < \frac{\psi(q)}{q^2}\Big\}.$$

Using the fact that $\psi$ is decreasing, we have that for any $n$

$$\mathcal{C}_1^\varepsilon \cap \mathcal{S}_2^*(\psi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1,p_2)\in \mathbb{Z}^2} \Big\{(x,y) \in \mathcal{C}_1^\varepsilon : \Big|x - \frac{p_1}{q}\Big| \cdot \Big|y - \frac{p_2}{q}\Big| < \frac{\psi(2^t)}{(2^t)^2}\Big\}. \tag{27}$$

If $t \in \mathbb{N}$, $(x,y) \in \mathcal{C}_1^\varepsilon$, $q \in \mathbb{N}$ with $2^t \leqslant q < 2^{t+1}$ and $\Big|x - \frac{p_1}{q}\Big| \cdot \Big|y - \frac{p_2}{q}\Big| < \frac{\psi(2^t)}{(2^t)^2}$ for some $(p_1,p_2) \in \mathbb{Z}^2$, then there is a unique integer $m$ such that

$$2^{m-1}\,\frac{\sqrt{2\,\psi(2^t)}}{2^t} \leqslant \Big|x - \frac{p_1}{q}\Big| < 2^m\,\frac{\sqrt{2\,\psi(2^t)}}{2^t}.$$

Therefore for this number $m$ we also have that

$$\left|y-\frac{p_2}{q}\right|<2^{-m}\frac{\sqrt{2\,\psi(2^t)}}{2^t}\ .$$

The upshot of this is that

$$\mathcal{C}_1^\varepsilon\cap\mathcal{S}_2^*(\psi)\subset\bigcup_{t=n}^{\infty}\ \bigcup_{2^t\leqslant q<2^{t+1}}\ \bigcup_{(p_1,p_2)\in\,\mathbb{Z}^2}\ \bigcup_{m=-\infty}^{+\infty}\mathcal{C}_1^\varepsilon\cap S(q,p_1,p_2,m)\ , \tag{28}$$

where

$$S(q,p_1,p_2,m)=\Big\{(x,y)\in\mathbb{R}^2:\left|x-\frac{p_1}{q}\right|<2^m\frac{\sqrt{2\,\psi(2^t)}}{2^t},\quad \left|y-\frac{p_2}{q}\right|<2^{-m}\frac{\sqrt{2\,\psi(2^t)}}{2^t}\Big\}$$

and $t$ is uniquely defined by $2^t\leqslant q<2^{t+1}$. The aim is to show that the Lebesgue measure $|\ .\ |_{\mathcal{C}_1^\varepsilon}$ of the R.H.S. of (28) tends to zero as $n\to\infty$. Since for each $n$ the R.H.S. of (28) is a cover for $\mathcal{C}_1^\varepsilon\cap\mathcal{S}_2^*(\psi)$, it follows that $|\mathcal{C}_1^\varepsilon\cap\mathcal{S}_2^*(\psi)|_{\mathcal{C}_1^\varepsilon}=0$ as required. To proceed, we consider two cases. Namely, case (a): $m\in\mathbb{Z}$ such that

$$2^{-|m|}\ \geqslant\ t\sqrt{\psi(2^t)} \tag{29}$$

and case (b): $m\in\mathbb{Z}$ such that

$$2^{-|m|}\ \leqslant\ t\sqrt{\psi(2^t)}\ \ . \tag{30}$$

**Case (a)** : First, observe that (29) together with (26) implies that

$$t\ \geq\ 2\,|m|\quad . \tag{31}$$

Next, it is a simple mater to see that

$$|\,\mathcal{C}_1^\varepsilon\cap S(q,p_1,p_2,m)|_{\mathcal{C}_1^\varepsilon}\ \ll\ 2^{-|m|}\ \frac{\sqrt{\psi(2^t)}}{2^t}\quad . \tag{32}$$

The implied constant depends on only $\varepsilon$ and is therefore irrelevant to the rest of the argument.

