Metric Diophantine approximation and 'absolutely friendly' measures

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Abstract

Let $W(\psi)$ denote the set of ψ-well approximable points in $\mathbb{R}^d$ and let K be a compact subset of $\mathbb{R}^d$ which supports a measure μ. In this short note, we show that if μ is an 'absolutely friendly' measure and a certain μ–volume sum converges then $\mu(W(\psi) \cap K) = 0$. The result obtained is in some sense analogous to the convergence part of Khintchines classical theorem in the theory of metric Diophantine approximation. The class of absolutely friendly measures is a subclass of the friendly measures introduced in [2] and includes measures supported on self similar sets satisfying the open set condition. We also obtain an upper bound result for the Hausdorff dimension of $W(\psi) \cap K$.

1 Introduction

1.1 The problem and results

The classical result of Dirichlet in the theory of Diophantine approximation states that for any point $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbf{R}^d$, there exist infinitely many $(\mathbf{p}, q) \in \mathbf{Z}^d \times \mathbf{N}$ such that

$$\max_{1 \le i \le d} |x_i - p_i/q| \le q^{-(d+1)/d} \quad .$$

Given a real, positive decreasing function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$, a point $\mathbf{x} \in \mathbf{R}^d$ is said to be ψ–well approximable if the above inequality remains valid with the right hand side replaced with $\psi(q)$. We will denote by $W(\psi)$ the set of all such points; that is

$$W(\psi) := \{\mathbf{x} \in \mathbf{R}^d : \max_{1 \le i \le d} |x_i - p_i/q| \le \psi(q) \text{ for infinitely many } (\mathbf{p}, q) \in \mathbf{Z}^d \times \mathbf{N}\} \ .$$

A straightforward application of the Borel-Cantelli lemma from probability theory yields the following statement.

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Lemma 1

$$|W(\psi)|_d = 0 \qquad \text{if} \qquad \sum_{r=1}^{\infty} (r\,\psi(r))^d \ < \ \infty \ \ .$$

Thus, if the above sum converges then almost every (with respect to d-dimensional Lebesgue measure) point $\mathbf{x} \in \mathbf{R}^d$ is not ψ–well approximable. For $\tau \geq 0$, consider the function $\psi_\tau : r \to r^{-\tau}$ and write $W(\tau)$ for $W(\psi_\tau)$. In view of Dirichlet's result, $W(\tau) = \mathbf{R}^d$ for $\tau \leq (d+1)/d$. However, in view of the above lemma we have that $|W(\tau)|_d = 0$ for $\tau > (d+1)/d$.

Now, let K be a compact subset of $\mathbf{R}^d$ which supports a non-atomic, finite measure μ and let

$$W_K(\psi) \ := \ K \cap W(\psi) \ \ .$$

In short, the problem is to determine conditions on μ and ψ under which $\mu(W_K(\psi)) = 0$; i.e. μ-almost every point $\mathbf{x} \in \mathbf{R}^d$ is not ψ–well approximable. Note that $\mu(W_K(\psi)) = \mu(W(\psi))$ since μ is supported on K. For the motivation behind the problem we refer the reader to [2, 3, 4].

In [2], Kleinbock, Lindenstrauss and Weiss introduce the notion of a 'friendly' measure and show that if μ is friendly then $\mu(W_K(\tau)) = 0$ for $\tau > (d+1)/d$ [1]. They also show that the class of friendly measures include (i) volume measures on non-degenerate manifolds and (ii) measures supported on self similar sets satisfying the open set condition. In full generality, the definition of friendly is rather technical and will not be reproduce here – see §2 of [2].

Our aim is to obtain a statement more in line with Lemma 1 which also implies that $\mu(W_K(\tau)) = 0$ for $\tau > (d+1)/d$. To achieve this we impose conditions on μ which are stronger than those of friendly. Nevertheless, measures supported on self similar sets satisfying the open set condition are still included – see §1.2. Unfortunately, volume measures on non-degenerate manifolds and not included.

