Diophantine approximation and badly approximable sets

Simon Kristensen [a,1] Rebecca Thorn [b] Sanju Velani [c,*,2]

[a] *School of Mathematics, The University of Edinburgh, James Clerk Maxwell Building, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, UK*

[b] *School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK*

[c] *Department of Mathematics, University of York, Heslington, York YO10 5DD, UK*

Abstract

Let (X, d) be a metric space and (Ω, d) a compact subspace of X which supports a non-atomic finite measure m. We consider 'natural' classes of badly approximable subsets of Ω. Loosely speaking, these consist of points in Ω which 'stay clear' of some given set of points in X. The classical set **Bad** of 'badly approximable' numbers in the theory of Diophantine approximation falls within our framework as do the sets $\mathbf{Bad}(i, j)$ of simultaneously badly approximable numbers. Under various natural conditions we prove that the badly approximable subsets of Ω have full Hausdorff dimension. Applications of our general framework include those from number theory (classical, complex, p-adic and formal power series) and dynamical systems (iterated function schemes, rational maps and Kleinian groups).

Key words: Diophantine approximation, Hausdorff dimension, badly approximable elements, dynamical systems.

1991 MSC: 11J83, 11J61, 28A80, 37C45, 37F

Dedicated to Lalji and Manchaben on their 70 plus birthdays

* Corresponding author
Email addresses: Simon.Kristensen@ed.ac.uk (Simon Kristensen), r.e.thorn@qmul.ac.uk (Rebecca Thorn), slv3@york.ac.uk (Sanju Velani).

1 William Gordon Seggie Brown Fellow

2 Royal Society University Research Fellow

1 Introduction

1.1 The setup and the problem

Let (X, d) be a metric space and (Ω, d) a compact subspace of X which contains the support of a non-atomic finite measure m. Let $\mathcal{R} = \{R_\alpha \in X : \alpha \in J\}$ be a family of subsets R_α of X indexed by an infinite, countable set J. The sets R_α will be referred to as *resonant sets*. Next, let $\beta : J \to \mathbb{R}^+ : \alpha \to \beta_\alpha$ be a positive function on J. To avoid pathological situations within our framework, we shall assume that the number of $\alpha \in J$ with β_α bounded above is finite – thus β_α tends to infinity as α runs through J. Given a real, positive function $\rho : \mathbb{R}^+ \to \mathbb{R}^+ : r \to \rho(r)$ such that $\rho(r) \to 0$ as $r \to \infty$ and that ρ is decreasing for r large enough, consider the set

$$\mathbf{Bad}^*(\mathcal{R}, \beta, \rho) := \{x \in \Omega : \exists\, c(x) > 0 \text{ such that } d(x, R_\alpha) \geq c(x)\rho(\beta_\alpha) \\ \text{for all } \alpha \in J\},$$

where $d(x, R_\alpha) := \inf_{a \in R_\alpha} d(x, a)$. Loosely speaking, in the case that the resonant sets are points, $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ consists of points in Ω which 'stay clear' of 'ρ-balls' centred at resonant points. Notice that since the number of $\alpha \in J$ with β_α bounded above is finite and ρ is eventually decreasing, the number of $\alpha \in J$ with $\rho(\beta_\alpha) \geq \varepsilon > 0$ is finite. In view of this, without loss of generality we shall assume that the $\sup_{\alpha \in J} \rho(\beta_\alpha)$ is finite. Otherwise, if $\rho(\beta_\alpha)$ can get arbitrarily large, then trivially $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho) = \emptyset$ – recall that Ω is compact and so is bounded.

The set $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ is easily seen to be a generalization of the classical set $\mathbf{Bad}$ of badly approximable numbers. Recall, a real number x is said to be badly approximable if there exists a constant $c(x) > 0$ such that $|x - p/q| \geq c(x)/q^2$ for all rational p/q. A result of Jarník [10] states that the Hausdorff dimension of $\mathbf{Bad}$ is maximal; i.e. $\dim \mathbf{Bad} = 1$. Our initial aim is to find a suitably general framework which allows us to conclude that $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho) = \dim \Omega$; that is to say that the set of badly approximable points in Ω is of maximal dimension. To a certain extent, this paper complements [3] in which a general framework for establishing measure theoretic laws for 'well approximable' sets is established.

A few words about our chosen notation are in order. In the above setup and its generalization in §2, the sets of badly approximable elements will be denoted by $\mathbf{Bad}^*$ followed by the appropriate variables in brackets. In applications we define a set, usually denoted by $\mathbf{Bad}$ with appropriate arguments, and show that this set may be realized as a specialization of a general set $\mathbf{Bad}^*$.

1.2 The conditions on the setup

Throughout, a ball $B(c,r)$ with centre c and radius r is defined to be the set $\{x \in X : d(c,x) \leq r\}$. Thus all balls will be assumed to be closed unless stated otherwise and by definition a ball is a subset of X. The following conditions on the measure m and the function ρ will play a central role in our work.

(A) There exist strictly positive constants δ and r_0 such that for $c \in \Omega$ and $r \leq r_0$

$$a\, r^{\delta} \leq \ \ m(B(c,r)) \ \ \leq b\, r^{\delta} \ ,$$

where $0 < a \leq 1 \leq b$ are constants independent of the ball.

It is easily verified that if the measure m supported on Ω is of type (A) then $\dim \Omega = \delta$. Trivially, this implies that $\dim X \geq \delta$. See §3 for the details.

(B) For $k > 1$ sufficiently large and any integer $n \geq 1$,

$$\lambda^l(k) \leq \frac{\rho(k^n)}{\rho(k^{n+1})} \leq \lambda^u(k)$$

where λ^l and λ^u are lower and upper bounds depending only on k such that $\lambda^l(k) \to \infty$ as $k \to \infty$.

Note that this condition on ρ is satisfied by any function satisfying the following 'regularity' condition. There exist a constant $k > 1$ such that for r sufficiently large

$$\lambda^l \ \leq \ \frac{\rho(r)}{\rho(k\, r)} \ \leq \ \lambda^u \quad ,$$

where $1 < \lambda^l \leq \lambda^u$ are constants independent of r but may depend on k.

1.3 The result

First some useful notation. For any $k > 1$ let $B_n := \{x \in \Omega : d(c,x) \leq \rho(k^n)\}$ denote a generic closed ball of radius $\rho(k^n)$ with centre c in Ω and for $\theta \in \mathbb{R}^+$, let $\theta B_n := \{x \in \Omega : d(c,x) \leq \theta\rho(k^n)\}$ denote the ball B_n scaled by θ. Notice, that by definition any generic ball B_n is a subset of Ω. Also, for $n \geq 1$ let $J(n) := \{\alpha \in J : k^{n-1} \leq \beta_\alpha < k^n\}$.

Theorem 1 *Let (X, d) be a metric space and (Ω, d, m) a compact measure subspace of X. Let the measure m and the function ρ satisfy conditions (A) and (B) respectively. For $k \geq k_0 > 1$, suppose there exists some $\theta \in \mathbb{R}^+$ so that for $n \geq 1$ and any ball B_n there exists a collection $\mathcal{C}(\theta B_n)$ of disjoint balls $2\theta B_{n+1}$ contained within θB_n satisfying*

$$\#\mathcal{C}(\theta B_n) \ \geq \ \kappa_1 \left(\frac{\rho(k^n)}{\rho(k^{n+1})} \right)^{\delta} \tag{1}$$

and

$$\# \left\{ 2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : \min_{\alpha \in J(n+1)} d(c, R_\alpha) \leq 2\theta \rho(k^{n+1}) \right\} \ \leq \ \kappa_2 \left(\frac{\rho(k^n)}{\rho(k^{n+1})} \right)^{\delta} , \tag{2}$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$. Then

$$\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho) = \delta \quad .$$

Remarks:

(i) In applications, the 'scaling factor' θ is usually dependent on k – see the basic example below. For k sufficiently large, it is always possible to find the collection $\mathcal{C}(\theta B_n)$ satisfying condition (1) – see §3 for the details. Finally, note that in the case that the resonant sets are points $\dim(\cup_{\alpha \in J} R_\alpha) = 0$ and the hypothesis that $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$ is trivially satisfied. This follows from the fact that the indexing set J is countable.

(ii) We suspect that Theorem 1 can be established using Schmidt games [20] – a standard mechanism in the subject to prove such full dimension results. However, we will deduce the result from a more general one (Theorem 2 below) which we have not been able to prove using Schmidt games. In fact, it is not at all clear that the Schmidt games mechanism is even applicable.

(iii) Here and in subsequent theorems, we consider families of general resonant sets R_α. However, in all the applications considered in §5, the resonant sets are assumed to be points. There are natural problems of the same type where this is not the case. For example, when considering the classical problem of approximation of systems of linear forms over $\mathbb{R}$ the resonant sets are affine spaces in $\mathbb{R}^n$ (see [21]). For reasons which will be explained in the final part of §2.3, our results are not immediately applicable to this situation. In a forthcoming paper [13], we will treat this aspect and related problems.

1.4 The basic example: Bad

Let $I = [0,1]$ and consider the set

$$\mathbf{Bad}_I := \{x \in [0,1] : |x - p/q| > c(x)/q^2 \text{ for all rationals } \ p/q \ \ (q > 0)\} \ .$$

This is the classical set **Bad** of badly approximable numbers restricted to the unit interval. Clearly, it can be expressed in the form $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ with $\rho(r) := r^{-2}$ and

$$X = \Omega := [0,1] \ , \quad J := \{(p,q) \in \mathbb{N} \times \mathbb{N}\backslash\{0\} : p \leq q\} \ ,$$

$$\alpha := (p,q) \in J \ , \quad \beta_\alpha := q \ , \quad R_\alpha := p/q \ .$$

The metric d is of course the standard Euclidean metric; $d(x,y) := |x - y|$. Thus in this basic example, the resonant sets R_α are simply rational points p/q and the function ρ clearly satisfies condition (B). With reference to our framework, let the measure m be one–dimensional Lebesgue measure on I. Thus, $\delta = 1$ and m clearly satisfies condition (A).

We show that the conditions of Theorem 1 are satisfied for this basic example. The existence of the collection $\mathcal{C}(\theta B_n)$, where B_n is an arbitrary closed interval of length $2\,k^{-2n}$ follows immediately from the following simple observation. For any two distinct rationals p/q and p'/q' with $k^n \leq q, q' < k^{n+1}$ we have that

$$\left| \frac{p}{q} - \frac{p'}{q'} \right| \geq \frac{1}{qq'} > k^{-2n-2} \ .$$

Thus, any interval θB_n with $\theta := \frac{1}{2} k^{-2}$ contains at most one rational p/q with $k^n \leq q < k^{n+1}$. Let $\mathcal{C}(\theta B_n)$ denote the collection of intervals $2\theta B_{n+1}$ obtained by subdividing θB_n into intervals of length $2k^{-2n-4}$ starting from the left hand side of θB_n. Clearly

$$\#\mathcal{C}(\theta B_n) \ \geq \ [k^2/2] \ > \ k^2/4 \ = \ \text{r.h.s. of (1) with } \ \kappa_1 := 1/4 \ .$$

Also, in view of the above observation, for k sufficiently large

$$\text{l.h.s. of (2)} \ \leq \ 1 \ < \ k^2/8 \ = \text{r.h.s. of (2) with } \kappa_2 := 1/8 \ .$$

The upshot of this is that Theorem 1 implies that

$$\dim \ \mathbf{Bad}_I = \ 1 \ \ .$$

In turn, since **Bad** is a subset of $\mathbb{R}$, this implies that $\dim \mathbf{Bad} = 1$ – the classical result of Jarník [10].

2 A more general framework

We now consider a more general framework in which the 'badly approximable' set consists of points avoiding 'rectangular' neighborhoods of resonant sets rather than simply 'balls'.

Let (X,d) be the product space of t metric spaces (X_i,d_i) and let (Ω,d) be a compact subspace of X which contains the support of a non-atomic finite measure m. As before, let $\mathcal{R}=\{R_\alpha \in X : \alpha \in J\}$ be a family of subsets R_α of X indexed by an infinite, countable set J. Thus, each resonant set R_α can be split into its t components $R_{\alpha,i}\subset (X_i,d_i)$. As before, let $\beta : J \to \mathbb{R}^+ : \alpha \to \beta_\alpha$ be a positive function on J and assume that the number of $\alpha \in J$ with β_α bounded above is finite.

For each $1\le i\le t$, let $\rho_i : \mathbb{R}^+ \to \mathbb{R}^+ : r \to \rho_i(r)$ be a real, positive function such that $\rho_i(r)\to 0$ as $r\to\infty$ and that ρ_i is decreasing for r large enough. Furthermore, assume that $\rho_1(r)\ge\rho_2(r)\ge\cdots\ge\rho_t(r)$ for r large – the ordering is irrelevant. Given a resonant set R_α, let

$$F_\alpha(\rho_1,\ldots,\rho_t) := \{x\in X \ : \ d_i(x_i,R_{\alpha,i})\le \rho_i(\beta_\alpha) \text{ for all } 1\le i\le t\} \ ,$$

denote the 'rectangular' $(\rho_1,\ldots,\rho_t)$-neighborhood of R_α and consider the set

$$\begin{aligned}\mathbf{Bad}^*(\mathcal{R},\beta,\rho_1,\ldots,\rho_t) := \{x\in\Omega : \exists\, c(x)>0 \text{ such that} \\ x\notin c(x)\,F_\alpha(\rho_1,\ldots,\rho_t) \text{ for all } \alpha\in J\} \ .\end{aligned}$$

Thus, $x\in \mathbf{Bad}^*(\mathcal{R},\beta,\rho_1,\ldots,\rho_t)$ if there exists a constant $c(x)>0$ such that for all $\alpha\in J$,

$$d_i(x_i,R_{\alpha,i}) \ \ge \ c(k)\,\rho_i(\beta_\alpha) \quad (1\le i\le t) \ .$$

Clearly, $\mathbf{Bad}^*(\mathcal{R},\beta,\rho_1,\ldots\rho_t)$ is precisely the set $\mathbf{Bad}^*(\mathcal{R},\beta,\rho)$ of §1.1 in the case $t=1$. The overall aim of this section is to find a suitably general framework which gives a lower bound for the Hausdorff dimension of $\mathbf{Bad}^*(\mathcal{R},\beta,\rho_1,\ldots,\rho_t)$. We shall assume that $\sup_{\alpha\in J}\rho_i(\beta_\alpha)$ is finite for each i without loss of generality – otherwise $\mathbf{Bad}^*(\mathcal{R},\beta,\rho_1,\ldots,\rho_t)=\emptyset$ and there is nothing to prove.