Given $t$ and $m$, let $N(t,m)$ denote the number of triples $(q,p_1,p_2)$ with $2^t\leqslant q<2^{t+1}$ such that $\mathcal{C}_1^\varepsilon\cap S(q,p_1,p_2,m)\neq\emptyset$. Suppose that $\mathcal{C}_1^\varepsilon\cap S(q,p_1,p_2,m)\neq\emptyset$. Then for some $(x,y)\in\mathcal{C}_1^\varepsilon$ and $\theta_1,\theta_2$ satisfying $-1<\theta_1,\theta_2<1$, we have that

$$x\ =\ \frac{p_1}{q}\ +\ \theta_1\,2^{|m|}\ \frac{\sqrt{2\,\psi(2^t)}}{2^t}\ ,\qquad y\ =\ \frac{p_1}{q}\ +\ \theta_2\,2^{|m|}\ \frac{\sqrt{2\,\psi(2^t)}}{2^t}\quad .$$

Hence

$$\begin{aligned}
1\ &=\ x^2+y^2\\
&=\ \sum_{i=1}^{2}\left(\frac{p_i}{q}\ +\ \theta_i\,2^{|m|}\frac{\sqrt{2\,\psi(2^t)}}{2^t}\right)^2\\
&=\ \frac{p_1^2+p_2^2}{q^2}\ +\ \frac{p_1\theta_1+p_2\theta_2}{q}\ \ 2^{|m|+1}\ \ \frac{\sqrt{2\,\psi(2^t)}}{2^t}\ +\ (\theta_1^2+\theta_2^2)\ 2^{2|m|+1}\ \frac{\psi(2^t)}{2^{2t}}\ .
\end{aligned}$$

It follows that

$$| \; q^2 - p_1^2 + p_2^2 \; | \;\ll\; q \;\; \max\{|p_1|,|p_2|\}\; 2^{|m|-t} \; \sqrt{\psi(2^t)} \;\; + \;\; q^2 \, 2^{2|m|-2t} \, \psi(2^t) \;\; .$$

On dividing both sides of the inequality by $q+\sqrt{p_1^2+p_2^2}$ and using the fact that $\sqrt{p_1^2+p_2^2} \geqslant \max\{|p_1|,|p_2|\}$ we obtain that

$$| \; q - \sqrt{p_1^2+p_2^2} \; | \;\ll\; 2^t \, U_{m,t} \; (1+U_{m,t}) \; ,$$

where

$$U_{m,t} \;:=\; 2^{-t} \, 2^{|m|} \, \sqrt{\psi(2^t)} \;\overset{(29)}{\leq}\; 2^{-t} \, t^{-1} \;<\; 1 \; .$$

Thus

$$| \; q - \sqrt{p_1^2+p_2^2} \; | \;\ll\; 2^{|m|} \, \sqrt{\psi(2^t)} \;\overset{(29)}{\leq}\; t^{-1} \; . \tag{33}$$

The upshot of this is that there exists an absolute constant $c>0$ such that

$$N(t,m) \;\ll\; \sum_{2^t \leqslant q < 2^{t+1}} \;\; \sum_{|q-\sqrt{n}| < c 2^{|m|} \sqrt{\psi(2^t)}} r(n) \; .$$

Now set $Q = 2^t$ and $\Psi = c2^{|m|}\sqrt{\psi(2^t)}$. In view of (26) and (33) we have that (16) is satisfied for all sufficiently large $t$, independently of $m$. Hence, (17) implies that

$$N(t,m) \;\ll\; 2^{|m|} \, 2^{2t} \, \sqrt{\psi(2^t)} \; , \tag{34}$$

where the implied constant is independent of both $t$ and $m$.

It now follows, via (32) and (34) that the Lebesgue measure $| \, . \, |_{\mathcal{C}_1^\varepsilon}$ of the R.H.S. of (28) restricted to case (a) is bounded above by

$$\begin{aligned}
\sum_{t=n}^{\infty} \; \sum_{m \in \mathrm{Case(a)}} 2^{-|m|} \, \frac{\sqrt{\psi(2^t)}}{2^t} \times 2^{|m|} \, 2^{2t} \sqrt{\psi(2^t)} \;\; &\ll \;\; \sum_{t=n}^{\infty} \; \sum_{m \in \mathrm{Case(a)}} 2^t \, \psi(2^t) \\
&\overset{(31)}{\ll} \;\; \sum_{t=n}^{\infty} \; t \, 2^t \psi(2^t) \\
&\asymp \;\; \sum_{q=2^n}^{\infty} \; \psi(q) \, \log q \;\; .
\end{aligned}$$