Let $B(x,r)$ be a ball in $\mathbf{R}^d$ with centre x and radius r. The measure μ is said to be *doubling* if there exist strictly positive constants D and r_0 such that

$$\mu(B(x,2r)) \ \leq \ D\,\mu(B(x,r)) \qquad \forall\, x \in K \quad \forall\, r < r_0 \ .$$

The following notion of 'absolutely decaying' is essentially taken from [2]. Let L denote a generic $(d-1)$–dimensional hyperplane of $\mathbf{R}^d$ and let $\mathrm{L}^{(\epsilon)}$ denote its ϵ-neighborhood. We say that μ is *absolutely α-decaying* if there exist strictly positive constants C, α, r_0 such that for any hyperplane L and any $\epsilon > 0$

$$\mu\left(B(x,r) \cap \mathrm{L}^{(\epsilon)}\right) \ \leq \ C\,\left(\frac{\epsilon}{r}\right)^{\alpha} \mu(B(x,r)) \qquad \forall\, x \in K \quad \forall\, r < r_0 \ .$$

In the case $d = 1$, the hyperplane L is simply a point $a \in \mathbf{R}$ and $\mathrm{L}^{(\epsilon)}$ is the ball $B(a,\epsilon)$ centred at a of radius ϵ.

[1]They actually prove their result in the multiplicative framework.

Remark. Let $B(a,r)$ be a ball in $\mathbf{R}^d$. A straightforward geometric argument shows that if μ is absolutely α-decaying, then for any $\epsilon < 1/4$

$$\mu\left(B(a,\epsilon r)\right) \ \leq \ C\, 2^{\alpha}\, \epsilon^{\alpha} \ \mu(B(a,r)) \qquad \forall\ a \in \mathbf{R}^d \quad \forall\ r < r_0 \ . \tag{1}$$

This essentially corresponds to the condition on μ imposed in [3, 4]. Note that in the case $d = 1$, condition (1) is equivalent to absolutely α-decay.

Definition A measure μ is said to be *absolutely α–friendly* if it is doubling and absolutely α-decaying.

We prove the following analogue of Lemma 1.

Theorem 1 *Let K be a compact subset of $\mathbf{R}^d$ equipped with an absolutely α–friendly measure μ. Then*

$$\mu(W_K(\psi)) = 0 \qquad \text{if} \qquad \sum_{r=1}^{\infty} r^{\alpha\frac{d+1}{d}-1}\psi(r)^{\alpha} \ < \ \infty \quad .$$

Remark. In the case when $d = 1$, it is possible to remove the condition that μ is doubling from the definition of absolutely α–friendly; i.e. all that is required is that μ is absolutely α–decaying – see §4.1. With this in mind, the above theorem restricted to $d = 1$ is identical to that established in [3]. Thus, Theorem 1 constitutes the natural higher dimensional analogue of [3]. The above theorem should also be compared with Theorem 9 of [4].

Note that in the case that $\psi_\tau : r \to r^{-\tau}$ and $\tau > \frac{d+1}{d}$,

$$\sum_{r=1}^{\infty} r^{\alpha\frac{d+1}{d}-1}\psi_\tau(r)^{\alpha} \ := \ \sum_{r=1}^{\infty} r^{-1-\alpha(\tau-\frac{d+1}{d})} \ < \ \infty$$

and so Theorem 1 implies that $\mu(W_K(\tau)) = 0$ whenever μ is absolutely α–friendly. More to the point, consider the function $\psi : r \to r^{-\frac{d+1}{d}}(\log r)^{-\beta}$ where $\beta > 1/\alpha$. Then

$$\sum_{r=1}^{\infty} r^{\alpha\frac{d+1}{d}-1}\psi(r)^{\alpha} \ := \ \sum_{r=1}^{\infty} r^{-1}(\log r)^{-\alpha\beta} \ < \ \infty \ ,$$

and Theorem 1 implies that $\mu(W_K(\psi)) = 0$ whenever μ is absolutely α–friendly.