2.1 The conditions on the general framework

Given $l_1,\ldots,l_t\in\mathbb{R}^+$ and $c\in\Omega$ let

$$F(c;l_1,\ldots,l_t) \ := \ \{x\in X \ : \ d_i(x_i,c_i)\le l_i \text{ for all } \ 1\le i\le t\} \ ,$$

denote the closed 'rectangle' centred at c with 'sidelengths' determined by $l_1, \ldots, l_t$. Also, for any $k > 1$ and $n \in \mathbb{N}$, let F_n denote a generic rectangle $F(c; \rho_1(k^n), \ldots, \rho_t(k^n)) \cap \Omega$ in Ω centred at a point c in Ω. As before, $B(c, r)$ is a closed ball with centre c and radius r. The following conditions on the measure m and the functions ρ_i will play a central role in our general framework. The first two are reminiscent of conditions (A) and (B) of §1.2.

(A*) There exists a strictly positive constant δ such that for any $c \in \Omega$

$$\liminf_{r \to 0} \frac{\log m(B(c, r))}{\log r} \ = \ \delta \ .$$

It is easily verified that if the measure m supported on Ω is of type (A*) then $\dim \Omega \geq \delta$ [6, Proposition 4.9] and so $\dim X \geq \delta$. Clearly condition (A) of §1.2 implies (A*).

(B*) For $k > 1$ sufficiently large, any integer $n \geq 1$ and any $i \in \{1, \ldots, t\}$,

$$\lambda_i^l(k) \leq \frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \leq \lambda_i^u(k),$$

where λ_i^l and λ_i^u are lower and upper constants such that $\lambda_i^l(k) \to \infty$ as $k \to \infty$.

Clearly, this is just condition (B) of §1.2 imposed on each function ρ_i.

(C*) There exist constants $0 < a \leq 1 \leq b$ and $l_0 > 0$ such that

$$a \ \leq \ \frac{m(F(c; l_1, \ldots, l_t))}{m(F(c'; l_1, \ldots, l_t))} \ \leq \ b,$$

for any $c, c' \in \Omega$ and any $l_1, \ldots, l_t \leq l_0$.

This condition implies that rectangles of the same size centred at points of Ω have comparable m measure.

(D*) There exist strictly positive constants D and l_0 such that

$$\frac{m(2\, F(c; l_1, \ldots, l_t))}{m(F(c; l_1, \ldots, l_t))} \ \leq \ D,$$

for any $c \in \Omega$ and any $l_1, \ldots, l_t \leq l_0$.

This condition simply says that the measure m is 'doubling' with respect to rectangles. In terms of achieving our aim of obtaining a lower bound for $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots \rho_t)$, the above four conditions are rather natural. The following final condition is in some sense the only genuine technical condition and is not particularly restrictive.

(E*) For $k > 1$ sufficiently large and any integer $n \geq 1$

$$\frac{m(F_n)}{m(F_{n+1})} \;\geq\; \lambda(k) \;,$$

where λ is a constant such that $\lambda(k) \to \infty$ as $k \to \infty$.

2.2 The general result

Recall, that $F_n := \{x \in \Omega : \; d_i(x_i, c_i) \leq \rho_i(k^n) \;\; \text{for all } 1 \leq i \leq t\}$ is a generic rectangle with centre c in Ω and 'sidelengths' determined by $\rho_i(k^n)$ and for $\theta \in \mathbb{R}^+$, θF_n is the rectangle F_n scaled by θ. Also, for $n \geq 1$ let $J(n) := \{\alpha \in J : k^{n-1} \leq \beta_\alpha < k^n\}$.

Theorem 2 *Let (X, d) be the product space of the metric spaces (X_1, d_1), $\ldots, (X_t, d_t)$ and let (Ω, d, m) be a compact measure subspace of X. Let the measure m and the functions ρ_i satisfy conditions (A*) to (E*). For $k \geq k_0 > 1$, suppose there exists some $\theta \in \mathbb{R}^+$ so that for $n \geq 1$ and any rectangle F_n there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1}$ contained within θF_n satisfying*

$$\#\mathcal{C}(\theta F_n) \;\geq\; \kappa_1 \, \frac{m(\theta F_n)}{m(\theta F_{n+1})} \tag{3}$$

and

$$\#\left\{ 2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : \min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \leq 2\theta\rho_i(k^{n+1}) \;\; \textit{for any } 1 \leq i \leq t \right\} \leq \kappa_2 \, \frac{m(\theta F_n)}{m(\theta F_{n+1})} \;. \tag{4}$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$. Then

$$\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots \rho_t) \geq \delta \;\; .$$

Remarks: For k sufficiently large, it is always possible to find the collection $\mathcal{C}(\theta F_n)$ satisfying condition (3). Clearly, the lower bound for $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ in Theorem 1 is an immediate consequence of Theorem 2. To see this, simply note that if $t = 1$ then the rectangles F_n are balls B_n and if conditions (A) and (B) are satisfied then trivially so are the conditions (A*) to (E*). In fact, if condition (A*) is replaced by the stronger condition (A) in the above theorem, then we are able to conclude that $\dim \mathbf{Bad}(\mathcal{R}, \beta, \rho_1, \ldots \rho_t) = \delta$ – see below.

We now consider an extremely useful specialization of the above general framework in which the space Ω is a product space equipped with a product measure.

Theorem 3 *For $1 \leq i \leq t$, let (X_i, d_i) be a metric space and (Ω_i, d_i, m_i) be a compact measure subspace of X_i where the measure m_i satisfies condition (A) with exponent δ_i. Let (X, d) be the product space of the spaces (X_i, d_i) and let (Ω, d, m) be the product measure space of the measure spaces (Ω_i, d_i, m_i). Let the functions ρ_i satisfy condition (B*). For $k \geq k_0 > 1$, suppose there exists some $\theta \in \mathbb{R}^+$ so that for $n \geq 1$ and any rectangle F_n there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1}$ contained within θF_n satisfying*

$$\#\mathcal{C}(\theta F_n) \;\geq\; \kappa_1 \prod_{i=1}^{t} \left(\frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \right)^{\delta_i} \tag{5}$$

and

$$\# \left\{ 2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : \min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \leq 2\theta \rho_i(k^{n+1}) \;\; \textit{for any } 1 \leq i \leq t \right\}$$
$$\leq \kappa_2 \prod_{i=1}^{t} \left(\frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \right)^{\delta_i} \quad , \tag{6}$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose $\dim(\cup_{\alpha \in J} R_\alpha) < \sum_{i=1}^{t} \delta_i$. Then

$$\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots \rho_t) = \sum_{i=1}^{t} \delta_i \quad .$$

The deduction of Theorem 3 from Theorem 2 is relatively straightforward and hinges on the following simple observation. Since m is the product measure of the measures m_i and the latter satisfy condition (A) with exponents δ_i $(1 \leq i \leq t)$, we have for any $c \in \Omega$ and any $l_1, \ldots, t_t \leq l_0$ that

$$a^t \;\leq\; \frac{m(F(c; l_1, \ldots, l_t))}{\prod_{i=1}^{t} l_i^{\delta_i}} \;\leq\; b^t. \tag{7}$$

It follows that conditions (C*) and (D*) are trivially satisfied as is condition (A) with $\delta := \sum_{i=1}^{t} \delta_i$. Recall, that (A) implies (A*). Also, (7) together with (B*) implies that condition (E*) is satisfied. Thus, Theorem 2 implies the desired lower bound estimate for $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots \rho_t)$. The complementary upper bound estimate is a simple consequence of the fact that m satisfies (A). If m satisfies (A), then $\dim \Omega = \delta$ [6, Proposition 4.9] and since $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) \subseteq \Omega$ the upper bound follows.

2.3 The general basic example: $\mathbf{Bad}(i,j)$

For $i,j \geq 0$ with $i+j=1$, denote by $\mathbf{Bad}(i,j)$ the set of (i,j)-badly approximable pairs $(x_1,x_2) \in \mathbb{R}^2$; that is $(x_1,x_2) \in \mathbf{Bad}(i,j)$ if there exists a positive constant $c(x_1,x_2)$ such that for all $q \in \mathbb{N}$

$$\max\{\ ||qx_1||^{1/i}\ ,\ ||qx_2||^{1/j}\ \}\ >\ c(x_1,x_2)\ q^{-1} \quad ,$$

where $||\cdot||$ denotes the distance of a real number to the nearest integer. In the case $i=j=1/2$, the set under consideration is simply the standard set of badly approximable pairs. If $i=0$ we identify the set $\mathbf{Bad}(0,1)$ with $\mathbb{R}\times\mathbf{Bad}$ where $\mathbf{Bad}$ is the set of badly approximable numbers. That is, $\mathbf{Bad}(0,1)$ consists of pairs (x_1,x_2) with $x_1 \in \mathbb{R}$ and $x_2 \in \mathbf{Bad}$. The roles of x_1 and x_2 are reversed if $j=0$. Recently [18], it has been shown that $\dim \mathbf{Bad}(i,j) = 2$. We now show that this result is in fact a simple consequence of Theorem 3.

Let $\mathbf{Bad}_{I^2}(i,j) := \mathbf{Bad}(i,j) \cap I^2$ where $I^2 := [0,1]\times[0,1]$. Without loss of generality assume that $i \leq j$. Clearly, it can be expressed in the form $\mathbf{Bad}^*(\mathcal{R},\beta,\rho_1,\rho_2)$ with $\rho_1(r) := r^{-(1+i)}$, $\rho_2(r) := r^{-(1+j)}$ and

$$X = \Omega := I^2\ ,\quad J := \{((p_1,p_2),q) \in \mathbb{N}^2\times\mathbb{N}\backslash\{0\} : p_1,p_2 \leq q\}\ ,$$

$$\alpha := ((p_1,p_2),q) \in J\ ,\quad \beta_\alpha := q\ ,\quad R_\alpha := (p_1/q, p_2/q)\ .$$

Furthermore, $d_1 = d_2$ is the standard Euclidean metric on I and $m_1 = m_2$ is one–dimensional Lebesgue measure on I. By definition, the metric d on I^2 is the product metric $d_1 \times d_1$ and the measure $m := m_1 \times m_1$ is simply two–dimensional Lebesgue measure on I^2.

We show that the conditions of Theorem 3 are satisfied for this basic example. Clearly the functions ρ_1,ρ_2 satisfy condition (B*) and the measures m_1,m_2 satisfy condition (A) with $\delta_1=\delta_2=1$. We now need to establish the existence of the collection $\mathcal{C}(\theta F_n)$, where F_n is an arbitrary closed rectangle of size $2k^{-n(1+i)} \times 2k^{-n(1+j)}$. To start with, note that $m(\theta F_n) = 4\theta^2 k^{-3n}$. Now assume there are at least three rational points $(p_1/q,p_2/q), (p_1'/q',p_2'/q')$ and $(p_1''/q'',p_2''/q'')$ with

$$k^n \leq q,q',q'' < k^{n+1}$$

lying within θF_n. Suppose for the moment that they do not lie on a line and form the triangle Δ sub-tended by them. Twice the area of the triangle Δ is equal to the absolute value of the determinant

$$\det := \begin{vmatrix} 1 & p_1/q & p_2/q \\ 1 & p_1'/q' & p_2'/q' \\ 1 & p_1''/q'' & p_2''/q'' \end{vmatrix} .$$

Then, in view of the denominator constraint, it follows that

$$2 \times m(\Delta) \geq \frac{1}{qq'q''} > k^{-3(n+1)} .$$

Now put

$$\theta := 2^{-1}(2k^3)^{-1/2}.$$

Then $m(\Delta) > m(\theta F_n)$ and this is impossible since $\Delta \subset \theta F_n$. The upshot of this is that the triangle in question can not exist. Thus, if there are two or more rational points with $k^n \leq q < k^{n+1}$ lying within θF_n then they must lie on a line $\mathcal{L}$.