The above comparability follows from the fact that $\psi$ is an approximating function and therefore decreasing. In view of (25)

$$\sum_{q=2^n}^{\infty} \; \psi(q) \, \log q \;\; \to \;\; 0 \qquad \text{as} \qquad n \to \infty \; ,$$

and so the Lebesgue measure $| \, . \, |_{\mathcal{C}_1^\varepsilon}$ of the R.H.S. of (28) restricted to case (a) tends to zero as $n \to \infty$.

**Case (b)** : In view of (30), we have that

$$S(q,p_1,p_2,m) \;\subset\; S'(q,p_1) \times [0,1] \;\cup\; [0,1] \times S'(q,p_2) \tag{35}$$

where

$$S'(q,p) \;=\; \Big\{ s \in [0,1] : \Big| s - \frac{p}{q} \Big| < \frac{2\,t\,\psi(2^t)}{2^t} \Big\} \tag{36}$$

Thus, the R.H.S. of (28) restricted to case (b) is contained in the following set:

$$\bigcup_{t=n}^{\infty} \;\bigcup_{2^t \leqslant q < 2^{t+1}} \;\bigcup_{(p_1,p_2) \in \mathbb{Z}^2} \mathcal{C}_1^{\varepsilon} \;\cap\; (\, S'(q,p_1) \times [0,1] \cup [0,1] \times S'(q,p_2)\,) \;\; . \tag{37}$$

It is readily verified that for any choice of $p_1, p_2$ and $q$ appearing in (37),

$$|\, \mathcal{C}_1^{\varepsilon} \;\cap\; (\, S'(q,p_1) \times [0,1] \cup [0,1] \times S'(q,p_2)\,)\, |_{\mathcal{C}_1^{\varepsilon}} \;\ll\; t\,\psi(2^t)/2^t \;\; .$$

The implied constant depends only on $\varepsilon$ and is therefore irrelevant. Furthermore, for a fixed $t$ and $q$ in (37) the number of $(p_1,p_2) \in \mathbb{Z}^2$ for which the sets

$$\mathcal{C}_1^{\varepsilon} \;\cap\; (\, S'(q,p_1) \times [0,1] \cup [0,1] \times S'(q,p_2)\,)$$

are non-empty and disjoint is $\ll q$. It now follows that the Lebesgue measure $|\;.\;|_{\mathcal{C}_1^{\varepsilon}}$ of the set given by (37) is bounded above by

$$\sum_{t=n}^{\infty} t\,\psi(2^t) \;\asymp\; \sum_{q=2^n}^{\infty} \psi(q) \log q \;\;\to\;\; 0 \qquad \text{as } n \to \infty \; .$$

Hence the Lebesgue measure $|\;.\;|_{\mathcal{C}_1^{\varepsilon}}$ of the R.H.S. of (28) restricted to case (b) tends to zero as $n \to \infty$.

The upshot of cases (a) and (b) is that the Lebesgue measure $|\;.\;|_{\mathcal{C}_1^{\varepsilon}}$ of the R.H.S. of (28) tends to zero as $n \to \infty$ and so

$$|\mathcal{C}_1^{\varepsilon} \cap \mathcal{S}_2^*(\psi)|_{\mathcal{C}_1^{\varepsilon}} \;=\; 0 \;\; .$$

This completes the proof of Theorem 3.

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# 5 Proof of Theorem 2

The divergence part of Theorem 2 is a consequence of Theorem 1. Thus we proceed with establishing the convergence part of Theorem 2. To a certain degree the proof of this follows the same line of argument as the proof of Theorem 3. In particular, it suffices to establish the convergent statement of the theorem for the unit circle, the parabola and the hyperbola. As in the proof of Theorem 3, we consider the case of the unit circle $\mathcal{C}_1$ only and leave the hyperbola and parabola to the reader.