It will be evident from the proof that all that is actually required in establishing the theorem is that the doubling and absolutely α-decaying inequalities are satisfied at μ–almost every point in K. Also the relevance of hyperplanes in the definition of absolutely α-decaying will become crystal clear from our proof of the theorem. Essentially, on the real line $\mathbf{R}$ an interval I_n of length $1/4n^2$ can contain at most one rational p/q with $n \leq q < 2n$. This follows from the trivial observation that if $n \leq q, q' < 2n$ then $|p/q - p'/q'| \geq 1/qq' > 1/4n^2$; i.e. the distance between two such rationals is strictly greater than the length of I_n. The higher dimension analogue of this is the following. Let B_n be a ball in $\mathbf{R}^d$ of radius $c/n^{(d+1)/d}$ where c is a sufficiently small constant dependent only on d. Then any rational points $\mathbf{p}/q$ lying within B_n

with $n \leq q < 2n$ must lie on a single $(d-1)$–dimensional hyperplane L. This is the key observation on which the proofs of Theorems 1 and 2 hinge.

We now turn our attention to determining an upper bound for $\dim W_K(\psi)$ – the Hausdorff dimension of $W_K(\psi)$. For $s \geq 0$, let $\mathcal{H}^s$ denote the s–dimensional Hausdorff measure - see §2.

Theorem 2 *Let K be a compact subset of $\mathbf{R}^d$ equipped with an absolutely α–friendly measure μ. Furthermore, suppose there exist positive constants a, b, δ and r_o such that*

$$a\,r^{\delta} \;\leq\; \mu(B(x,r)) \;\leq\; b\,r^{\delta} \qquad \forall\, x \in K \quad \forall\, r < r_0\ . \tag{2}$$

Then, for $s \leq \delta$

$$\mathcal{H}^s(W_K(\psi)) = 0 \qquad \text{if} \qquad \sum_{r=1}^{\infty} r^{\alpha\frac{d+1}{d}-1}\psi(r)^{\alpha+s-\delta} \;<\; \infty \quad .$$

Remark 1: Note that (2) imposed on μ trivially implies that μ is doubling. Furthermore, if $\delta > d-1$ then (2) together with a straightforward geometric argument implies that μ is absolutely α–decaying with $\alpha := \delta - (d-1) > 0$. Thus, if $\delta > d-1$ the hypothesis that μ is absolutely α–friendly is in fact redundant from the statement of Theorem 2.

Remark 2: If K supports a measure μ satisfying (2), then $\dim K = \delta$ and moreover that $0 < \mathcal{H}^{\delta}(K) < \infty$ – see [1] for the details. Now, since $W_K(\psi)$ is a subset of K we have that $\dim W_K(\psi) \leq \delta$ and so $\mathcal{H}^s(W_K(\psi)) = 0$ for any $s > \delta$. Thus, the condition $s \leq \delta$ in the statement of the theorem can be assumed without any loss of generality.

Given a real, positive decreasing function ψ, the *lower order* λ_ψ of $1/\psi$ is defined by

$$\lambda_\psi \;:=\; \liminf_{r\to\infty} \frac{-\log\psi(r)}{\log r}\ ,$$

and indicates the growth of the function $1/\psi$ 'near' infinity. Note that λ_ψ is non-negative since ψ is a decreasing function. A simple consequence of Theorem 2 is the following statement.

Corollary 1 *Let K be a compact subset of $\mathbf{R}^d$ equipped with an absolutely α–friendly measure μ satisfying (2). Then, for $\lambda_\psi \geq (d+1)/d$*

$$\dim W_K(\psi) \;\leq\; \delta \,-\, \alpha\left(1 - \tfrac{d+1}{\lambda_\psi\, d}\right) \quad .$$

As a special case we obtain the following statement.

Corollary 2 *Let K be a compact subset of $\mathbf{R}^d$ equipped with an absolutely α–friendly measure μ satisfying (2). Then, for $\tau \geq (d+1)/d$*

$$\dim W_K(\tau) \;\leq\; \delta \,-\, \alpha\left(1 - \tfrac{d+1}{\tau\, d}\right) \quad .$$

Note that for $\tau > (d+1)/d$ we have that $\dim W_K(\tau) < \delta$. Since μ is comparable to $\mathcal{H}^\delta$ restricted to K, it follows that $\mu(W_K(\tau)) = 0$.