Starting from a 'corner' of the rectangle θF_n, partition θF_n into rectangles $2\theta F_{n+1}$ of size $4k^{-(n+1)(1+i)} \times 4k^{-(n+1)(1+j)}$ and denote by $\mathcal{C}(\theta F_n)$ the collection of rectangles $2\theta F_{n+1}$ obtained. Trivially

$$\#\mathcal{C}(\theta F_n) \ \geq \ \left[\frac{2\theta k^{-n(1+i)}}{4\theta k^{-(n+1)(1+i)}}\right] \left[\frac{2\theta k^{-n(1+j)}}{4\theta k^{-(n+1)(1+j)}}\right] \ \geq \frac{k^3}{16} .$$

In view of the above 'triangle' argument we have that

$$\# \left\{ 2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : \min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \leq 2\theta \rho_i(k^{n+1}) \ \text{ for all } 1 \leq i \leq t \right\}$$

$$\leq \ \# \left\{ 2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset \right\} \ ,$$

where $\mathcal{L}$ is any line passing through θF_n. Recall, that we are assuming that $i \leq j$. A simple geometric argument ensures that for k sufficiently large

$$\# \left\{ 2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset \right\} \ \leq \ \left[\frac{2\theta k^{-n(1+j)}}{4\theta k^{-(n+1)(1+j)}}\right] = \left[\frac{k^{1+j}}{2}\right]$$

$$\leq \ k^{1+j} \ \leq \ k^3/32 .$$

The upshot of this is that the collection $\mathcal{C}(\theta F_n)$ satisfies the required conditions and Theorem 3 implies that

$$\dim \mathbf{Bad}_{I^2}(i,j) \ = \ 2 .$$

In turn, since $\mathbf{Bad}(i,j)$ is a subset of $\mathbb{R}^2$, this implies that $\dim \mathbf{Bad}(i,j) = 2$.

In [18], the stronger result that $\dim \mathbf{Bad}(i,j) \cap \mathbf{Bad}(1,0) \cap \mathbf{Bad}(0,1) = 2$ is established; i.e. the set of pairs (x_1, x_2) with x_1 and x_2 both badly approximable numbers and an (i,j)-badly approximable pair has full dimension. In §5.1, we obtain a much more general result and remark on a beautiful conjecture of W.M. Schmidt. In full generality, Schmidt's conjecture states that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(i',j') \neq \emptyset$. It is a simple exercise to show that if Schmidt's conjecture is false for some pairs (i,j) and (i',j') then Littlewood's conjecture in simultaneous Diophantine approximation is true.

We now turn our attention to the natural generalization of $\mathbf{Bad}(i,j)$ to higher dimensions. For any N-tuple of real numbers $i_1, \ldots, i_N \geq 0$ such that $\sum i_r = 1$, denote by $\mathbf{Bad}(i_1, \ldots, i_N)$ the set of points $(x_1, \ldots, x_N) \in \mathbb{R}^N$ for which there exists a positive constant $c(x_1, \ldots, x_N)$ such that for any $q \in \mathbb{N}$,

$$\max\{ \, ||qx_1||^{1/i_1} \, , \ldots , \, ||qx_N||^{1/i_N} \, \} \ > \ c(x_1, \ldots, x_N) \ q^{-1}.$$

Clearly, the two-dimensional argument can easily be modified to show that

$$\dim \mathbf{Bad}(i_1, \ldots, i_N) = N \ .$$

The key modification is the following lemma which naturally extends the main feature of the 'triangle' argument in dimension two to a 'simplex' one in dimension N.

Lemma 4 *(Simplex Lemma) Let $N \geq 1$ be an integer and $k > 1$ be a real number. Let $E \subseteq \mathbb{R}^N$ be a convex set of N-dimensional Lebesgue measure*

$$|E| \ \leq \ (N!\,)^{-1} k^{-(N+1)} \ .$$

Suppose that E contains $N+1$ rational points $(p_i^{(1)}/q_i, \ldots, p_i^{(N)}/q_i)$ with $1 \leq q_i < k$, where $0 \leq i \leq N$. Then these rational points lie in some hyperplane.

Proof. Suppose to the contrary that this is not the case. In that case, the rational points $(p_i^{(1)}/q_i, \ldots, p_i^{(N)}/q_i)$ where $0 \leq i \leq N$ are distinct. Consider the N-dimensional simplex Δ subtended by them; i.e. an interval when $N = 1$, a triangle when $N = 2$, a tetrahedron when $N = 3$ and so on. Clearly, Δ is a subset of E since E is convex. The volume of the simplex $|\Delta|$ times N factorial is equal to the absolute value of the determinant

$$\det := \begin{vmatrix} 1 & p_0^{(1)}/q_0 & \cdots & p_0^{(N)}/q_0 \\ 1 & p_1^{(1)}/q_1 & \cdots & p_1^{(N)}/q_1 \\ \vdots & \vdots & & \vdots \\ 1 & p_N^{(1)}/q_N & \cdots & p_N^{(N)}/q_N \end{vmatrix} .$$

As this determinant is not zero, it follows from the assumption made on the q_i that

$$N! \times |\Delta| = |\det| \geq \frac{1}{q_0 q_1 \cdots q_N} > k^{-(N+1)} .$$

Consequently, $|\Delta| > (N!)^{-1} k^{-(N+1)} \geq |E|$. This contradicts the fact that $\Delta \subseteq E$. $\square$

Remarks:

(i) The Simplex Lemma should be viewed as the higher dimensional generalization of the following simple fact already exploited in the §1.4: on the real line $\mathbb{R}$ an interval I_k of length $1/k^2$ can contain at most one rational p/q with $1 \leq q < k$. This follows from the trivial observation that if $1 \leq q, q' < k$ then $|p/q - p'/q'| \geq 1/qq' > 1/k^2$.

(ii) Our general setup will be applied to settings other than subsets of $\mathbb{R}^N$ (see §5). In most of these, an analogue of the Simplex Lemma will be required. In these settings we will either give a complete proof or sketch the argument required in two dimensions; i.e. the analogue of the 'triangle' argument. Based on the proof of the Simplex Lemma in $\mathbb{R}^N$, it should then be obvious how to extend the $N = 2$ argument to higher dimensions. In short, within this paper the main ideas are always exposed on establishing a given N-dimensional statement in the $N = 2$ case. The proof in higher dimensions requires no new ideas. Thus in all the various applications of our general framework, for the sake of both clarity and notation we shall stick to $N = 2$ in proofs.

(iii) The 'triangle' argument (or variants thereof) described above is critical in most of the applications considered in this paper (see §5). To some extent this is the reason why our main results cannot be directly applied to the problem of badly approximable systems of linear forms. In this case the resonant sets R_α are affine spaces and although the 'triangle' or more generally the 'simplex' approach remains the main ingredient it requires deeper considerations in the geometry of numbers to successfully execute it. We will return to this and other aspects of the linear forms theory in a forthcoming paper [13].

3 Preliminaries

In this short section we define Hausdorff measure and dimension in order to establish some notation and then describe a method for obtaining lower bounds for the dimension.

Suppose Ω is a non–empty subset of (X, d). For $\rho > 0$, a countable collection $\{B_i\}$ of balls in X with radii $r_i \leq \rho$ for each i such that $\Omega \subset \bigcup_i B_i$ is called a ρ-cover for Ω. Clearly such a cover always exists for totally bounded metric spaces. Let s be a non-negative number and define

$$\mathcal{H}^s_\rho(\Omega) \ = \ \inf \left\{ \sum_i r_i^s \ : \{B_i\} \text{ is a } \rho\text{-cover of } \Omega \right\} \ ,$$

where the infimum is over all ρ-covers. The *s-dimensional Hausdorff measure* $\mathcal{H}^s(\Omega)$ of Ω is defined by

$$\mathcal{H}^s(\Omega) := \lim_{\rho \to 0} \mathcal{H}^s_\rho(\Omega) \ = \ \sup_{\rho > 0} \mathcal{H}^s_\rho(\Omega)$$

and the Hausdorff dimension $\dim \Omega$ of a set Ω by

$$\dim \Omega \ := \ \inf \{s : \mathcal{H}^s(\Omega) = 0\} = \sup \{s : \mathcal{H}^s(\Omega) = \infty\} \, .$$

In particular when s is an integer $\mathcal{H}^s$ is comparable to s-dimensional Lebesgue measure. For further details see [6,16]. A general and classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set Ω is the following mass distribution principle (see e.g. [6, page 55]).

Lemma 5 *(Mass Distribution Principle) Let μ be a probability measure supported on a subset Ω of (X, d). Suppose there are positive constants c and r_0 such that*

$$\mu(B) \leq c \, r^s \ ,$$

for any ball B with radius $r \leq r_0$. Then $\mathcal{H}^s(\Omega) \geq 1/c$. In particular, we have that $\dim \Omega \geq s$.

The following rather simple covering result will be crucial to our proof of Theorem 2.

Lemma 6 *(Covering Lemma) Let (X, d) be the product space of the metric spaces $(X_1, d_1), \ldots, (X_t, d_t)$ and $\mathcal{F}$ be a finite collection of 'rectangles' $F := F(c; l_1, \ldots, l_t)$ with $c \in X$ and $l_1, \ldots, l_t$ fixed. Then there exists a disjoint sub-collection $\{F_m\}$ such that*

$$\bigcup_{F \in \mathcal{F}} F \ \subset \ \bigcup_m 3F_m \ .$$

Proof. Let S denote the set of centres c of the rectangles in $\mathcal{F}$. Choose $c(1) \in S$ and for $k \geq 1$,

$$c(k+1) \ \in \ S \setminus \bigcup_{m=1}^{k} 2\,F(c(m); l_1, \ldots, l_t)$$

as long as $S \setminus \bigcup_{m=1}^{k} 2\,F(c(m); l_1, \ldots, l_t) \neq \emptyset$. Since $\#S$ is finite, there exists $k_1 \leq \#S$ such that

$$S \ \subset \ \bigcup_{m=1}^{k_1} 2\,F(c(m); l_1, \ldots, l_t) \ .$$

By construction, any rectangle $F(c; l_1, \ldots, l_t)$ in the original collection $\mathcal{F}$ is contained in some rectangle $3\,F(c(m); l_1, \ldots, l_t)$ and since $d_i(c_i(m), c_i(n)) > 2l_i$ for each $1 \leq i \leq t$ the chosen rectangles $F(c(m); l_1, \ldots, l_t)$ are clearly disjoint.

$\square$

We end this section by making use of the covering lemma to establish the following assertion made in §2.2. The result is extremely useful when it comes to applying our theorems – see §5. With reference to Theorem 2, it guarantees the existence of a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles with the necessary cardinality.

Lemma 7 *Let (X, d) be the product space of the metric spaces $(X_1, d_1), \ldots,$ (X_t, d_t) and let (Ω, d, m) be a compact measure subspace of X. Let the measure m and the functions ρ_i satisfy conditions (B*) to (D*). Let k be sufficiently large. Then for any $\theta \in \mathbb{R}^+$ and for any rectangle F_n $(n \geq 1)$ there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1}$ contained within θF_n satisfying (3) of Theorem 2.*

Proof. Begin by choosing k large enough so that for any $i \in \{1, \ldots, t\}$,

$$\frac{\rho_i(k^n)}{\rho_i(k^{n+1})} \ \geq 4. \tag{8}$$

That this is possible follows from the fact that $\lambda_i^l(k) \to \infty$ as $k \to \infty$ (condition (B*)). Take an arbitrary rectangle F_n and let $l_i(n) := \theta \rho_i(k^n)$. Thus $\theta F_n := F(c; l_1(n), \ldots, l_t(n))$. Consider the rectangle $T_n \subset \theta F_n$ where

$$T_n := F(c; l_1(n) - 2l_1(n+1), \ldots, l_t(n) - 2l_t(n+1)) \ .$$

Note that in view of (8) we have that $T_n \supset \frac{1}{2}\theta F_n$. Now, cover T_n by rectangles $2\theta F_{n+1}$ with centres in $\Omega \cap T_n$. By construction, these rectangles are contained in θF_n and in view of the covering lemma there exists a disjoint sub-collection $\mathcal{C}(\theta F_n)$ such that

$$T_n \subset \bigcup_{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n)} 6\theta F_{n+1} \quad .$$

Using that fact that rectangles of the same size centred at points of Ω have comparable m measure (condition (C*)), it follows that

$$a\, m(\tfrac{1}{2}\theta F_n) \;\leq\; m(T_n) \leq \#\mathcal{C}(\theta F_n)\; b\, m(6\theta F_{n+1}) \quad .$$

Using that fact that the measure m is doubling on rectangles (condition (D*)), so that $m(\frac{1}{2}\theta F_n) \geq D^{-1} m(\theta F_n)$ and $m(6\theta F_{n+1}) \leq m(8\theta F_{n+1}) \leq D^3 m(\theta F_{n+1})$, it follows that

$$\#\mathcal{C}(\theta F_n) \;\geq\; \frac{a}{bD^4}\;\frac{m(\theta F_n)}{m(\theta F_{n+1})} \quad .$$

□

Remark. Clearly, with reference to Theorem 1, the above lemma guarantees the existence of the collection $\mathcal{C}(\theta B_n)$ satisfying (1).

4 Proof of Theorem 2

The overall strategy is as follows. For any k sufficiently large we construct a Cantor-type set $\mathbf{K}_{c(k)}$ such that $\mathbf{K}_{c(k)}$ with at most a finite number of points removed is a subset of $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t)$. Next, we construct a measure μ supported on $\mathbf{K}_{c(k)}$ with the property that for any ball A with radius $r(A)$ sufficiently small

$$\mu(A) \;\ll\; r(A)^{\delta - \epsilon(k)} \;;$$

where $\epsilon(k) \to 0$ as $k \to \infty$. Hence, by construction and the mass distribution principle we have that

$$\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) \;\geq\; \dim \mathbf{K}_{c(k)} \;\geq\; \delta - \epsilon(k) \; .$$

Now suppose that $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) < \delta$. Then, $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) = \delta - \eta$ for some $\eta > 0$. However, by choosing k large enough so that $\epsilon(k) < \eta$ we obtain a contradiction and thereby the lower bound result follows.