Let $\mathcal{C} = \mathcal{C}_1$ – the unit circle, and notice that it suffices to prove the theorem for every arc of $\mathcal{C}_1$ given by $\mathcal{C}_1^{\varepsilon} = \{(x,y) \in \mathcal{C}_1 : \varepsilon < x, y < 1 - \varepsilon\}$ with $\varepsilon > 0$. For the sake of convenience, let $\psi := \psi_1$ and $\phi := \psi_2$. It is clear that

$$\mathcal{S}_2(\psi,\phi) \;\subset\; \mathcal{S}_2(\psi^*,\psi_*) \cup \mathcal{S}_2(\psi_*,\psi^*)$$

where

$$\psi_* = \min\{\psi, \phi\} \ , \qquad \psi^* = \max\{\psi, \phi\}.$$

Since $\psi^*\psi_* = \psi\phi$, we have that $\sum \psi^*(q)\psi_*(q) < \infty$. Thus to prove the theorem it suffices to prove that both the sets $\mathcal{C}_1^\varepsilon \cap \mathcal{S}_2(\psi^*, \psi_*)$ and $\mathcal{C}_1^\varepsilon \cap \mathcal{S}_2(\psi_*, \psi^*)$ are of Lebesgue measure $| \ . \ |_{\mathcal{C}_1^\varepsilon}$ zero. We will consider one of these two sets – the other case is similar. Thus, without loss of generality we assume that

$$\psi(q) \ \geqslant \ \phi(q) \quad \text{for all } q \ \ .$$

Since $\sum_{q=1}^{\infty} \psi(q)\,\phi(q) < \infty$ and both $\psi$ and $\phi$ are decreasing we have that $\psi(q)\phi(q) < q^{-1}$ for all sufficiently large $q$. Hence

$$\phi(q) \ \leqslant \ q^{-1/2} \qquad \text{for sufficiently large } q \ .$$

Further, we can assume that

$$\psi(q) \geqslant q^{-2/3} \qquad \text{for all } q \in \mathbb{N} \ \ . \tag{38}$$

To see this, consider the auxiliary function $\widetilde{\psi} : q \to \widetilde{\psi}(q) := \max\{\psi(q), q^{-2/3}\}$. Clearly, $\widetilde{\psi}$ is an approximating function, satisfies (38) and

$$\mathcal{S}_2(\psi, \phi) \subset \mathcal{S}_2(\widetilde{\psi}, \phi) \ .$$

Moreover,

$$\begin{aligned}\sum_{q=1}^{\infty} \widetilde{\psi}(q)\,\phi(q) \leqslant \ & \sum_{q=1}^{\infty} \psi(q)\phi(q) + \sum_{q=1}^{\infty} q^{-2/3}\phi(q) \\ \ll \ & \sum_{q=1}^{\infty} \psi(q)\phi(q) + \sum_{q=1}^{\infty} q^{-2/3}q^{-1/2} \ < \ \infty \ .\end{aligned}$$

Thus, it suffices to prove the convergence part of Theorem 2 with $\psi := \psi_1$ replaced $\widetilde{\psi}$. Hence, without loss of generality, (38) can be assumed.

In analogy to (27), it is readily verified that for any $n \geq 1$

$$\mathcal{C}_1^\varepsilon \cap \mathcal{S}_2(\psi, \phi) \ \subset \ \bigcup_{t=n}^{\infty} \ \bigcup_{2^t \leqslant q < 2^{t+1}} \ \bigcup_{(p_1,p_2) \in \mathbb{Z}^2} \mathcal{C}_1^\varepsilon \cap S_2(p_1, p_2, q) \ \ , \tag{39}$$

where

$$S_2(p_1, p_2, q) = \left\{ (x, y) \in \mathbb{R}^2 : \left| x - \frac{p_1}{q} \right| < \frac{\psi(2^t)}{2^t}, \ \ \left| y - \frac{p_2}{q} \right| < \frac{\phi(2^t)}{2^t} \right\}$$

and $t$ is uniquely defined by $2^t \leqslant q < 2^{t+1}$. Next, in analogy to (32), we verify that

$$|\, \mathcal{C}_1^\varepsilon \cap S_2(q, p_1, p_2)|_{\mathcal{C}_1^\varepsilon} \ \ll \ \frac{\phi(2^t)}{2^t} \ . \tag{40}$$

Again, the implied constant depends only on $\varepsilon$ and is therefore irrelevant to the rest of the argument. For $t$ fixed , let $N(t)$ denote the number of triples $(q, p_1, p_2)$ with $2^t \leqslant q < 2^{t+1}$

such that $\mathcal{C}_1^\varepsilon \cap S(q,p_1,p_2) \neq \emptyset$. On modifying the argument used to establish (33) and (34) in the proof of Theorem 3, one obtains that

$$N(t) \ \ll \ 2^{2t}\psi(2^t) \ . \tag{41}$$

It is worth stressing that the argument within the proof of Theorem 3 is much simplified in the current situation due to the absence of the additional parameter $m$.