A general remark: For $d \geq 2$, it is highly unlikely that either Theorem 1 or Theorem 2 are ever sharp. For instance, take the case that $K := [0,1]^d$ and μ is d–dimensional Lebesgue measure. It is easily verified that μ is absolutely α–friendly with $\alpha = 1$. Thus, Theorem 1 implies that $|W_K(\psi)|_d = 0$ whenever

$$\sum_{r=1}^{\infty} r^{\frac{d+1}{d}-1}\,\psi(r) \;<\; \infty \quad .$$

So when $d = 1$ this coincides with the Lemma 1. However, for $d \geq 2$ the above statement is weaker than that of the lemma. In view of Khintchines theorem one knows that the lemma is sharp; that is to say that if the sum in the lemma diverges then not only is $|W_K(\psi)|_d > 0$ but it is of full measure. It is probable that the theorems of this paper are sharp in the case $d = 1$.

1.2 The main example

The following statement which combines Theorems 2.2 and 8.1 of [2], shows that a large class of fractal measures are absolutely α–friendly and satisfy (2).

Theorem KLW *Let $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$ be an irreducible family of contracting self similarity maps of $\mathbf{R}^d$ satisfying the open set condition and let μ be the restriction of $\mathcal{H}^\delta$ to its attractor K where $\delta := \dim K$. Then μ is absolutely α–friendly and satisfies (2).*

Thus for the natural measures associated with self similar sets satisfying the open set condition, Theorems 1 and 2 are applicable. The simpliest examples of such sets include regular Cantor sets, the Sierpiński gasket and the von Kock curve. All the terminology except for 'irreducible' is pretty much standard – see for example [1, Chp.9]. The notion of irreducible introduced in [2, §2] avoids the natural obstruction that there is a finite collection of proper affine subspaces of $\mathbf{R}^d$ which is invariant under $\{\mathbf{S}_1, \ldots, \mathbf{S}_k\}$.

2 Hausdorff measures and dimension

In this short section we define Hausdorff measure and dimension for completeness and in order to establish some notation. For $\rho > 0$, a countable collection $\{B_i\}$ of Euclidean balls in $\mathbf{R}^d$ of radii $r_i \leq \rho$ for each i such that $X \subset \bigcup_i B_i$ is called a ρ-cover for X. Let s be a non-negative number and define

$$\mathcal{H}^s_\rho(X) \;=\; \inf\left\{\sum_i r_i^s \;\; : \{B_i\} \text{ is a } \rho\text{−cover of } X\right\} \;,$$

where the infimum is taken over all possible ρ-covers of X. The *s-dimensional Hausdorff measure* $\mathcal{H}^s(X)$ of X is defined by

$$\mathcal{H}^s(X) = \lim_{\rho\to 0} \mathcal{H}^s_\rho(X) = \sup_{\rho>0} \mathcal{H}^s_\rho(X)$$

and the *Hausdorff dimension* $\dim X$ of X by

$$\dim X = \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s(X) = \infty\} \ .$$

Further details and alternative definitions of Hausdorff measure and dimension can be found in [1].

3 A covering lemma

The following rather simple covering result will be used at various stages during the proof of our theorems.

Covering Lemma *Let (Ω, d) be a metric space and $\mathcal{B}$ be a finite collection of balls with common radius $r > 0$. Then there exists a disjoint sub-collection $\{B_i\}$ such that*

$$\bigcup_{B\in\mathcal{B}} B \ \subset \ \bigcup_i 3B_i \ .$$

Proof : Let S denote the set of centres of the balls in $\mathcal{B}$. Choose $c_1 \in S$ and for $k \geq 1$,

$$c_{k+1} \ \in \ S \setminus \bigcup_{i=1}^{k} B(c_i, 2r)$$

as long as $S \setminus \bigcup_{i=1}^{k} B(c_i, 2r) \neq \emptyset$. Since $\#S$ is finite, there exists $k_1 \leq \#S$ such that

$$S \ \subset \ \bigcup_{i=1}^{k_1} B(c_i, 2r) \ .$$

By construction, any ball $B(c, r)$ in the original collection $\mathcal{B}$ is contained in some ball $B(c_i, 3r)$ and since $d(c_i, c_j) > 2r$ the chosen balls $B(c_i, r)$ are clearly disjoint.