4.1 The Cantor-type set $\mathbf{K}_{c(k)}$

Choose k_0 sufficiently large so that for $k \geq k_0$, $\rho_i(k)$ $(1 \leq i \leq t)$ is decreasing and the hypotheses of the theorem are valid. Now fix some $k \geq k_0$ and suppose that

$$\{\alpha \in J : \beta_\alpha < k\} \; = \; \emptyset \; . \tag{9}$$

Define $\mathcal{F}_1$ to be any rectangle θF_1 of radius $\theta\rho(k)$ and centre c in Ω. The idea is to establish, by induction on n, the existence of a collection $\mathcal{F}_n$ of disjoint rectangles θF_n such that $\mathcal{F}_n$ is nested in $\mathcal{F}_{n-1}$; that is, each rectangle θF_n in $\mathcal{F}_n$ is contained in some rectangle θF_{n-1} of $\mathcal{F}_{n-1}$. Also, any θF_n in $\mathcal{F}_n$ will have the property that for all points $x \in \theta F_n$, for all $i \in \{1, \ldots, t\}$ and for all $\alpha \in J$ with $\beta_\alpha < k^n$,

$$d_i(x, R_{\alpha,i}) \; \geq \; c(k)\, \rho_i(\beta_\alpha), \tag{10}$$

where the constant

$$c(k) \; := \; \min_{1 \leq i \leq t} (\theta / \lambda_i^u(k))$$

is dependent on k but is independent of n. Then, since the rectangles θF_n of $\mathcal{F}_n$ are closed, nested and the space Ω is compact, any limit point in θF_n will satisfy (10) for all α in J with $\beta_\alpha \geq k$. In particular, we put

$$\mathbf{K}_{c(k)} \; := \; \bigcap_{n=1}^{\infty} \mathcal{F}_n \; .$$

By construction, we have that $\mathbf{K}_{c(k)}$ is a subset of $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t)$ under the assumption (9).

The induction. For $n = 1$, (10) is trivially satisfied for $\mathcal{F}_1 = \theta F_1$ since we are assuming (9). Given $\mathcal{F}_n$ satisfying (10) we wish to construct a nested collection $\mathcal{F}_{n+1}$ for which (10) is satisfied for $n+1$. Consider any rectangle $\theta F_n \subset \mathcal{F}_n$. We construct a 'local' collection $\mathcal{F}_{n+1}(\theta F_n)$ of disjoint rectangles θF_{n+1} contained in θF_n so that for any point $x \in \theta F_{n+1}$ the condition given by (10) is satisfied for $n+1$. Given that any rectangle θF_{n+1} of $\mathcal{F}_{n+1}(\theta F_n)$ is to be nested in θF_n, it is enough to show that for any point $x \in \theta F_{n+1}$ the inequalities

$$d_i(x_i, R_{\alpha,i}) \; \geq \; c(k)\, \rho_i(\beta_\alpha) \quad (1 \leq i \leq t)$$

are satisfied for $\alpha \in J$ with $k^n \leq \beta_\alpha < k^{n+1}$; i.e. with $\alpha \in J(n+1)$.

For k sufficiently large, by the hypotheses of the theorem, there exists a disjoint sub-collection $\mathcal{G}(\theta F_n)$ of $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1} \subset \theta F_n$ with

$$\#\mathcal{G}(\theta F_n) \; = \; \left[\kappa \, \frac{m(\theta F_n)}{m(\theta F_{n+1})} \right] \qquad\qquad \kappa := \min\{1, \tfrac{1}{2}(\kappa_1 - \kappa_2)\} \; , \tag{11}$$

and such that for any rectangle $2\theta F_{n+1} \subset \mathcal{G}(\theta F_n)$ with centre c

$$\min_{\alpha \in J(n+1)} d_i(c_i, R_{\alpha,i}) \; \geq \; 2\,\theta\, \rho_i(k^{n+1}) \; .$$

Clearly, by choosing k large enough we can ensure that $\#\mathcal{G}(\theta F_n) > 1$ – this makes use of conditions (D*) and (E*). Now let

$$\mathcal{F}_{n+1}(\theta F_n) \; := \; \{\theta F_{n+1} : 2\theta F_{n+1} \subset \mathcal{G}(\theta F_n)\} \; .$$

Thus the rectangles of $\mathcal{F}_{n+1}(\theta F_n)$ are precisely those of $\mathcal{G}(\theta F_n)$ but scaled by a factor 1/2. Then, by construction for any $x \in \theta F_{n+1} \subset \mathcal{F}_{n+1}(\theta F_n)$ and $1 \leq i \leq t$

$$\begin{aligned} d_i(x_i, R_{\alpha,i}) \ \geq \ \theta \rho_i(k^{n+1}) \ = \ \theta \rho_i(k^n) \, \frac{\rho_i(k^{n+1})}{\rho_i(k^n)} \ &\geq \ \frac{\theta}{\lambda_i^u(k)} \, \rho_i(\beta_\alpha) \\ &\geq \ c(k) \ \rho_i(\beta_\alpha). \end{aligned}$$

Here we have made use of condition (B*) and the fact that $\rho_i(k)$ is decreasing for $k \geq k_0$ and that $\alpha \in J(n+1)$. Finally let

$$\mathcal{F}_{n+1} := \bigcup_{\theta F_n \in \mathcal{F}_n} \mathcal{F}_{n+1}(\theta F_n) \ .$$

This completes the proof of the induction step and so the construction of the Cantor-type set

$$\mathbf{K}_{c(k)} \ := \ \bigcap_{n=1}^{\infty} \mathcal{F}_n \ ,$$

where $c(k) := \min_{1 \leq i \leq t}(\theta/\lambda_i^u(k))$ and k is sufficiently large.

Note, that in view of (11) we have that for $n \geq 2$

$$\begin{aligned} \#\mathcal{F}_n = \#\mathcal{F}_{n-1} \ \times \ \#\mathcal{F}_n(\theta F_{n-1}) \ &= \ \prod_{m=2}^{n} \#\mathcal{F}_m(\theta F_{m-1}) \\ &\geq \prod_{m=2}^{n} \frac{\kappa}{2} \ \frac{m(\theta F_{m-1})}{m(\theta F_m)} \ = \ \left(\frac{\kappa}{2}\right)^{n-1} \ \frac{m(\theta F_1)}{m(\theta F_n)}. \end{aligned} \tag{12}$$

4.2 The measure μ on $\mathbf{K}_{c(k)}$

We now describe a probability measure μ supported on the Cantor–type set $\mathbf{K}_{c(k)}$ constructed in the previous subsection. For any rectangle θF_n in $\mathcal{F}_n$ we attach a weight $\mu(\theta F_n)$ which is defined recursively as follows: for $n = 1$,

$$\mu(\theta F_1) \ := \ \frac{1}{\#\mathcal{F}_1} = 1$$

and for $n \geq 2$,

$$\mu(\theta F_n) \ := \ \frac{1}{\#\mathcal{F}_n(\theta F_{n-1})} \, \mu(\theta F_{n-1}) \quad (F_n \subset F_{n-1}) \ .$$

This procedure thus defines inductively a mass on any rectangle used in the construction of $\mathbf{K}_{c(k)}$. In fact a lot more is true – μ can be further extended to all Borel subsets A of Ω to determine $\mu(A)$ so that μ constructed as above actually defines a measure supported on $\mathbf{K}_{c(k)}$; see [6, Proposition 1.7]. We state this formally as a

Fact. *The probability measure* μ *constructed above is supported on* $\mathbf{K}_{c(k)}$ *and for any Borel subset* A *of* Ω

$$\mu(A) \; = \; \inf \sum_{F \in \mathcal{F}} \mu(F) \; .$$

The infimum is over all coverings $\mathcal{F}$ *of* A *by rectangles* $F \in \{\mathcal{F}_n : n \geq 1\}$.

Notice that, in view of (12), we simply have that

$$\mu(\theta F_n) = \frac{1}{\#\mathcal{F}_n} \quad (n \geq 1) \; .$$

4.3 A lower bound for $\dim \mathbf{K}_{c(k)}$

Let A be an arbitrary ball with centre a not necessarily in Ω and of radius $r(A) < \theta\rho_*(k^{n_0})$ where $\rho_*(r) := \max_{1 \leq i \leq t} \rho_i(r)$ and n_0 is to be determined later. We now determine an upper bound for $\mu(A)$ in terms of its radius. Choose $n \geq n_0$ so that

$$\theta\rho_*(k^{n+1}) \; < \; r(A) \; \leq \; \theta\rho_*(k^n) \; .$$

Without loss of generality, assume that $A \cap \mathbf{K}_{c(k)} \neq \emptyset$ since otherwise there is nothing to prove. Clearly

$$\mu(A) \; \leq \; \mathcal{N}_{n+1}(A) \times \mu(\theta F_{n+1})$$

where

$$\mathcal{N}_{n+1}(A) \; := \; \#\{\theta F_{n+1} \subset \mathcal{F}_{n+1} : \theta F_{n+1} \cap A \neq \emptyset\} \; .$$

If $\theta F_{n+1} \cap A \neq \emptyset$, then $\theta F_{n+1} \subset 3\,A$ since $r(A) \geq \theta\rho_i(k^{n+1})$ for $1 \leq i \leq t$. The balls in $\mathcal{F}_{n+1}$ are disjoint and have comparable m measure (condition (C*)), thus

$$\mathcal{N}_{n+1}(A) \; \leq \; \frac{m(3A)}{a\, m(\theta F_{n+1})} \; .$$

It follows by (12), that

$$\mu(A) \; \leq \; \frac{m(3A)}{a\, m(\theta F_{n+1})} \times \frac{1}{\#\mathcal{F}_{n+1}} \quad \leq \quad \frac{m(3A)}{a\, m(\theta F_1)} \left(\frac{2}{\kappa}\right)^n \; .$$

Using the fact that $\rho_*(k^n) \leq \lambda_*^l(k)^{-(n-1)}\rho_*(k)$, it is easily verified that

$$\frac{1}{a\, m(\theta F_1)} \left(\frac{2}{\kappa}\right)^n \; < \; \left(\frac{1}{\theta\rho_*(k^n)}\right)^{\epsilon(k)}$$

for

$$n \ \geq \ n_1 \ := \ \left[4 + \frac{\log \frac{(\theta\, \rho_*(k))^{\epsilon(k)}}{a\, m(\theta F_1)}}{\log \frac{2}{\kappa}} \right] \qquad \text{and} \qquad \epsilon(k) \ := \ \frac{4\, \log \frac{2}{\kappa}}{\log \lambda_*^l(k)} \ .$$

Hence,

$$\mu(A) \ \leq \ m(3A) \times (\theta \rho_*(k^n))^{-\epsilon(k)} \ .$$

Since $A \cap \mathbf{K}_{c(k)} \neq \emptyset$, there exists some point $x \in A \cap \Omega$. Moreover, $3A \subset B(x, 4\, r(A))$ which together with condition (A*) implies that

$$m(3A) \leq m(B(x, 4\, r(A))) \leq r(A)^{\delta - \epsilon(k)}$$

for $r(A) \leq r_0 := r_0(\epsilon(k))$. Now $\rho_*(r) \to 0$ as $r \to \infty$, so $\theta \rho_*(k^n) < r_0$ for $n \geq n_2$. Thus, for $n \geq n_0 := \max\{n_1, n_2\}$

$$\mu(A) \ \leq \ r(A)^{\delta - \epsilon(k)} \times (\theta \rho_*(k^n))^{-\epsilon(k)} \ .$$

On using the fact that $r(A) \leq \theta \rho_*(k^n)$, we obtain that

$$\mu(A) \ \leq \ r(A)^{\delta - 2\epsilon(k)} \ .$$

This together with the mass distribution principle implies that

$$\dim \mathbf{K}_{c(k)} \ \geq \ \delta - 2\epsilon(k) \ .$$

Note that since $\epsilon(k) \to 0$ as $k \to \infty$ we have that $\dim \mathbf{K}_{c(k)} \to \delta$ as $k \to \infty$.

4.4 Completion of proof

Recall, that $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$. Now suppose that

$$\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) \ < \ \delta \ .$$

It follows that $\max\{\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t), \dim(\cup_{\alpha \in J} R_\alpha)\} = \delta - \eta$ for some $\eta > 0$. Fix some k sufficiently large so that $2\, \epsilon(k) < \eta$. Then,

$$\dim \mathbf{K}_{c(k)} \ \geq \ \delta - 2\epsilon(k) \ > \ \delta - \eta \ .$$

By construction, for any point $x \in \mathbf{K}_{c(k)}$ we have for all $\alpha \in J$ with $\beta_\alpha \geq k$ that

$$d_i(x_i, R_{\alpha,i}) \ \geq \ c(k)\, \rho_i(\beta_\alpha) \quad (1 \leq i \leq t) \ .$$

Now let $J_k := \{\alpha \in J : \beta_\alpha < k\}$. If (9) is true for our fixed k then $J_k = \emptyset$ and clearly $\mathbf{K}_{c(k)} \subseteq \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t)$. In turn, $\dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) \geq \dim \mathbf{K}_{c(k)} > \delta - \eta$ and we have a contradiction. So suppose, $J_k \neq \emptyset$ and let

$\mathcal{R}_k := \{R_\alpha : \alpha \in J_k\}$. For any fixed k the number of elements in J_k is finite. So, if $x \notin R_k$ then there exists a constant $c'(x) > 0$ such that for all $\alpha \in J_k$,

$$d_i(x_i, R_{\alpha,i}) \ \geq \ c'(k)\, \rho_i(\beta_\alpha) \quad (1 \leq i \leq t) \ .$$

Thus, for $x \in \mathbf{K}_{c(k)} \backslash \mathcal{R}_k$ and $\alpha \in J$,

$$d_i(x_i, R_{\alpha,i}) \ \geq \ c^*(k)\, \rho_i(\beta_\alpha) \quad (1 \leq i \leq t) \ ,$$

where $c^*(x) := \min\{c(k), c'(x)\}$. It follows that

$$\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) \supseteq \mathbf{K}_{c(k)} \backslash \mathcal{R}_k \ ,$$

and since $\dim \mathcal{R}_k < \dim \mathbf{K}_{c(k)}$ we have that

$$\begin{aligned} \dim \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \ldots, \rho_t) \ &\geq \ \dim(\mathbf{K}_{c(k)} \backslash \mathcal{R}_k) \\ &= \ \dim \mathbf{K}_{c(k)} \ \geq \ \delta - 2\epsilon(k) \ > \ \delta - \eta \ . \end{aligned}$$

This is a contradiction and completes the proof of Theorem 2. □

5 Applications

5.1 Intersecting sets with $\mathbf{Bad}(i_1, \ldots, i_N)$

Let $\mathbf{Bad}(i_1, \ldots, i_N)$ be the set of $(i_1, \ldots, i_N)$-badly approximable N-tuples in $\mathbb{R}^N$ as defined in §2.3 and $\mathbf{Bad}(N) := \mathbf{Bad}(i_1, \ldots, i_N)$ with $i_1 = \ldots = i_N = 1/N$. Thus $\mathbf{Bad}(1)$ is simply the set $\mathbf{Bad}$ of badly approximable real numbers. Let Ω be a compact subset of $\mathbb{R}^N$. The problem is to determine conditions on Ω under which

$$\mathbf{Bad}_\Omega(i_1, \ldots, i_N) \ := \ \Omega \cap \mathbf{Bad}(i_1, \ldots, i_N)$$

is of full dimension; i.e. $\dim \mathbf{Bad}_\Omega(i_1, \ldots, i_N) = \dim \Omega$. Recall, that the '2-dimensional' argument of §2.3 can easily be extended to show that $\dim \mathbf{Bad}(i_1, \ldots, i_N) = N$.