The upshot of the above inclusions and estimates is that

$$\begin{aligned}
|\, \mathcal{C}_1^\varepsilon \cap \mathcal{S}_2(\psi,\phi)|_{\mathcal{C}_1^\varepsilon} &\overset{(39)}{\ll} \sum_{t=n}^{\infty} \ \sum_{2^t \leqslant q < 2^{t+1}} \ \bigcup_{(p_1,p_2)\in\, \mathbb{Z}^2} |\mathcal{C}_1^\varepsilon \cap S_2(q,p_1,p_2)| \\
&\overset{(40)}{\ll} \sum_{t=n}^{\infty} N(t)\, \frac{\phi(2^t)}{2^t} \\
&\overset{(41)}{\ll} \sum_{t=n}^{\infty} 2^t\psi(2^t)\phi(2^t) \\
&\asymp \sum_{q=2^n}^{\infty} \psi(q)\phi(q)
\end{aligned}$$

Since $\sum_{q=2^n}^{\infty} \psi(q)\phi(q) < \infty$, we have that $\sum_{q=2^n}^{\infty} \psi(q)\phi(q) \to 0$ as $n \to \infty$. Thus,

$$|\, \mathcal{C}_1^\varepsilon \cap \mathcal{S}_2(\psi,\phi)|_{\mathcal{C}_1^\varepsilon} = 0 \ .$$

This completes the proof of the theorem.

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# 6 Proofs of Theorems 4 – 6

## 6.1 Proof of Theorem 4

The statement of the theorem will follow on establishing the upper and lower bounds for the dimension separately. Without loss of generality we can assume that $f$ satisfies (13) on $I_0$ (see [3, §5] if necessary) and that $v_1 \geqslant v_2$. In view of the latter, our aim is to show that

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(v_1,v_2) \ = \ \frac{2-v_2}{1+v_1} \ .$$

*The upper bound.* For a point $\mathbf{p}/q \in \mathbb{Q}^2$, define

$$\sigma(\mathbf{p}/q) := \{(x,y)\in\mathbb{R}^2 : |x-p_1/q| < q^{-v_1-1}, \ |y-p_2/q| < q^{-v_2-1}\}.$$

In view of (13) and the fact that $I_0$ is a bounded interval we have that $f'$ is bounded on $I_0$ and so $|\mathcal{C}_f \cap \sigma(\mathbf{p}/q)| \ll q^{-v_1-1}$. Clearly, if $\sigma(\mathbf{p}/q)\cap\mathcal{C}_f \neq \emptyset$ then the distance of $\mathbf{p}/q$ from $\mathcal{C}_f$ is at most a constant times $q^{-1-v_2}$. Let $\varepsilon > 0$. In view of (15) we have that for $t$ sufficiently large the number $\mathbf{p}/q \in \mathbb{Q}^2$ with $2^t \leqslant q < 2^{t+1}$ and $\sigma(\mathbf{p}/q) \cap \mathcal{C}_f \neq \emptyset$ is at most $2^{t(2+\varepsilon-v_2)}$. Now let

$$\eta \ := \ \frac{2-v_2+2\varepsilon}{1+v_1} \ .$$

Then

$$\begin{aligned}\sum_{\mathbf{p}/q\in\mathbb{Q}^2 : \mathcal{C}_f\cap\sigma(\mathbf{p}/q)\neq\emptyset} \operatorname{diam}(\mathcal{C}_f \cap \sigma(\mathbf{p}/q))^\eta \ &= \ \sum_{t=0}^{\infty} \sum_{\mathbf{p}/q\in\mathbb{Q}^2,\, \mathcal{C}_f\cap\sigma(\mathbf{p}/q)\neq\emptyset,\, 2^t\leqslant q<2^{t+1}} \operatorname{diam}(\mathcal{C}_f \cap \sigma(\mathbf{p}/q))^\eta \\ &\ll \ \sum_{t=0}^{\infty} 2^{t(2+\varepsilon-v_2)} \cdot 2^{t(-1-v_1)\eta} \ = \ \sum_{t=0}^{\infty} 2^{-t\varepsilon} \ < \ \infty \ .\end{aligned}$$