♠

4 Proof of Theorem 1

Step 1: *Preliminaries.* We are assuming that $\sum r^{\alpha\frac{d+1}{d}-1}\psi(r)^\alpha$ converges and since ψ is monotonic, it follows that

$$\sum_{n=1}^{\infty} \left(2^{n\frac{d+1}{d}}\psi(2^n)\right)^\alpha < \infty \ . \tag{3}$$

Next notice, that without loss of generality we can assume that

$$\psi(2^n) \ < \ c\, 2^{-n\frac{d+1}{d}} \tag{4}$$

for any $c > 0$ and n sufficiently large. This is easy to see. Suppose on the contrary that there exists a sequence $\{n_i\}$ such that $\psi(2^{n_i}) \geq c\, 2^{-n_i\frac{d+1}{d}}$. Then

$$\sum_{n=1}^{\infty} \left(2^{n\frac{d+1}{d}}\psi(2^n)\right)^\alpha \ \geq \ \sum_{i=1}^{\infty} \left(2^{n_i\frac{d+1}{d}}\psi(2^{n_i})\right)^\alpha \ \geq \ c^\alpha \sum_{n=1}^{\infty} 1 \ = \ \infty \ \ ,$$

and this contradicts (3).

Step 2: *The balls* D_n. For $n \in \mathbf{N}$, let D_n denote a generic ball with centre in K and of radius

$$r_n := \tfrac{1}{6}\left(\tfrac{1}{\kappa\, d!}\right)^{\frac{1}{d}} 2^{-\frac{d+1}{d}(n+1)} \quad .$$

Here $\kappa := \kappa(d)$ is the volume (d–dimensional Lebesgue measure) of a ball of radius one in $\mathbf{R}^d$. In view of the covering lemma and the fact that K is compact, there exists a finite, disjoint collection $\mathcal{D}_n$ of balls D_n with centers in K such that

$$\bigcup_{\mathcal{D}_n} 3\mathrm{D}_n \ \supset \ K \quad .$$

Note that since μ is doubling, we have that

$$\sum_{\mathcal{D}_n} \mu(3\mathrm{D}_n) \ \leq \ \sum_{\mathcal{D}_n} \mu(4\mathrm{D}_n) \ \leq \ D^2 \sum_{\mathcal{D}_n} \mu(\mathrm{D}_n) \ = \ D^2 \mu\left(\overset{\circ}{\bigcup_{\mathcal{D}_n}} \mathrm{D}_n\right) \ \leq \ D^2 \mu(K) \quad . \tag{5}$$

Next, consider a ball $3\mathrm{D}_n$ where $\mathrm{D}_n \in \mathcal{D}_n$. Suppose there is a rational point $\mathbf{p}/q := (p^{(1)}/q, \ldots, p^{(d)}/q)$ such that

$$B(\mathbf{p}/q, \sqrt{d}\,\psi(q)) \cap \ 3\mathrm{D}_n \neq \emptyset \qquad \text{and} \qquad 2^n \leq q < 2^{n+1} \ . \tag{6}$$