To begin with, we address the above problem for the set $\mathbf{Bad}_\Omega(N) = \Omega \cap \mathbf{Bad}(N)$ in the case that Ω supports an 'absolutely α-decaying' measure that satisfies condition (A).

The notion of an 'absolutely decaying' measure was introduced in [11]. The following restrictive definition, exploited in [19], serves our purpose. Let Ω be a compact subset of $\mathbb{R}^N$ which supports a non-atomic, finite measure m. Let $\mathcal{L}$ denote a generic hyperplane of $\mathbb{R}^N$ and let $\mathcal{L}^{(\epsilon)}$ denote its ϵ-neighborhood. We say that m is *absolutely α-decaying* if there exist strictly positive constants C, α, r_0 such that for any hyperplane $\mathcal{L}$, any $\epsilon > 0$, any $x \in \Omega$ and any $r < r_0$,

$$m\left(B(x,r) \cap \mathcal{L}^{(\epsilon)}\right) \ \leq \ C \left(\frac{\epsilon}{r}\right)^{\alpha} m(B(x,r)) \ .$$

In the case $N = 1$, the hyperplane $\mathcal{L}$ is simply a point $a \in \mathbb{R}$ and $\mathcal{L}^{(\epsilon)}$ is the ball $B(a, \epsilon)$ centred at a of radius ϵ. Also note that in this case, if the measure m satisfies condition (A) with exponent δ then m is automatically absolutely δ-decaying.

Theorem 8 *Let Ω be a compact subset of $\mathbb{R}^N$ which supports a measure m satisfying condition (A) and which in addition is absolutely α-decaying for some $\alpha > 0$. Then*

$$\dim \mathbf{Bad}_{\Omega}(N) = \dim \Omega \ .$$

Proof. With reference to §1, the set $\mathbf{Bad}_{\Omega}(N)$ can be expressed in the form $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ with $\rho(r) := r^{-(1+\frac{1}{N})}$ and

$$X = (\mathbb{R}^N, d) \ , \quad J := \{((p_1, \ldots, p_N), q) \in \mathbb{N}^N \times \mathbb{N} \backslash \{0\}\} \ ,$$

$$\alpha := ((p_1, \ldots, p_N), q) \in J \ , \quad \beta_\alpha := q \ , \quad R_\alpha := (p_1/q, \ldots, p_N/q) \ .$$

Here d is standard sup metric on $\mathbb{R}^N$; $d(x,y) := \max\{d(x_1,y_1), \ldots, d(x_N,y_N)\}$. Thus balls $B(c,r)$ in $\mathbb{R}^N$ are genuinely cubes of sidelength $2r$.

We show that the conditions of Theorem 1 are satisfied. Clearly the function ρ satisfies condition (B) and we are given that the measure m supported on Ω satisfies condition (A). Also, since the resonant sets are points the condition that $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$ is satisfied. We need to establish the existence of the disjoint collection $\mathcal{C}(\theta B_n)$ of balls (cubes) $2\theta B_{n+1}$ where B_n is an arbitrary ball of radius $k^{-n(1+\frac{1}{N})}$ with centre in Ω. In view of Lemma 7, there exists a disjoint collection $\mathcal{C}(\theta B_n)$ such that

$$\#\mathcal{C}(\theta B_n) \ \geq \ \kappa_1 \, k^{(1+\frac{1}{N})\delta} \ ; \tag{13}$$

i.e. (1) of Theorem 1 holds. We now verify that (2) is satisfied for any such collection.

We consider two cases.

Case 1: $N = 1$. The trivial argument of §1.4 shows that any interval θB_n with $\theta := \frac{1}{2}k^{-2}$ contains at most one rational p/q with $k^n \leq q < k^{n+1}$; i.e. $\alpha \in J(n+1)$. Thus, for k sufficiently large

$$\text{l.h.s. of (2)} \;\leq\; 1 \;< \tfrac{1}{2} \times \text{r.h.s. of (13)} \,.$$

Hence (2) is trivially satisfied and Theorem 1 implies the desired result.

Case 2: $N \geq 2$. We shall prove the theorem in the case that $N = 2$. There are no difficulties and no new ideas are required in extending the proof to higher dimensions, especially in view of the Simplex Lemma (see §2.3).

Suppose that there are three or more rational points $(p_1/q, p_2/q)$ with $k^n \leq q < k^{n+1}$ lying within the ball/square θB_n. Now put $\theta := 2^{-1}(2k^3)^{-1/2}$. Then the 'triangle' argument of §2.3 (where m is Lebesgue measure) implies that the rational points must lie on a line $\mathcal{L}$ passing through θB_n. It follows that

$$\begin{aligned}
\text{l.h.s. of (2)} &\leq \# \left\{2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : 2\theta B_{n+1} \cap \mathcal{L} \neq \emptyset\right\} \\
&\leq \# \left\{2\theta B_{n+1} \subset \mathcal{C}(\theta B_n) : 2\theta B_{n+1} \subset \mathcal{L}^{(\epsilon)}\right\} && \text{for } \epsilon := 8\theta k^{-(n+1)\frac{3}{2}} \\
&\leq \frac{m(\theta B_n \cap \mathcal{L}^{(\epsilon)})}{m(2\theta B_{n+1})} && \text{the balls } 2\theta B_{n+1} \text{ are disjoint} \\
&\leq a^{-1} b\, C\ 8^{\alpha}\, 2^{-\delta}\, k^{\frac{2}{3}(\delta-\alpha)} && m \text{ is absolutely } \alpha\text{-decaying} \\
&< \tfrac{1}{2} \times \text{ r.h.s. of (13)} && \text{for } k \text{ sufficiently large.}
\end{aligned}$$

Hence (2) is satisfied and Theorem 1 implies the desired result. □

The following statement which combines Theorems 2.2 and 8.1 of [11], shows that a large class of fractal measures are absolutely α-decaying and satisfy condition (A).

Theorem 9 *Let $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ be an irreducible family of contracting self similarity maps of $\mathbb{R}^N$ satisfying the open set condition and let m be the restriction of $\mathcal{H}^\delta$ to its attractor K where $\delta := \dim K$. Then m is absolutely α-decaying and satisfies condition (A).*

The simplest examples of such sets include regular Cantor sets, the Sierpiński gasket and the von Koch curve. All the terminology except for 'irreducible' is pretty much standard – see for example [6, Chp.9]. The notion of irreducible introduced in [11, §2] avoids the natural obstruction that there is a finite collection of proper affine subspaces of $\mathbb{R}^N$ which is invariant under $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$. More recently, the class of examples regarding absolutely α-decaying measures has been extended by Urbański [23,24].

In view of Theorem 9, the following statement is a simple consequence of Theorem 8. It has also been independently established by Kleinbock & Weiss [11, Theorem 10.3] and [12]. In fact, Theorem 8 is also derived in [12] by an alternative approach.

Corollary 10 *Let $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ be an irreducible family of contracting self similarity maps of $\mathbb{R}^N$ satisfying the open set condition and let m be the restriction of $\mathcal{H}^\delta$ to its attractor K where $\delta := \dim K$. Then*

$$\dim(K \cap \mathbf{Bad}(N)) = \dim K \ .$$

We now consider the more general problem of determining conditions on Ω under which $\dim \mathbf{Bad}_\Omega(i_1, \ldots, i_N) = \dim \Omega$. Under the hypotheses of Theorem 2, by modifying the definition of 'absolutely decaying' to accommodate 'rectangles' it is clearly possible to obtain an analogue of the 'abstract' theorem (Theorem 8) for $\mathbf{Bad}_\Omega(i_1, \ldots, i_N)$. We have decided against establishing such a statement in this paper. The reason for this is simple. We are currently unable to prove the existence of a natural class of sets satisfying the more general 'rectangular' hypotheses. Nevertheless, in the special case that Ω is a product space we are able to prove the following statement.

Theorem 11 *For $1 \leq j \leq N$, let Ω_j be a compact subset of $\mathbb{R}$ which supports a measure m_j satisfying condition (A) with exponent δ_j. Let Ω denote the product set $\Omega_1 \times \ldots \times \Omega_N$. Then, for any N-tuple $(i_1, \ldots, i_N)$ with $i_j \geq 0$ and $\sum_{j=1}^N i_j = 1$,*

$$\dim \mathbf{Bad}_\Omega(i_1, ..., i_N) = \dim \Omega \ .$$

A simple application of the above theorem leads to following result.

Corollary 12 *Let K_1 and K_2 be regular Cantor subsets of $\mathbb{R}$. Then*

$$\dim\left((K_1 \times K_2) \cap \mathbf{Bad}(i,j)\right) \ = \ \dim(K_1 \times K_2) \ = \ \dim K_1 \ + \ \dim K_2 \ .$$

Proof of Theorem 11. Without loss of generality assume that $N \geq 2$. The case that $N = 1$ is covered by Theorem 8. For the sake of clarity, as with the proof of Theorem 8, we shall restrict our attention to the case $N = 2$.

Recall that since $\Omega_j \subset \mathbb{R}$ and m_j satisfies (A), then m_i is automatically absolutely δ_j-decaying. A relatively straightforward argument shows that $m := m_1 \times m_2$ is absolutely α-decaying on Ω with $\alpha := \min\{\delta_1, \delta_2\}$. In fact this trivially follows from the following general fact - see [11, §9].

Fact: *For $2 \leq j \leq N$, if each m_j is absolutely α_j-decaying on Ω_j, then $m := m_1 \times \ldots \times m_N$ is absolutely α-decaying on $\Omega = \Omega_1 \times \ldots \times \Omega_N$ with $\alpha = \min\{\alpha_1, \ldots, \alpha_N\}$.*

Now let us write $\mathbf{Bad}(i,j)$ for $\mathbf{Bad}(i_1, i_2)$ and without loss of generality assume that $i < j$. The case $i = j$ is already covered by Theorem 4 since m is absolutely α-decaying on Ω and clearly satisfies condition (A). The set $\mathbf{Bad}_\Omega(i,j)$ can be expressed in the form $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2)$ with $\rho_1(r) = r^{-(1+i)}$, $\rho_2(r) = r^{-(1+j)}$ and

$$X = \mathbb{R}^2 \ , \quad \Omega := \Omega_1 \times \Omega_2 \ , \quad J := \{((p_1, p_2), q) \in \mathbb{N}^2 \times \mathbb{N} \setminus \{0\}\} \ ,$$

$$\alpha := ((p_1, p_2), q) \in J \ , \quad \beta_\alpha := q \ , \quad R_\alpha := (p_1/q, p_2/q) \ .$$

With reference to Theorem 3, the functions ρ_1, ρ_2 satisfy condition (B*) and the measures m_1, m_2 satisfy condition (A). Also note that $\dim(\cup_{\alpha \in J} R_\alpha) = 0$ since the union in question is countable. We need to establish the existence of the collection $\mathcal{C}(\theta F_n)$, where F_n is an arbitrary closed rectangle of size $2k^{-n(1+i)} \times 2k^{-n(1+j)}$ with centre c in Ω. In view of Lemma 7, there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1} \subset \theta F_n$ such that

$$\#\mathcal{C}(\theta F_n) \ \geq \ \kappa_1 \, k^{(1+i)\delta_1} k^{(1+j)\delta_2}; \tag{14}$$

i.e. (5) of Theorem 3 is satisfied. We now verify that (6) is satisfied for any such collection. With $\theta = 2^{-1}(2k^3)^{-1/2}$, the 'triangle' argument or equivalently the Simplex Lemma of §2.3 implies that

$$\text{l.h.s. of (6)} \ \leq \ \#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset\} \ , \tag{15}$$

where $\mathcal{L}$ is a line passing through θF_n. Consider the thickening $T(\mathcal{L})$ of $\mathcal{L}$ obtained by placing rectangles $4\theta F_{n+1}$ centred at points of $\mathcal{L}$; that is, by 'sliding' a rectangle $4\theta F_{n+1}$, centred at a point of $\mathcal{L}$, along $\mathcal{L}$. Then, since the rectangles $2\theta F_{n+1} \subset \mathcal{C}(\theta F_n)$ are disjoint,

$$\begin{aligned} &\#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset\} \\ &\qquad \leq \#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \subset T(\mathcal{L})\} \\ &\qquad \leq \ \frac{m(T(\mathcal{L}) \cap \theta F_n)}{m(2\theta F_{n+1})}. \end{aligned} \tag{16}$$

Without loss of generality we can assume that $\mathcal{L}$ passes through the centre of θF_n. To see this, suppose that $m(T(\mathcal{L}) \cap \theta F_n) \neq 0$ since otherwise there is nothing to prove. Then, there exists a point $x \in T(\mathcal{L}) \cap \theta F_n \cap \Omega$ such that

$$T(\mathcal{L}) \cap \theta F_n \subset 2\theta F_n' \cap T'(\mathcal{L}') \ .$$

Here F_n' is the rectangle of size $k^{-n(1+i)} \times k^{-n(1+j)}$ centred at x, $\mathcal{L}'$ is the line parallel to $\mathcal{L}$ passing through x and $T'(\mathcal{L}')$ is the thickening obtained by 'sliding' a rectangle $8\theta F_{n+1}$ centred at x, along $\mathcal{L}'$. Then the following argument works just as well on $2\theta F_n' \cap T'(\mathcal{L}')$.