By the Hausdorff–Cantelli Lemma [4, p. 68], $\dim \mathcal{C}_f \cap \mathcal{S}_2(v_1, v_2) \leqslant \eta$. As $\varepsilon > 0$ is arbitrary,

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(v_1, v_2) \ \leqslant \ \frac{2-v_2}{1+v_1} \ . \tag{42}$$

*The lower bound.* Firstly, with reference to Lemma 3 let $\psi(t) := \frac{1}{2}t^{-v_2}$ and $u(t) := t^\varepsilon$ where $\varepsilon > 0$ is arbitrary. Thus $\Phi(t) = \frac{1}{2}t^{-1-v_2}$ and $\rho(t) := \frac{1}{2}t^{-2+v_2+\varepsilon}$. Since $0 < v_2 = \min(v_1, v_2) < 1$, the approximating function $\psi$ satisfies (14) and it follows that $(\mathbb{Q}^2_{\mathcal{C}}(\Phi), \beta)$ is locally ubiquitous with respect to $\rho$. Next, let $\Phi(t) := \frac{1}{2}t^{-1-v_1}$. Then Lemma 2 implies that

$$\dim \Lambda\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right) \ \geqslant \ \min\left\{1, \left|\limsup_{t\to\infty} \frac{\log \frac{1}{2}2^{t(-(2-v_2-\varepsilon))}}{\log \frac{1}{2}2^{t(-(1+v_1))}}\right|\right\} \ = \ \frac{2-v_2-\varepsilon}{1+v_1}.$$

As $\varepsilon > 0$ can be made arbitrarily small, we have that $\dim \Lambda\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right) \geqslant \frac{2-v_2}{1+v_1}$. Finally, it is readily verified that

$$\widetilde{\Lambda}\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right) := \{(x, f(x)) : x \in \Lambda\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right)\} \ \subset \ \mathcal{C}_f \cap \mathcal{S}_2(v_1, v_2).$$

Hence

$$\dim \mathcal{C}_f \cap \mathcal{S}_2(v_1, v_2) \ \geq \ \dim \widetilde{\Lambda}\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right) \ = \ \dim \Lambda\left(\mathcal{R}_{\mathcal{C}}(\Phi), \beta, \Psi\right) \ \geq \ \frac{2-v_2}{1+v_1} \ .$$

The equality here is justified by the fact that the map $x \mapsto (x, f(x))$ is locally bi-Lipshitz.

♠

## 6.2 Proof of Theorem 5

Let $m = \dim \mathcal{M}$. Since $\mathcal{M}$ is a Lipshitz manifold in $\mathbb{R}^n$, there exists a local parameterization of $M$ of the form $\mathbf{f} = (x_1, \ldots, x_m, f_{m+1}, \ldots, f_n)$ where $\mathbf{f}$ is an invertible continuous map of $x_1, \ldots, x_m$ defined on $\mathbb{R}^m$ such that $\mathbf{f}^{-1}$ satisfies the Lipshitz condition. It is easy to verify that any point on $\mathcal{M}$ with $x_1 \in \mathcal{S}_1^*(v)$ belongs to $\mathcal{S}_n^*(v)$. Therefore,

$$B := \mathbf{f}(\mathcal{S}_1^*(v) \times \mathbb{R}^{m-1}) \ \subset \ \mathcal{S}_n^*(v) \cap \mathcal{M} \ \ .$$