By (4) and using the fact that ψ is decreasing, it follows that for n sufficiently large $\mathbf{p}/q \in 6\mathrm{D}_n$. Now assume that there are $d+1$ or more such rational points satisfying (6). Take any $d+1$ such rationals; $\mathbf{p}_0/q_0, \mathbf{p}_1/q_1, \ldots, \mathbf{p}_d/q_d$. In view of the denominator constraint, the rational points are necessarily distinct. Suppose for the moment that they do not lie on a (d–1)–dimensional hyperplane and form the d–dimensional simplex Δ sub-tended by them; i.e. an interval when $d = 1$, a triangle when $d = 2$ and a tetrahedron when $d = 3$. The volume (d–dimensional Lebesgue measure) of the simplex Δ times d factorial is equal to the absolute value of the determinant

$$\det \ := \ \begin{vmatrix} 1 & p_0^{(1)}/q_0 & \cdots & p_0^{(d)}/q_0 \\ 1 & p_1^{(1)}/q_1 & \cdots & p_1^{(d)}/q_1 \\ \vdots & & & \\ 1 & p_d^{(1)}/q_d & \cdots & p_d^{(d)}/q_d \end{vmatrix} \ .$$

Then, by (6)

$$d! \ \times \ |\Delta|_d \ \geq \ \frac{1}{q_0 q_1 \cdots q_d} > 2^{-(d+1)(n+1)} \ .$$

Trivially,

$$|\, 6\mathrm{D}_n \,|_d \ = \ \kappa\,(6\,r_n)^d \ := \ \tfrac{1}{d!}\ 2^{-(d+1)(n+1)} \quad .$$

Thus $|\Delta|_d > |6\mathrm{D}_n|_d$ and this is impossible since $\Delta \subset 6\mathrm{D}_n$. The upshot of this is that the d–dimensional simplex Δ can not exist and so if there are $d+1$ or more rational points satisfying (6) then they must lie on a $(d-1)$–dimensional hyperplane $\mathrm{L} := \mathrm{L}(\mathrm{D}_n)$ passing through the ball $3\mathrm{D}_n$. In the event that there are no more than d

rational points satisfying (6), the existence of such a hyperplane is obvious – of course it is not unique if the number of rational points is less than d. Thus, associated with each ball $\mathrm{D}_n \in \mathcal{D}_n$ there is a $(d-1)$–dimensional hyperplane $\mathrm{L} := \mathrm{L}(\mathrm{D}_n)$ containing all rational points satisfying (6). Note that in the case $d = 1$, any hyperplane L is simply a point.

Step 3: *The finale.* For $n \in \mathbf{N}$, let

$$A_n \ := \bigcup_{2^n \leq q < 2^{n+1}} \ \bigcup_{\mathbf{p} \in \mathbf{Z}^d} B\left(\mathbf{p}/q, \sqrt{d}\,\psi(q)\right) \ .$$

By definition, $W_K(\psi) \subset \limsup_{n\to\infty} A_n \cap K$. It follows via Step 2 and the fact that ψ is decreasing, that for n sufficiently large

$$\begin{aligned}
\mu(A_n) \ := \ \mu(A_n \cap K) \ &= \ \mu\left(A_n \cap \bigcup_{\mathcal{D}_n} 3\mathrm{D}_n\right) \\
&\leq \ \sum_{\mathcal{D}_n} \mu\left(3\mathrm{D}_n \cap \mathrm{L}^{(\epsilon)}\right) && \epsilon := \sqrt{d}\,\psi(2^n) \quad \mathrm{L} := \mathrm{L}(\mathrm{D}_n) \\
&\ll \ \left(2^{n\frac{d+1}{d}}\psi(2^n)\right)^{\alpha} \ \sum_{\mathcal{D}_n} \mu(3\mathrm{D}_n) && \mu \ \text{ is absolutely } \alpha\text{–decaying} \\
&\ll \ \left(2^{n\frac{d+1}{d}}\psi(2^n)\right)^{\alpha} \ \mu(K) && \text{by (5).}
\end{aligned}$$

Hence, by (3)

$$\sum \mu(A_n \cap K) \ = \ \sum \mu(A_n) \ \ll \ \sum \left(2^{n\frac{d+1}{d}}\psi(2^n)\right)^{\alpha} \ < \ \infty$$

and the Borel-Cantelli lemma implies that $\mu(\limsup_{n\to\infty} A_n) = 0$. Thus, $\mu(W_K(\psi))$ is zero as required.