Let Δ denote the slope of the line $\mathcal{L}$ and assume that $\Delta \geq 0$. The case $\Delta < 0$ can be dealt with similarly. By moving the rectangle θF_n to the origin, straightforward geometric considerations lead to the following facts:

(F1)

$$T(\mathcal{L}) = \mathcal{L}^{(\epsilon)} \quad \text{where} \quad \epsilon := \frac{4\theta \left(k^{-(n+1)(1+j)} + \Delta k^{-(n+1)(1+i)}\right)}{\sqrt{1+\Delta^2}}$$

(F2) $T(\mathcal{L}) \cap \theta F_n \subset F(c; l_1, l_2)$ where $F(c; l_1, l_2)$ is the rectangle with the same centre c as F_n and of size $2l_1 \times 2l_2$ with

$$l_1 := \frac{\theta}{\Delta} \left(k^{-n(1+j)} + 4k^{-(n+1)(1+j)} + \Delta k^{-(n+1)(1+i)}\right)$$

$$l_2 := \theta k^{-n(1+j)} \quad .$$

We now estimate the right hand side of (16) by considering two cases. Throughout, let a_i, b_i denote the constants associated with the measure m_i and condition (A) and let

$$\varpi := 3 \left(\frac{4 b_1 b_2}{\kappa_1 \, a_1 a_2 2^{\delta_1+\delta_2}} \right)^{1/\delta_1} \quad .$$

Case (i): $\Delta \geq \varpi k^{-n(1+j)}/k^{-n(1+i)}$. In view of (F2) above, we trivially have that

$$m(\theta F_n \cap T(\mathcal{L})) \ \leq \ m(F(c; l_1, l_2)) \ \leq \ b_1 \, b_2 \, l_1^{\delta_1} \, l_2^{\delta_2} \ .$$

It follows that

$$
\begin{aligned}
\frac{m(T(\mathcal{L})\cap\theta F_n)}{m(2\theta F_{n+1})} &\le \frac{b_1b_2 l_1^{\delta_1} l_2^{\delta_2}}{a_1a_2(2\theta)^{\delta_1+\delta_2}\, k^{-(n+1)(1+j)\delta_1}\, k^{-(n+1)(1+i)\delta_2}} \\
&\le \frac{b_1b_2}{a_1a_22^{\delta_1+\delta_2}}\left(\frac{1}{\varpi}+\frac{1}{\varpi k^{1+j}}+\frac{1}{k^{1+i}}\right)^{\delta_1}\ k^{(1+j)\delta_1}\, k^{(1+i)\delta_2} \\
&\le \frac{b_1b_2}{a_1a_22^{\delta_1+\delta_2}}\left(\frac{3}{\varpi}\right)^{\delta_1}\ k^{(1+j)\delta_1}\, k^{(1+i)\delta_2} \\
&= \frac{\kappa_1}{4}\ k^{(1+j)\delta_1}k^{(1+i)\delta_2}\ .
\end{aligned}
$$

Case (ii): $0\le\Delta<\varpi k^{-n(1+j)}/k^{-n(1+i)}$. By the covering lemma of §3, there exists a collection $\mathcal{B}_n$ of disjoint balls B_n with centres in $\theta F_n\cap\Omega$ and radii $\theta k^{-n(1+j)}$ such that

$$
\theta F_n\cap\Omega\ \subset\ \bigcup_{B_n\in\mathcal{B}_n} 3B_n\ .
$$

Since $i<j$, it is easily verified that the disjoint collection $\mathcal{B}_n$ is contained in $2\theta F_n$ and thus $\#\mathcal{B}_n\le m(2\theta F_n)/m(B_n)$. It follows that

$$
\begin{aligned}
m(\theta F_n\cap\, T(\mathcal{L})) &\le m\left(\cup_{B_n\in\mathcal{B}_n}3B_n\cap\, T(\mathcal{L})\right) \\
&\le \#\mathcal{B}_n\ m(3B_n\cap T(\mathcal{L})) \\
&\le \frac{m(2\theta F_n)}{m(B_n)}\ m\left(3B_n\cap\mathcal{L}^{(\epsilon)}\right) && \text{by (F1) above} \\
&\le m(2\theta F_n)\,\frac{m(3B_n)}{m(B_n)}\ \left(\frac{\epsilon}{3\theta k^{-n(i+j)}}\right)^{\alpha} && m \text{ is absolutely } \alpha\text{-decaying.}
\end{aligned}
$$

Now notice that

$$
\frac{\epsilon}{3\theta k^{-n(i+j)}}\ \le\ \frac{4}{3}\left(k^{-(1+j)}+\varpi k^{-(1+i)}\right)\ .
$$

Hence, for k sufficiently large we have that

$$
\frac{m(T(\mathcal{L})\cap\theta F_n)}{m(2\theta F_{n+1})}\ \le\ \frac{\kappa_1}{4}\ k^{(1+j)\delta_1}k^{(1+i)\delta_2}\ .
$$

On combining the above two cases, we have that

$$
\text{l.h.s. of (6)}\le\frac{m(T(\mathcal{L})\cap\theta F_n)}{m(2\theta F_{n+1})}\ \le\ \frac{\kappa_1}{4}\ k^{(1+j)\delta_1}k^{(1+i)\delta_2}\ =\ \frac{1}{4}\times\ \text{l.h.s. of (14)}\ .
$$

Hence (6) is satisfied and Theorem 3 implies the desired result.

□

The argument used to establish Theorem 11 can be adapted in the obvious manner to prove a slightly more general result.

Theorem 13 *For $1 \leq j \leq N$, let Ω_j be a compact subset of $\mathbb{R}^{d_j}$ which supports an absolutely α_j-decaying measure m_j satisfying condition (A) with exponent δ_j. Let Ω denote the product set $\Omega_1 \times \ldots \times \Omega_N$. Then, for any N-tuple $(i_1, \ldots, i_N)$ with $i_j \geq 0$ and $\sum_{j=1}^N d_j\, i_j = 1$,*

$$\dim \mathbf{Bad}_\Omega(\underbrace{i_1, \ldots, i_1}_{d_1 \text{ times}};\ \underbrace{i_2, \ldots, i_2}_{d_2 \text{ times}};\ \ldots;\ \underbrace{i_N, \ldots, i_N}_{d_N \text{ times}}) \ = \ \dim \Omega \ = \ \sum_{j=1}^{N} \delta_j \ .$$

The following is a simple consequence of Theorem 9 and Theorem 13.

Corollary 14 *For $1 \leq j \leq N$, let K_j be the attractor of a finite irreducible family of contracting self similarity maps of $\mathbb{R}^{d_j}$ satisfying the open set condition. Let m_j be the restriction of $\mathcal{H}^{\delta_j}$ to K_j where $\delta_j = \dim K_j$. Let K denote the 'product attractor' $K_1 \times \ldots \times K_N$. Then, for any N-tuple $(i_1, \ldots, i_N)$ with $i_j \geq 0$ and $\sum_{j=1}^N d_j\, i_j = 1$,*

$$\dim(K \cap \mathbf{Bad}(\underbrace{i_1, \ldots, i_1}_{d_1 \text{ times}};\ \underbrace{i_2, \ldots, i_2}_{d_2 \text{ times}};\ \ldots;\ \underbrace{i_N, \ldots, i_N}_{d_N \text{ times}})) \ = \ \dim K \ .$$

As an application of Corollary 14 we obtain the following statement which to some extent is more illuminating – even this special case appears to be new.

Corollary 15 *Let $V \subset \mathbb{R}^2$ be the von Koch curve and $K \subset \mathbb{R}$ be the middle third Cantor set. Then, for any positive i and j with $2\,i + j = 1$*

$$\dim\left((V \times K) \cap \mathbf{Bad}(i, i, j)\right) \ = \ \dim(V \times K) \ = \ \frac{\log 8}{\log 3} \ .$$

5.1.1 Remarks related to Schmidt's conjecture

In §2.3, we mentioned the result that $\dim(\mathbf{Bad}(i, j) \cap \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)) = 2$. This can easily be obtained via Theorem 11. To see this, first of all notice that $\mathbf{Bad} \times \mathbf{Bad} = \mathbf{Bad}(1, 0) \cap \mathbf{Bad}(0, 1)$. For $M \geq 2$, let $F_M := \{x \in [0, 1] : x := [a_1, a_2, \ldots] \text{ with } a_i \leq M \text{ for all } i\}$. Thus F_M is the set of real numbers in the unit interval with partial quotients bounded above by M. By definition F_M is a compact subset of **Bad** and moreover it is well known that F_M supports a measure m_M which satisfies condition (A) with exponent δ_M with $\delta_M \to 1$ as $M \to \infty$. Now let $\Omega := F_M \times F_M$, then Theorem 11 implies that

$$\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1,0) \cap \mathbf{Bad}(0,1)) \ \geq \ \dim(\mathbf{Bad}_\Omega(i,j)) \ = \ 2\delta_M \ .$$

On letting $M \to \infty$, we obtain that $\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1,0) \cap \mathbf{Bad}(0,1)) \geq 2$. The complementary upper bound result is trivial since the set in question is a subset of $\mathbb{R}^2$.

Recall, that Schmidt's conjecture [22] states that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(i',j') \neq \emptyset$. In fact, Schmidt stated this conjecture in the simpler situation when $i = j' = 1/3$ and $i' = j = 2/3$. Even this specific and symmetric case is unsolved. In order to illustrate a possible approach towards the conjecture via the results of this paper we consider the special case of $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2)$. Suppose for the moment that we could find a compact set $\Omega \subseteq \mathbf{Bad}(i,j)$ with a measure m satisfying condition (A) for some $\delta > 1$. Let $\rho(r) = r^{-3/2}$. Using Lemma 7 together with the 'triangle' argument or equivalently the Simplex Lemma of §2.3, we may construct collections $\mathcal{C}(\theta B_n)$ as in the statement of Theorem 1. The condition that $\delta > 1$ is used to ensure (2). This leads to the following enticing statement:

If there exists a compact subset Ω of $\mathbf{Bad}(i,j)$ which supports a measure m satisfying condition (A) with exponent $\delta > 1$, then

$$\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2)) \geq \delta \ .$$

Clearly, this would imply that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2) \neq \emptyset$. Regarding the above statement, it is not particularly difficult to prove the existence of a compact subset Ω supporting a measure m satisfying condition (A) with $\delta < 1$. However, from this we are not able to deduce that $\dim(\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2)) \geq \delta$ or even that $\mathbf{Bad}(i,j) \cap \mathbf{Bad}(1/2,1/2) \neq \emptyset$.

5.2 Rational Maps

In this section we consider the 'badly approximable' analogue of the 'shrinking target' problem introduced in [8] for expanding rational maps. Let T be an expanding rational map (degree ≥ 2) of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

and $J(T)$ be its Julia set. For any $z_0 \in J(T)$ consider the set

$$\mathbf{Bad}_{z_0}(J) := \{z \in J(T) \ : \exists\, c(z) > 0 \text{ such that } T^n(z) \notin B(z_0, c(z)) \text{ for any } n \in \mathbb{N}\} \ .$$

Clearly, the forward orbit of points in $\mathbf{Bad}_{z_0}(J)$ are not dense in $J(T)$. Now let m be Sullivan measure and $\delta = \dim J(T)$. Thus m is a non-atomic, δ-conformal probability measure supported on $J(T)$ and since T is expanding it satisfies condition (A). Moreover, m is equivalent to δ-dimensional Hausdorff measure $\mathcal{H}^\delta$ – see [8,9] for the details. In view of the 'Khintchine type' result for expanding rational maps (see, for example [3, §8.4]) it is easily verified that $\mathcal{H}^\delta(\mathbf{Bad}_{z_0}(J)) = 0 = m(\mathbf{Bad}_{z_0}(J))$. Nevertheless, the set $\mathbf{Bad}_{z_0}(J)$ is large in that it is of maximal dimension.

Theorem 16

$$\dim \mathbf{Bad}_{z_0}(J) \ = \ \delta \ .$$

This result is not new and has been established by numerous people (see e.g. [2]). However, we give a short proof which indicates the versatility and generality of our framework and results.

