0590313

UNIVERSITY OF YORK

BA, BSc and MMath Examinations 2006
MATHEMATICS
Metric Spaces

Time Allowed: $1\frac{1}{2}$ hours.

Answer three questions.
Please write your answers in ink; pencil is acceptable for graphs and diagrams. Do not use red ink.
Standard calculators will be provided.
The marking scheme shown on each question is indicative only.

1 (of 4). Suppose X is a non-empty set and that $d : X \times X \to \mathbb{R}$. What properties must d satisfy, in order for it to be a *metric* on X? If $x_0 \in X$ and $r > 0$, define the *open ball* $B(x_0, r)$. **[8]**

Show that the function defined by

$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric (the *discrete metric*) on X. **[5]**

If d is a metric on X and $A \subseteq X$, define the interior $\operatorname{int}(A)$, the exterior $\operatorname{ext}(A)$ and the boundary ∂A in the metric space (X, d). Show that A and its complement A^c have the same boundary. **[8]**

Define the terms *open* and *closed*, as applied to the set A in (X, d). Show that, under the discrete metric, all sets are both open and closed, and have empty boundary. **[9]**

2 (of 4). Suppose (X, d) is a metric space, $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $x \in X$. What does it mean to say that x_n *converges* to x as $n \to \infty$? Show that if, as $n \to \infty$, $x_n \to x$ and $x_n \to y$ then $x = y$. **[10]**

Suppose a sequence $(f_n)_{n\in\mathbb{N}}$ converges to a limit f in the metric space $C[a, b]$ (continuous, real-valued functions on the interval $[a, b]$), with the uniform metric

$$d_\infty(f, g) = \sup_{t\in[a,b]} |f(t) - g(t)| = \max_{t\in[a,b]} |f(t) - g(t)|.$$

Show that f_n also converges pointwise to f; that is, that for each $t \in [a, b]$ we have $f_n(t) \to f(t)$ in $\mathbb{R}$. **[4]**

Consider the following sequences in $C[0, 1]$:

$$g_n(t) = \frac{\sin(nt)}{n};$$
$$h_n(t) = \frac{nt}{1 + (nt)^2}.$$

Indicate briefly why both of these sequences converge pointwise to 0, and determine whether or not each one converges to 0 in the uniform metric. **[16]**

3 (of 4). Suppose (X, d) is a metric space and $f : X \to X$ is a function. What does it mean to say that a real number $k \geq 0$ is a *Lipschitz constant* for f? Show that if f is a Lipschitz function (that is, one for which a Lipschitz constant exists) then f is continuous. **[10]**

If X is the interval $[a, b]$ with the standard metric, f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$, use the Mean Value Theorem to show that the smallest Lipschitz constant for f is

$$\max_{t \in [a,b]} |f'(t)|.$$ **[5]**

State without proof the *Contraction Mapping Theorem* **[6]**
and use it to show that there is exactly one solution in the interval $[0, 1]$ to the equation $t = 1 - \cos(1 - t)$. **[6]**

Starting from $t_0 = 0.25$, use a calculator to find the next three steps in the iteration converging to this solution. **[3]**

4 (of 4). What is meant by a *subsequence* of a sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d)? Show that if $x_n \to x \in X$ as $n \to \infty$ then any subsequence of $(x_n)_{n \in \mathbb{N}}$ also converges to x. **[9]**

In $\mathbb{R}$ with the standard metric, give examples of:

(a) a sequence with no convergent subsequence;

(b) a divergent sequence with a convergent subsequence

(and indicate why your examples have these properties). **[8]**

What does it mean to say that a metric space (X, d) is *sequentially compact?* Show that the following spaces are *not* sequentially compact:

(a) the open interval $(0, 1) \subseteq \mathbb{R}$ with the standard metric;

(b) the set

$$\{f : [0, 1] \to \mathbb{R} : \ f \text{ is continuous and } |f(t)| \leq 1 \text{ for all } t \in [0, 1]\}$$

with the uniform metric d_∞ (defined in Question 2). **[13]**

 End of examination.

1. The three properties of a metric are:

M1 $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$ [2 Marks]

M2 $d(x,y) = d(y,x)$ for all $x, y \in X$ [2 Marks]

M3 $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. [2 Marks]

The ball is defined by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$ [2 Marks]

M1 and M2 are clear from the definition. [1 Mark]
M3 is clear in the case $x = y$, since then the LHS is zero and the RHS is non-negative. In the case that $x = y$, so the LHS is 1, we must have either $x \neq z$ or $z \neq y$, so the RHS is either 1 or 2, showing that M3 is true in this case too. [4 Marks]

The interior of A is the set

$$\text{int}(A) = \{x \in X : B(x,r) \subseteq A \text{ for some } r > 0\}$$ [2 Marks]

The exterior of A is the set

$$\text{ext}(A) = \{x \in X : B(x,r) \subseteq A^c \text{ for some } r > 0\}$$ [2 Marks]

The boundary of A is the set

$$\partial A = \{x \in X : \text{ for all } r > 0,\ B(x,r) \text{ intersects both } A \text{ and } A^c\}$$ [2 Marks]

A and A^c have the same boundary because the definition of ∂A is symmetric in A and A^c. [2 Marks]

A is open if it contains none of its boundary ($A \cap \partial A = \emptyset$), [2 Marks]
and closed if it contains all of its boundary ($\partial A \subseteq A$). [2 Marks]