♠

4.1 The case when $d = 1$ revisited

Clearly the above proof contains the case when $d = 1$. However, it is possible to give a more direct proof of a stronger statement which does not assume that μ is doubling – see the remark straight after the statement of Theorem 1. Although the proof below is basically the same as that in [3], we have decided to include a sketch in order to bring out the true nature of the 'simplex/determinate' argument and the role of hyperplanes when $d \geq 2$ in the proof above. In the $d = 1$ case, the 'simplex/determinate' argument reduces to the following. Consider rationals p/q with $2^n \leq q < 2^{n+1}$. For any two such rationals, notice that

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| \ \geq \ \frac{1}{qq'} \ > \ 2^{-2(n+1)} \ := \ 2\,r_n \ .$$

Thus, any interval of length $2r_n$ can contain at most one rational. In particular, the intervals $B(p/q, r_n)$ are disjoint.

Now let A_n be as in Step 3. By definition, $W_K(\psi) = \limsup_{n\to\infty} A_n \cap K$. Then, in view of (4) and the fact that ψ is decreasing we have that for n sufficiently large

$$\begin{aligned} \mu(A_n) &\leq \sum_{2^n \leq q < 2^{n+1}} \sum_{p \in \mathbf{Z}} \mu(B(p/q, \epsilon\, r_n)) && \epsilon\, r_n := \psi(2^n) \\ &\ll \left(2^{2n}\psi(2^n)\right)^{\alpha} \sum_{2^n \leq q < 2^{n+1}} \sum_{p \in \mathbf{Z}} \mu(B(p/q, r_n)) && \text{by (1)} \\ &\leq \left(2^{2n}\psi(2^n)\right)^{\alpha} \ \mu(K) && \text{by disjointness.} \end{aligned}$$

The Borel-Cantelli lemma implies the desired statement. ♠

5 Proof of Theorem 2

To a certain extent the proof of Theorem 2 is similar to that of Theorem 1.

Step 1: *Preliminaries.* Without loss of generality we can assume that $\psi(r) \to 0$ as $r \to \infty$. Suppose that this was not the case. Then $W_K(\psi) = K$ by Dirichlet's theorem and so $\mathcal{H}^s(W_K(\psi)) > 0$ for any $s \leq \delta$ - see Remark 2 straight after the statement of Theorem 2.

Without loss of generality we can assume that $s > \delta - \alpha$. If this where not the case then the sum in the statement of the theorem cannot possibly converge.

Since ψ is monotonic, the convergence of the sum in the statement of the theorem is equivalent to

$$\sum_{n=1}^{\infty} 2^{n\alpha\frac{d+1}{d}} \psi(2^n)^{\alpha+s-\delta} < \infty \ . \tag{7}$$

Finally, notice that since $s > \delta - \alpha$, we can assume (4) without any loss of generality. Otherwise, (7) would be contradicted.

Step 2: *A good ρ–cover for $W_K(\psi)$.* For $n \in \mathbf{N}$, let $\mathcal{D}_n$ be the disjoint collection of balls D_n as defined in Step 2 of §4. Since the collection is disjoint and μ satisfies (2), we have that for n sufficiently large

$$\#\mathcal{D}_n \times 2^{-\frac{d+1}{d}(n+1)\delta} \asymp \sum_{\mathcal{D}_n} \mu(\mathrm{D}_n) \ = \ \mu\Bigg(\bigcup_{\mathcal{D}_n}^{\circ} \mathrm{D}_n\Bigg) \ \leq \ \mu(K) \ \ .$$