Proof of Theorem 16. In view of the bounded distortion property for expanding maps [8, Proposition 1], we can rewrite $\mathbf{Bad}_{z_0}(J)$ in terms of points in the Julia set which 'stay clear' of balls centred around the backward orbit of the selected point z_0:

$$\mathbf{Bad}_{z_0}(J) \equiv \{z \in J(T) \ : \exists\, c(z) > 0 \text{ such that } z \notin B(y, c(z)|(T^n)'(y)|^{-1}) \text{ for any } (y,n) \in I\} \ ,$$

where $I := \{(y,n) : n \in \mathbb{N} \text{ with } T^n(y) = z_0\}$. Also, since T is expanding, $J(T)$ can be thought of as a compact metric space with the usual metric on $\mathbb{C}$. It is now clear that $\mathbf{Bad}_{z_0}(J)$ can be expressed in the form $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ with $\rho(r) := r^{-1}$ and

$$X = \Omega := J(T) \ , \quad J := I \ , \quad \alpha := (y,n) \in I \ , \quad \beta_\alpha := |(T^n)'(y)| \ , \quad R_\alpha := y \ .$$

With reference to Theorem 1, Sullivan measure m and the function ρ satisfy condition (A) and (B) respectively. To deduce Theorem 16 from Theorem 1 we need to establish the existence of the disjoint collection $\mathcal{C}(\theta B_n)$ of balls $2\theta B_{n+1}$ where B_n is an arbitrary ball of radius k^{-n} with centre in Ω. In view of Lemma 7, for k sufficiently large, there exists a disjoint collection $\mathcal{C}(\theta B_n)$ such that

$$\#\mathcal{C}(\theta B_n) \ \geq \ \kappa_1 \, k^\delta \ ; \tag{17}$$

i.e. (1) of Theorem 1 holds. We now verify that (2) is satisfied for any such collection. First we recall a key result which is the second part of the statement of Lemma 8 in [9]. For ease of reference we keep the same notation and numbering of constants as in [9].

Constant Multiplicity: For $X \in \mathbb{R}^+$, let $P(X)$ denote the set of pairs $(y,n) \in I$ such that $f_n(y) - C_8 \leq X \leq f_{n+1}(y) + C_8$, where $f_n(y) := \log |(T^n)'(y)|$. Let $z \in J(T)$. Then there are no more than C_9 pairs $(y,n) \in P(X)$ such that $z \in B\left(y, C_{10}\,|(T^n)'(y)|^{-1}\right)$.

We are now in the position to verify (2) of Theorem 1. By definition $J(n+1) := \{(y,m) \in I : k^{n-1} \leq |(T^m)'(y)| < k^n\}$ and let $\theta := C_{10} k^{-1}$. It follows that

$$\begin{aligned} \text{l.h.s. of (2)} &\leq \#\{y \in \theta B_n : (y,m) \in J(n+1)\} \\ &\leq \#\{y \in B(c, C_{10}|(T^m)'(y)|^{-1}) : (y,m) \in J(n+1)\} \ , \qquad (18) \end{aligned}$$

where c is the centre of θB_n. Without loss of generality, assume that $|T'(z_0)| > 1$. Otherwise, since T is expanding we simply work with some higher iterate T^q of T for which $|(T^q)'(z_0)| > 1$. Then, the chain rule together with the above 'constant multiplicity' fact implies that the r.h.s. of (18) is $\ll C_9 \log k$. Hence, for k sufficiently large

$$\text{l.h.s. of (2)} \ \leq \ \tfrac{1}{2} \times \ \text{r.h.s. of (17)} \ .$$

Thus, (2) is easily satisfied and Theorem 1 implies Theorem 16. □

Remark: It is worth mentioning that our framework also yields (just as easily) the analogue of Theorem 16 within the Kleinian group setup. Briefly, let G be either a geometrically finite Kleinian group of the first kind or a convex co-compact group and let $\Lambda(G)$ denote its limit set. For these groups, Patterson measure supported on $\Lambda(G)$ satisfies condition (A) and plays the role of Sullivan measure. Then, it is not difficult to obtain the Kleinian group analogue of Theorem 16 via Theorem 1; i.e. the set of 'badly approximable' limit points is of full dimension – $\dim \Lambda(G)$.

5.3 Complex numbers

In this section we consider the badly approximable analogue of $\mathbf{Bad}(i_1, \ldots, i_N)$ in $\mathbb{C}^N$. Let $N \in \mathbb{N}$ and $i_1, \ldots, i_N \geq 0$ such that $i_1 + \cdots + i_N = 1$. Now define the set $\mathbf{Bad}_{\mathbb{C}}(i_1, \ldots, i_N)$ to consist of $z := (z_1, \ldots, z_N) \in \mathbb{C}^N$ for which there exists a constant $c(z) > 0$ such that for any $q, p_1, \ldots, p_N \in \mathbb{Z}[i]$, $q \neq 0$,

$$\max\{|qz_1 - p_1|^{1/i_1}, \ldots, |qz_N - p_N|^{1/i_N}\} \geq c(z)|q|^{-1} \ .$$

In the case $i_1 = \cdots = i_N = 1/N$, the corresponding set will be denoted by $\mathbf{Bad}_{\mathbb{C}}(N)$. Notice, that the role of the rationals in the real setup is replaced by ratios of Gaussian integers in the complex setup. We shall refer to the latter as Gaussian points.

The Hausdorff dimension of the set $\mathbf{Bad}_{\mathbb{C}}(N)$ has been studied in the past by various people using Kleinian groups [4], Riemannian geometry [7] and Schmidt's (α, β)-games [5]. Theorem 1 of this paper will also give the Hausdorff dimension of this set. In fact, our general framework enables us to find the dimension of $\mathbf{Bad}_{\mathbb{C}}(i_1, \ldots, i_N)$ intersected with direct products of sets supporting measures satisfying condition (A). As a consequence, the previously known results are extended to the 'rectangular' or 'weighted' form of simultaneous approximation in $\mathbb{C}^N$. The following statement is the 'complex' analogue of Theorem 11.

Theorem 17 *For $1 \leq j \leq N$, let Ω_j be a compact subset of $\mathbb{C}$ which supports a measure m_j satisfying condition (A) with exponent δ_j. Let Ω denote the product set $\Omega_1 \times \ldots \times \Omega_N$. Then, for any N-tuple $(i_1, \ldots, i_N)$ with $i_j \geq 0$ and $\sum_{j=1}^{N} i_j = 1$,*

$$\dim(\mathbf{Bad}_{\mathbb{C}}(i_1, \ldots, i_N) \cap \Omega) = \dim \Omega \ .$$

The following complex notion of absolutely decaying measures will be useful in proving the above theorem. Let Ω be a compact subset of $\mathbb{C}^N$ which supports a non-atomic, finite measure m. Let $\mathcal{L}$ denote a generic $(N-1)$-dimensional complex hyperplane of $\mathbb{C}^N$ and let $\mathcal{L}^{(\epsilon)}$ denote its ϵ-neighborhood. We say that m is *absolutely α-decaying* if there exist strictly positive constants C, α, r_0 such that for any complex hyperplane $\mathcal{L}$, any $\epsilon > 0$, any $z \in \Omega$ and any $r < r_0$,

$$m\left(B(z, r) \cap \mathcal{L}^{(\epsilon)}\right) \ \leq \ C \left(\frac{\epsilon}{r}\right)^{\alpha} m(B(z, r)) \ .$$

Note that if $N = 1$, so that Ω is a subset of $\mathbb{C}$, the complex hyperplane $\mathcal{L}$ is simply a point $a \in \mathbb{C}$ and $\mathcal{L}^{(\epsilon)}$ is the ball $B(a, \epsilon)$ centred at a of radius ϵ. Moreover, if the measure m satisfies condition (A) with exponent δ then m is automatically absolutely δ-decaying.

It is easy to verify that the statement of the 'Fact' in §5.1 regarding the product of absolutely decaying measures remains valid for the complex notion.

Proof of Theorem 17 *(Sketch)*. As usual we restrict our attention to the case $N = 2$ and write $\mathbf{Bad}_{\mathbb{C}}(i, j)$ for $\mathbf{Bad}_{\mathbb{C}}(i_1, i_2)$. Assume that $i \leq j$. Clearly, the set $\mathbf{Bad}_{\mathbb{C}}(i, j) \cap \Omega$ can be expressed in the form $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2)$ with $\rho_1(r) = r^{-(1+i)}$, $\rho_2(r) = r^{-(1+j)}$ and

$$X = (\mathbb{C}^2, d)\ ,\quad J := \{((p_1, p_2), q) \in \mathbb{Z}[i]^2 \times \mathbb{Z}[i]\backslash\{0\}\}\ ,$$

$$\alpha := ((p_1, p_2), q) \in J\ ,\quad \beta_\alpha := |q|\ ,\quad R_\alpha := (p_1/q, p_2, q)\ .$$

The metric d on $\mathbb{C}^2$ is the maximum of the coordinate metrics; i.e. $d((z_1, z_2), (z_1', z_2')) = \max\{d(z_1, z_1'), d(z_2, z_2')\}$. Also note that the measure $m := m_1 \times m_2$ is absolutely α-decaying on Ω with $\alpha := \min\{\delta_1, \delta_2\}$. This follows from the above discussion concerning the complex notion of absolutely decaying measures and their product.

With reference to Theorem 3, we need to establish the existence of the collection $\mathcal{C}(\theta F_n)$ where F_n is an arbitrary closed polydisc $B_{n,1} \times B_{n,2}$ with centre c in Ω. Here $B_{n,1}$ (resp. $B_{n,2}$) is a closed ball in $\mathbb{C}$ of radius $k^{-n(1+i)}$ (resp. $k^{-n(1+j)}$). In view of Lemma 7, there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of polydiscs $2\theta F_{n+1} \subset \theta F_n$ such that (5) of Theorem 3 is satisfied. We now verify that (6) is satisfied for any such collection by modifying the proof of Theorem 11 in the obvious manner. The only part which is not so obvious is the complex analogue of the 'triangle' argument of §2.3. For this suppose that θF_n is given and that there are at least three Gaussian points $(p_1/q, p_2/q), (p_1'/q', p_2'/q')$ and $(p_1''/q'', p_2''/q'')$ with

$$k^n \leq |q|, |q'|, |q''| < k^{n+1}$$

lying within θF_n. Suppose for the moment that they do not lie on a one–dimensional complex hyperplane (i.e. a complex line) $\mathcal{L}$ of $\mathbb{C}^2$ and consider the determinant

$$D = \det \begin{pmatrix} 1 & p_1/q & p_2/q \\ 1 & p_1'/q' & p_2'/q' \\ 1 & p_1''/q'' & p_2''/q'' \end{pmatrix} \neq 0\ .$$

Expanding the determinant in the first column and using the fact that the ring of Gaussian integers is a unique factorization domain, we find that

$$|D| > \frac{1}{k^{3(n+1)}}\ .$$

On the other hand, the absolute value of D can be at most twice the diameters of the two projections $\theta B_{n,1}$ and $\theta B_{n,2}$ of θF_n. That is

$$|D|\ \leq\ 2\ \frac{2\,\theta}{k^{n(i+1)}}\ \frac{2\,\theta}{k^{n(j+1)}}\ =\ \frac{8\theta^2}{k^{3n}}\ .$$

To see this, note that for $(z_1, z_2), (z_1', z_2'), (z_1'', z_2'') \in \theta F_n$

$$\left|\det\begin{pmatrix}1 & z_1 & z_2\\ 1 & z_1' & z_2'\\ 1 & z_1'' & z_2''\end{pmatrix}\right| = |(z_1-z_1')(z_2'-z_2'')+(z_1'-z_1'')(z_2'-z_2)|$$

$$\leq\ 2\times 2\theta\rho_1(k^n)\,2\theta\rho_2(k^n)\ .$$

Now with $\theta := (8k^3)^{-1/2}$, we obtain the desired contradiction. Thus, if there are two or more Gaussian points with $k^n \leq |q| < k^{n+1}$ lying within θF_n then they must lie on a complex line $\mathcal{L}$. It now follows that

$$\text{l.h.s. of (6)}\ \leq\ \#\{2\theta F_{n+1}\subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1}\cap\mathcal{L}\neq\emptyset\}\ .$$

This is the precise complex analogue of (15) and the proof can now be completed by modifying the proof of the real case (Theorem 11) in the obvious manner. We leave the details to the reader. □

It is worth mentioning that Theorem 17 can be generalized in the obvious manner to obtain the complex analogue of Theorem 13.

5.4 p-adic numbers

For a prime p, let $|\cdot|_p$ denote the p-adic absolute value and let $\mathbb{Q}_p$ denote the p-adic field. Furthermore, let $\mathbb{Z}_p$ denote the ring of p-adic integers. In this section we consider the badly approximable analogue of $\mathbf{Bad}(i_1,\dots,i_N)$ in $\mathbb{Z}_p^N$. Let $N\in\mathbb{N}$ and $i_1,\dots,i_N\geq 0$ such that $i_1+\dots+i_N=1$. Now define the set $\mathbf{Bad}_{\mathbb{Z}_p}(i_1,\dots,i_N)$ to consist of $x:=(x_1,\dots,x_N)\in\mathbb{Z}_p^N$ for which there exists a constant $c(x)>0$ such that

$$\max\{|qx_1-r_1|_p^{1/(1+i_1)},\dots,\ |qx_N-r_N|_p^{1/(1+i_N)}\}$$
$$\geq\ c(x)\,\max\{|r_1|,\dots,|r_N|,|q|\}^{-1}\ , \tag{19}$$

for all $((r_1,\dots,r_N),q)\in\mathbb{Z}^N\times\mathbb{Z}\setminus\{0\}$. In the case $i_1=\dots=i_N=1/N$, the corresponding set will be denoted by $\mathbf{Bad}_{\mathbb{Z}_p}(N)$.