Since $\mathbf{f}^{-1}$ is a Lipshitz map and $\mathbf{f}^{-1}(B) = \mathcal{S}_1^*(v) \times \mathbb{R}^{m-1}$, it follows that for $v \geq 1$

$$\begin{aligned} \dim \mathcal{S}_n^*(v) \cap \mathcal{M} \;\geqslant\; \dim B \;\geqslant\; \dim \mathcal{S}_1^*(v) \times \mathbb{R}^{m-1} \;&=\; m - 1 + \dim \mathcal{S}_1^*(v) \\ &=\; \dim \mathcal{M} - 1 + \frac{2}{1+v} \;. \end{aligned} \tag{43}$$

The fact that $\dim \mathcal{S}_1^*(v) = 2/(1+v)$ is the Jarník–Besicovitch theorem (see §1.1). ♠

### 6.3 Proof of Theorem 6

The lower bound is a trivial consequence of Theorem 5. Alternatively, it follows from Theorem 4 with $v_1 = \varepsilon$, $v_2 = v - \varepsilon$ and then letting $\varepsilon \to 0$.

To establish the complementary upper bound we fix $v > 1$ and without loss of generality assume that $f$ satisfies (13) on $I_0$. The case that $v = 1$ is trivial. Now fix $\varepsilon$ such that

$$0 \;<\varepsilon\;<\; \min\{1/(1+v),\, 1/5\} \qquad\qquad \text{and} \qquad\qquad v - \varepsilon \;>\; 1 \;\;.$$

The following inclusions readily follow from the definitions of $\mathcal{S}_2^*(v)$ and $\mathcal{S}_2(v_1, v_2)$ :

$$\mathcal{S}_2^*(v) \;\subset\; \mathcal{S}_2(v-\varepsilon, 0) \;\cup\; \mathcal{S}_2(0, v-\varepsilon) \;\cup\; \bigcup_{t=-t_0}^{t_0} \mathcal{S}_2(v_1(t), v_2(t)) \;\subset\; \mathcal{S}_2^*(v-\varepsilon) \;, \tag{44}$$

where $t_0 > 0$ is the unique positive integer satisfying $\frac{v}{2\varepsilon} - \frac{3}{2} \leq t_0 < \frac{v}{2\varepsilon} - \frac{1}{2}$ and

$$v_1(t) \;:=\; \tfrac{v}{2} - \tfrac{(2t+1)\varepsilon}{2} \qquad\qquad v_2(t) \;:=\; \tfrac{v}{2} + \tfrac{(2t-1)\varepsilon}{2} \;\;.$$

The required upper bound will follow on establishing the corresponding upper bounds for the sets 'between the inclusions' of (44). For this we will repeatedly apply Theorem 4.

First, consider the sets $\mathcal{S}_2(v_1(t), v_2(t))$ for $t \geqslant 0$ (the case $t < 0$ is similar). So, $v_1(t) \leq v_2(t)$. Assume for the moment that $v_1(t) < 1$. Then $v_1(t) \in (0,1)$ and since $v_1(t) + v_2(t) = v - \varepsilon > 1$ we have via Theorem 4 that

$$\dim\, \mathcal{S}_2(v_1(t), v_2(t)) \cap \mathcal{C}_f \;=\; \frac{2 - v_1(t)}{1 + v_2(t)} \;=\; \frac{2 - v_1(t)}{1 + v - \varepsilon - v_1(t)} \;\leq\; \frac{2}{1 + v - \varepsilon} \;\;.$$

Now suppose that $v_1(t) \geq 1$. It follows from the definition of $v_1(t)$ that $v > 2$. Trivially, $\mathcal{S}_2(v_1(t), v_2(t)) \subset \mathcal{S}_2(1-\varepsilon, v/2)$. Now, $v/2 > 1 - \varepsilon > 0$ and on applying Theorem 4 we have that

$$\dim\, \mathcal{S}_2(v_1(t), v_2(t)) \cap \mathcal{C}_f \;\leq\; \dim\, \mathcal{S}_2(1-\varepsilon, v/2) \cap \mathcal{C}_f \;=\; \frac{2+2\varepsilon}{2+v} \;\leqslant\; \frac{2}{1+v} \;.$$