Thus, for n sufficiently large

$$\#\mathcal{D}_n \ \ll \ 2^{\frac{d+1}{d}(n+1)\delta} \ \ . \tag{8}$$

Now put $\epsilon := \sqrt{d}\,\psi(2^n)$ and fix some ball $\mathrm{D}_n \in \mathcal{D}_n$. Let $\mathrm{L} := \mathrm{L}(\mathrm{D}_n)$ be the $(d-1)$–dimensional hyperplane associated with D_n – see Step 2 of §4. In view of the covering lemma, there exists a finite disjoint collection $\mathcal{C}(\mathrm{D}_n)$ of balls $B_n(\psi)$ with centers in K and common radius $\psi(2^n)$ such that

$$\bigcup_{\mathcal{C}(\mathrm{D}_n)} 3B_n(\psi) \ \supset \ 3\,\mathrm{D}_n \cap \mathrm{L}^{(\epsilon)} \cap K \tag{9}$$

and

$$\bigcup_{\mathcal{C}(\mathrm{D}_n)}^{\circ} B_n(\psi) \ \subset \ 6\,\mathrm{D}_n \cap \mathrm{L}^{(2\epsilon)}$$

The latter together with the fact that μ is absolutely α–decaying implies that

$$\begin{aligned}
\#\mathcal{C}(\mathrm{D}_n) \times \psi(2^n)^{\delta} \ \ &\asymp \ \ \sum_{\mathcal{C}(\mathrm{D}_n)} \mu(B_n(\psi)) \ = \ \mu\Bigg(\bigcup_{\mathcal{C}(\mathrm{D}_n)}^{\circ} B_n(\psi)\Bigg) \\
&\leq \ \ \mu(6\,\mathrm{D}_n \cap \mathrm{L}^{(2\epsilon)}) \ \ll \ \Big(2^{\frac{d+1}{d}(n+1)}\psi(2^n)\Big)^{\alpha} \ 2^{-\frac{d+1}{d}(n+1)\delta} \ .
\end{aligned}$$

Thus, for n sufficiently large

$$\#\mathcal{C}(\mathrm{D}_n) \ \ll \ \Big(2^{\frac{d+1}{d}(n+1)}\psi(2^n)\Big)^{\alpha-\delta} \quad . \tag{10}$$

Now with A_n defined as in Step 3 of §4, it follows via (9) that

$$\begin{aligned}
A_n \cap K \quad &= \quad \bigcup_{\mathrm{D}_n \in \mathcal{D}_n} 3\mathrm{D}_n \cap A_n \cap K \quad \subset \quad \bigcup_{\mathrm{D}_n \in \mathcal{D}_n} 3\mathrm{D}_n \cap \mathrm{L}^{(\epsilon)} \cap K \\
&\subset \quad \bigcup_{\mathrm{D}_n \in \mathcal{D}_n} \ \ \bigcup_{B_n(\psi) \in \mathcal{C}(\mathrm{D}_n)} 3\,B_n(\psi) \quad .
\end{aligned}$$

In particular, for each $k \in \mathbf{N}$ the collection

$$\{3\,B_n(\psi) \ : \ B_n(\psi) \in \mathcal{C}(\mathrm{D}_n), \mathrm{D}_n \in \mathcal{D}_n \text{ and } n = k, k+1, \cdots\} \quad ,$$

is a ρ–cover for $W_K(\psi)$ with $\rho = \rho(k) := 3\,\psi(2^k)$.

Step 3: *The finale.* Let $\rho = \rho(k) := 3\,\psi(2^k)$. Step 2 together with the definition of s–dimensional Hausdorff measure implies that

$$\mathcal{H}^s_{\rho}(W_K(\psi)) \ \leq \ \sum_{n=k}^{\infty} \ \ \sum_{\mathrm{D}_n \in \mathcal{D}_n} \ \sum_{B_n(\psi) \in \mathcal{C}(\mathrm{D}_n)} (3\,\psi(2^n))^s \quad .$$

Thus, in view of (8) and (10), it follows that for k sufficiently large

$$\mathcal{H}^s_{\rho}(W_K(\psi)) \ \ll \ \sum_{n=k}^{\infty} \ 2^{n\alpha\frac{d+1}{d}} \ \psi(2^n)^{\alpha+s-\delta} \ .$$

This together with (7) implies that

$$\mathcal{H}^s_{\rho}(W_K(\psi)) \ \to \ 0 \qquad \text{as} \qquad \rho \to 0 \quad (k \to \infty) \quad ,$$

and so $\mathcal{H}^s(W_K(\psi)) = 0$ as required.

♠

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