There are two points worth making when comparing the above set with the 'classical' set $\mathbf{Bad}(i_1,...,i_N)$. Firstly, the r.h.s of (19) in the p-adic setup is a function of $\max(|r_1|,\dots,|r_N|,|q|)$ rather than simply $|q|$. This is due to the fact that within the p-adic setup for any $x\in\mathbb{Z}_p^N$ and $q\in\mathbb{Z}$ there exists $r\in\mathbb{Z}^N$ such that the l.h.s of (19) can be made arbitrarily small. Thus, the set of $x\in\mathbb{Z}_p^N$ for which l.h.s of (19) $\geq c(x)\,|q|^{-1}$ is in fact empty and there is nothing to prove. Secondly, in the p-adic setup the 'weighting' factor occurring on the l.h.s of (19) is $1/(1+i_s)$ rather than $1/i_s$ $(1 \leq s \leq N)$. This is due to the fact that we approximate in terms of the p-adic absolute value on the left hand side, but measure the 'rate' of approximation in terms of the ordinary absolute value on the right hand side. Because of the arithmetical properties of the p-adic absolute value, we generally expect the 'rate' of the approximation to be better (see below).

Badly approximable p-adic numbers have in the past been studied by Abercrombie [1], who showed that $\mathbf{Bad}_{\mathbb{Z}_p}(1)$ has full Hausdorff dimension. In higher dimensions, the corresponding result for even the 'symmetric' set $\mathbf{Bad}_{\mathbb{Z}_p}(N)$ is unknown. Using the framework established in this paper, we are able to prove the following complete result.

Theorem 18

$$\dim \mathbf{Bad}_{\mathbb{Z}_p}(i_1, \ldots, i_N) = N \ .$$

Proof of Theorem 18 ***(Sketch).*** As in the preceding applications, we restrict our attention to the case $N = 2$ and write $\mathbf{Bad}_{\mathbb{Z}_p}(i,j)$ for $\mathbf{Bad}_{\mathbb{Z}_p}(i_1, i_2)$. Assume that $i \leq j$. Clearly the set $\mathbf{Bad}_{\mathbb{Z}_p}(i,j)$ can be expressed in the form $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2)$ with $\rho_1(x) := x^{-(1+i)}$, $\rho_2(x) := x^{-(1+j)}$ and

$$X = \Omega := \mathbb{Z}_p^2 = \mathbb{Z}_p \times \mathbb{Z}_p \ , \quad J := \{((r_1, r_2), q) \in \mathbb{Z}^2 \times \mathbb{Z} \setminus \{0\}\} \ ,$$

$$\alpha := ((r_1, r_2), q) \in J \ , \quad \beta_\alpha := \max\{|r_1|, |r_2|, |q|\}$$

$$R_\alpha := \{(x_1, x_2) \in \mathbb{Z}_p^2 : qx_1 = r_1, \ qx_2 = r_2\} \ .$$

Furthermore, $d = d_1 \times d_1$ where $d_1(x,y) := |x - y|_p$ is the p-adic metric on $\mathbb{Q}_p$ and $m := \mu \times \mu$ where μ is normalized Haar measure on $\mathbb{Q}_p$. Thus, $\mu(\mathbb{Z}_p) = 1$ and $\mu(B(x, p^{-t})) = p^{-t}$ for any $t \in \mathbb{N}$. Note that these are the only radii which make sense – if $p^{-t} \leq r < p^{-t+1}$, then $B(x,r) = B(x, p^{-t})$.

We take a moment to verify that the set $\mathbf{Bad}(i,j)$ is indeed equal to the set $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2)$. Fix $q \in \mathbb{Z} \setminus \{0\}$ and $(r_1, r_2) \in \mathbb{Z}^2$. Associated with the pair $((r_1, r_2), q)$ is the resonant point $R_{((r_1,r_2),q)} = (R_{(r_1,q)}, R_{(r_2,q)})$. First, note that $|qx_s - r_s|_p = |q|_p \ d_1(x_s, R_{(r_s,q)})$ for $s \in \{1, 2\}$. However, $|q|_p \leq 1$ and so clearly $\mathbf{Bad}(i,j) \subseteq \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2)$. Conversely, let $x \in \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2)$. We show that (19) is satisfied for r and q. If $(q,p) = 1$, then $|q|_p = 1$ and the inequality is immediate. If $p^t | q$ for some $t \in \mathbb{N}$, but either $(r_1, p) = 1$ or $(r_2, p) = 1$, the inequality is also satisfied. To see this, suppose that $(r_1, p) = 1$ and express $-r_1$ and qx_1 as power series in p. Clearly, the lowest exponent of p in the expansion of qx_1 it at least t, whereas the expansion of $-r_1$ has a term with exponent zero. Hence the sum of the two must have a term of exponent zero, and so $|qx_1 - r_1|_p = 1$ and we are done. In the remaining case, when p divides q, r_1 and r_2, we simply factor out the highest possible power of p in the left hand side of (19) and the problem reduces to one of the previous cases. Thus, $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_2) \subseteq \mathbf{Bad}(i, j)$.

With reference to Theorem 3, the functions ρ_1, ρ_2 satisfy condition (B*) and the measures $m_1 := \mu$ and $m_2 := \mu$ satisfy condition (A) with $\delta_1 = \delta_2 = 1$. We need to establish the existence of the collection $\mathcal{C}(\theta F_n)$ where F_n is an arbitrary closed rectangle of size $2k^{-n(1+i)} \times 2k^{-n(1+j)}$. Here, we take $k = p^s$ and $\theta = p^{-t}$ for some $s, t \in \mathbb{N}$ which will be chosen sufficiently large later on. In view of Lemma 7, there exists a disjoint collection $\mathcal{C}(\theta F_n)$ of rectangles $2\theta F_{n+1} \subset \theta F_n$ such that (5) of Theorem 3 is satisfied. We now verify that (6) is satisfied for any such collection. This follows by modifying the 'triangle' argument of §2.3 to the p-adic setting. So, let us assume that we have three resonant points (which by definition are rational points) $(r_1/q, r_2/q), (r_1'/q', r_2'/q')$ and $(r_1''/q'', r_2''/q'')$ lying in some rectangle θF_n with

$$k^n \leq \max_{1 \leq s,t,u \leq 2} \{|r_s|, |r_t'|, |r_u''|, |q|\} < k^{n+1}. \tag{20}$$

Suppose that they do not lie on a line. Then, they span a p-adic triangle Δ. By results in Lutz [14, Chapter I, §4], the Haar measure m of Δ is comparable to

$$\left| \det \begin{pmatrix} 1 & r_1/q & r_2/q \\ 1 & r_1'/q' & r_2'/q' \\ 1 & r_1''/q'' & r_2''/q'' \end{pmatrix} \right|_p \neq 0.$$

The determinant is a rational number with denominator $qq'q''$. As these are integers, the p-adic absolute value is ≤ 1. Hence, the absolute value of the determinant is bounded below by the p-adic absolute value of the enumerator:

$$N = r_1 r_2' q'' - r_2 r_1' q'' - r_1 q' r_2'' + r_2 q' r_1'' + q r_1' r_2'' - q r_2' r_1''.$$

This is an integer. In view of (20), we have that

$$|N| < 6k^{3n+3}.$$

We may assume without loss of generality that $N > 0$. Clearly, the p-adic valuation $v_p(N)$ (i.e. the number of times p divides N) satisfies

$$v_p(N) < \log_p(6k^{3n+3}).$$

But $|N|_p = p^{-v_p(N)}$ so that

$$|N|_p > p^{-\log_p(6k^{3n+3})} = 1/(6k^{3n+3}) \ .$$

Hence, there is a constant $C > 0$ such that $m(\Delta) > \ C/(6k^{3n+3})$. However, $\mu(\theta F_n) \leq \theta^2 k^{-3n}$ and on choosing $\theta^2 := p^{-2t} < C/(6k^3)$ we obtain the desired contradiction; i.e. by choosing t sufficiently large. Thus, it there are two or more resonant points satisfying (20) lying within θF_n then they must lie on a p-adic line $\mathcal{L}$. It now follows that

$$\text{l.h.s. of (6)} \ \leq \ \#\{2\theta F_{n+1} \subset \mathcal{C}(\theta F_n) : 2\theta F_{n+1} \cap \mathcal{L} \neq \emptyset\} \ .$$

A simple geometric argument, analogous to that employed in §2.3, ensures that the line $\mathcal{L}$ can not pass through more than $C' \, k^{j+1}$ of the $2\theta F_{n+1}$ rectangles. Here $C' > 0$ is a constant independent of k. On choosing $k := p^s$ sufficiently large (i.e. s large enough), we ensure that $C' \, k^{1+j} < \kappa_1 \ k^3$ which establishes (6) and thereby completes the proof of the theorem. □

Under suitable assumptions on subsets Ω_i of $\mathbb{Z}_p$ with measures satisfying condition (A), we can also obtain the p-adic analogues of Theorems 11 and 13. Of course, to achieve this, one also needs to assume the natural p-adic analogue of a measure being absolutely α-decaying.

5.5 Formal power series

Apart from the p-adics, badly approximable elements have been extensively studied over another locally compact ultra-metric field. Let $\mathbb{F}$ be the finite field with h elements. Thus, $h = p^r$ for some prime p and $r \in \mathbb{N}$. Now define

$$\mathbb{F}((X^{-1})) := \left\{ \sum_{i=-n}^{\infty} a_{-i} X^{-i} : n \in \mathbb{Z}, a_i \in \mathbb{F}, a_n \neq 0 \right\} \cup \{0\} \ ,$$

with an absolute value

$$\left\| \sum_{i=-n}^{\infty} a_{-i} X^{-i} \right\| := h^n, \quad \|0\| := 0 \ .$$

Under ordinary addition and multiplication, this is a locally compact field. The closed unit ball $I = \{x \in \mathbb{F}((X^{-1})) : \|x\| \leq 1\}$ is a compact subspace of this space.

In this section we consider the badly approximable analogue of $\mathbf{Bad}(i_1, \ldots, i_N)$ in I^N. Let $N \in \mathbb{N}$ and $i_1, \ldots, i_N \geq 0$ such that $i_1 + \cdots + i_N = 1$. Now define the set $\mathbf{Bad}_{\mathbb{F}((X^{-1}))}(i_1, \ldots, i_N)$ to consist of $x := (x_1, \ldots, x_N) \in \mathbb{F}((X^{-1}))^N$ for which there exists a constant $c(x) > 0$ such that

$$\max\{\|qx_1 - p_1\|^{1/i_1}, \ldots, \|qx_N - p_N\|^{1/i_N}\} \geq c(x) \, \|q\|^{-1}$$

for all $q, p_1, \ldots, p_N \in \mathbb{F}[X]$ $(q \neq 0)$. Note that in this setup, the polynomial ring $\mathbb{F}[X]$ plays the role of the integers. When $i_1 = \ldots = i_N = 1/N$, the corresponding set will be denoted by $\mathbf{Bad}_{\mathbb{F}((X^{-1}))}(N)$. Niederreiter and Vielhaber [17] have shown that the set $\mathbf{Bad}_{\mathbb{F}((X^{-1}))}(1)$ has full dimension. Using the framework established in this paper, we are able to obtain the complete result for the 'weighted' simultaneous set.

Theorem 19

$$\dim \mathbf{Bad}_{\mathbb{F}((X^{-1}))}(i_1, \ldots, i_N) = N \ .$$

Proof of Theorem 19 *(Sketch).* As usual, we restrict our attention to the case $N = 2$ and write $\mathbf{Bad}_{\mathbb{F}((X^{-1}))}(i, j)$ for $\mathbf{Bad}_{\mathbb{F}((X^{-1}))}(i_1, i_2)$. In view of the geometrical nature of our approach and the similarities between this situation and the preceding ones (in particular the p-adic case), we only outline the modifications needed to deal with the present situation in the briefest sense. The field $\mathbb{F}((X^{-1}))$ supports a Haar measure m satisfying $m(B(c, h^{-t})) = h^{-t}$ for all $t \in \mathbb{Z}$. As was the case in the p-adics, these are the only balls for which a calculation is needed. Let I denote the unit ball in this space. We set $X_1 = X_2 = \mathbb{F}((X^{-1}))$, $\Omega_1 = \Omega_2 = I$ with the metrics induced by the absolute value and Haar measure defined above. We let $J = \{((p_1, p_2), q) \in \mathbb{F}[X]^2 \times \mathbb{F}[X] \setminus \{0\}\}$ and for any $((p_1, p_2), q) \in J$, we let $\beta_{((p_1,p_2),q)} = \|q\|$. The resonant sets $R_{((p_1,p_2),q)} = (p_1/q, p_2/q)$. Finally, define functions $\rho_1(x) = x^{-(i+1)}$ and $\rho_2(x) = x^{-(j+1)}$. Clearly, the conditions of Theorem 3 are satisfied and the set $\mathbf{Bad}_{\mathbb{F}((X^{-1}))}(i, j) \cap I^2 = \mathbf{Bad}^*(\mathcal{R}, \beta, \rho_1, \rho_j)$.

We establish the collection $\mathcal{C}(\theta F_n)$ by Lemma 7. The triangle argument works in this setting by results of Mahler [15] to calculate the measures of the sets involved. Note that in this case, the lower bound on the denominator is the important feature in the argument, so the proof differs from the p-adic case in this respect. Finally, maximal number of rectangles in $\mathcal{C}(2\theta F_{n+1})$ with non-trivial intersection with the resulting 'line' is estimated by arguments as in the p-adic case. $\square$

As in the p-adic setup, under appropriate assumptions we can also obtain the formal power series analogues of Theorems 11 and 13. We have chosen to restrict ourselves to the simpler situation, as this already yields new results and illustrates the versatility of our framework.

Acknowledgments: SV would like to thank Dmitry Kleinbock and Barak Weiss for the many e-mail conversations regarding the development of our separate approaches. Also, he'd like to take this opportunity to say 'marra peeta' to Iona and Ayesha.

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