Next, we consider the set $\mathcal{S}_2(v-\varepsilon, 0)$ – the case of $\mathcal{S}_2(0, v-\varepsilon)$ is similar. By definition, $\mathcal{S}_2(v-\varepsilon, 0) = \mathcal{S}_1(v-\varepsilon) \times \mathbb{R}$ and so

$$\dim\, \mathcal{S}_2(v-\varepsilon, 0) \cap \mathcal{C}_f \;\leqslant\; \dim \mathcal{S}_1(v-\varepsilon) \;=\; \frac{2}{1+v-\varepsilon} \;.$$

The upshot is that

$$\dim \mathcal{S}_2^*(v) \cap \mathcal{C}_f \;\leq\; \max\left\{\frac{2}{1+v-\varepsilon}, \frac{2}{1+v}\right\} \;=\; \frac{2}{1+v-\varepsilon}\,,$$

and since $\varepsilon$ can be made arbitrarily small the required upper bound follows. ♠

# 7 Final remarks: the dual form of approximation

In view of Khintchine's transference principle [13], Theorem KM can be reformulated for the dual form of approximation:

**Theorem KM′** *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$. Then for any $v > 1$ for almost every point $(y_1, \ldots, y_n) \in \mathcal{M}$ the inequality*

$$\|a_1y_1 + \ldots + a_ny_n\| < \Pi_+(\mathbf{a})^{-v} \tag{45}$$

*has only finite number of solutions $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, where*

$$\Pi_+(\mathbf{a}) := \prod_{i=1}^{n} \max\{1, |a_i|\}\,.$$

The problems S1 and S2 considered in §1.2 above can therefore be reformulated for the dual form of approximation. Given an approximating function $\psi$, consider the inequality

$$\|a_1y_1 + \ldots + a_ny_n\| < \psi(\Pi_+(\mathbf{a}))\ . \tag{46}$$

Let

$$\mathcal{L}_n^*(\psi) := \{\mathbf{y} \in \mathbb{R}^n :\ (46) \text{ holds for infinitely many } \mathbf{a} \in \mathbb{Z}^n\}$$

and

$$\mathcal{L}_n^*(v) := \mathcal{S}^*(q \mapsto q^{-v}) := \{\mathbf{y} \in \mathbb{R}^n :\ (45) \text{ holds for infinitely many } \mathbf{a} \in \mathbb{Z}^n\}\,.$$

**Problem D1 :** Given a non-degenerate manifold $\mathcal{M} \subset \mathbb{R}^n$ and $v > 1$, what is the Hausdorff dimension of $\mathcal{L}^*(v) \cap \mathcal{M}$ ?

Note that above theorem of Kleinbock and Margulis only implies that the Lebesgue measure of $\mathcal{L}^*(v) \cap \mathcal{M}$ is zero.

**Problem D2 :** Given a non-degenerate manifold $\mathcal{M} \subset \mathbb{R}^n$ and an approximating function $\psi$, what is the weakest condition under which $\mathcal{L}^*(\psi) \cap \mathcal{M}$ is of Lebesgue measure zero ?

Regarding Problem D1 the following general lower bound can be established:

**Theorem 7** *Let $\mathcal{M}$ be arbitrary manifold in $\mathbb{R}^n$. Then for any $v > 1$*

$$\dim \mathcal{M} \cap \mathcal{L}_n^*(v) \;\geqslant\; \dim \mathcal{M} - 1 + \frac{2}{1+v}\,. \tag{47}$$

The proof of Theorem 7 follows the same line of reasoning as that of Theorem 5 and is left to the reader. It is highly likely that the inequality given by (47) is in fact an equality. For $n = 2$, that this is indeed the case is easily verified by modifying the arguments of [14]. However, the general case ($n \geq 3$) seems to be a difficult problem.

Regarding Problem D2 a general Khintchine-Groshev type theorem for convergence has been established in [5]. This states that in Theorem KM$'$ above one can replace (45) with (46) whenever

$$\sum_{h=1}^{\infty} \psi(h) \log^{n-1} h \ < \ \infty \ .$$

The divergence counterpart remains an open problem even for planar curves.

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Victor V. Beresnevich: Institute of Mathematics, Academy of Sciences of Belarus,
220072, Surganova 11, Minsk, Belarus.

e-mail: beresnevich@im.bas-net.by

Sanju L. Velani: Department of Mathematics, University of York,
Heslington, York, YO10 5DD, England.

e-mail: slv3@york.ac.uk