In the discrete metric d_0, we have

$$B(x_0, 1) = \{x \in X : d_0(x_0, x) < 1\} = \{x_0\}$$

If $x \in A$ then $B(x,1) = \{x\} \subseteq A$ and if $x \in A^c$ then $B(x,1) = \{x\} \subseteq A$. It follows that $\text{int}(A) = A$, that $\text{ext}(A) = A^c$ and that $\partial A = \emptyset$. [3 Marks]
Since the boundary is empty, we have both $A \cap \partial A = \emptyset$ and $\partial A \subseteq A$, so A is both open and closed. [2 Marks]

Remarks. *All familiar stuff. The definition of open and closed is unusual, of course, but this is the way I did it in lectures. If someone uses the more conventional approach, then of course they'll get full credit.*

$\boxed{\text{Total: 30 Marks}}$

2. x_n converges to x as $n \to \infty$ if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, x) < \varepsilon$. $\boxed{\text{3 Marks}}$

If both $x_n \to x$ and $x_n \to y$ then for any $\varepsilon > 0$ there exist $M, N \in \mathbb{N}$ such that if $n > M$ then $d(x_n, x) < \varepsilon/2$ and if $n > N$ then $d(x_n, y) < \varepsilon/2$. If $n > \max(M, N)$ then

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since this is true for all $\varepsilon > 0$, $d(x, y) \leq 0$. But $d(x, y) \geq 0$ by definition, so $d(x, y) = 0$ and hence $x = y$. $\boxed{\text{7 Marks}}$

For any given $t \in [a, b]$, we have

$$|f_n(t) - f(t)| \leq \sup_{t \in [a,b]} |f_n(t) - f(t)| = d_\infty(f_n, f)$$

It follows that if $f_n \to f$ in the uniform metric, then $f_n(t) \to f(t)$ for each t. $\boxed{\text{4 Marks}}$

Since $|\sin(nt)|/n \leq 1/n \to 0$, we have $g_n(t) \to 0$ for any t; also

$$h_n(t) = \frac{1/(nt)}{1/(nt)^2 + 1} \to 0$$

for any t. $\boxed{\text{4 Marks}}$

Calculate:

$$\begin{aligned} d_\infty(g_n, 0) &= \sup_{t \in [0,1]} |f_n(t) - 0| \\ &= \sup_{t \in [0,1]} \frac{|\sin(nt)|}{n} \\ &\leq \frac{1}{n} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which shows that $g_n \to 0$ in the metric d_∞. $\boxed{\text{5 Marks}}$

Notice that $h_n(t) = h_1(nt)$, and choose $t_0 \in (0,1)$, so $h_1(t_0) > 0$. Since $h_n(t_0/n) = h_1(t_0)$, we have

$$\begin{aligned} d_\infty(h_n, 0) &= \sup_{t\in[0,1]} |h_n(t) - 0| \\ &= \sup_{t\in[0,1]} h_n(t) \\ &\geq h_n(t_0/n) \\ &= h_1(t_0) \end{aligned}$$

This shows that $d_\infty(h_n, 0) \nrightarrow 0$ as $n \to \infty$, so h_n does not converge to 0 in the metric d_∞. (7 Marks)

Remarks. *All familiar ideas; but of course these particular examples are unseen.*

(Total: 30 Marks)

3. A Lipschitz constant is a constant $k \geq 0$ with the property that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$. (3 Marks)

If f has a Lipschitz constant of 0, then it is evidently constant, and hence continuous. If f has a Lipschitz constant $k > 0$, then for any $\varepsilon > 0$ we let $\delta = \varepsilon/k$, so if $d(x, y) < \delta$ then $d(f(x), f(y)) \leq kd(x, y) < k\delta = \varepsilon$, showing that f is uniformly continuous on X.

Let

$$k = \max_{t\in[a,b]} |f'(t)|.$$

If $s, t \in [a, b]$ and $s \neq t$ then, by the mean value theorem,

$$|f(s) - f(t)| = |f'(\tau)||s - t| \leq k|s - t|$$

where τ is some number between s and t. If $s = t$ then $|f(s) - f(t)| = 0 = k|s - t|$, so k is indeed a Lipschitz constant for f. (7 Marks)

To show that k is the minimal Lipschitz constant, choose some t_0 such that $|f'(t_0)| = k$. This means that

$$\frac{|f(t) - f(t_0)|}{|t - t_0|} \to k$$

as $t \to t_0$. In particular, for any $\varepsilon > 0$ we can choose t such that

$$\frac{|f(t) - f(t_0)|}{|t - t_0|} > k - \varepsilon$$

This shows that $k - \varepsilon$ is not a Lipschitz constant for f, so k is the minimal Lipschitz constant. (5 Marks)

The contraction mapping theorem states that if (X, d) is a complete metric space and $F : X \to X$ has a Lipschitz constant $k < 1$ then F has a unique fixed point in X. Moreover, for any $x_0 \in X$ the iterates $(F^{(n)}(x_0))_{n\in\mathbb{N}}$ converge to the fixed point as $n \to \infty$. $\boxed{\text{6 Marks}}$

Let $f(t) = 1-\cos(1-t)$. If $t \in [0, 1]$ then $\cos(t) \in [0, 1]$, so $1-\cos(1-t) \in [0, 1]$ also. This shows that f leaves invariant the complete metric space $[0, 1]$.

We have $f'(t) = \sin(1 - t)$. On the interval $[0, 1]$, $\sin$ is non-negative and increasing, so

$$\max_{t\in[0,1]} |\sin(1 - t)| = \sin(1)$$

This shows that f has Lipschitz constant $\sin(1) < 1$, and hence by the contraction mapping theorem a unique fixed in point in $[0, 1]$. $\boxed{\text{6 Marks}}$

Starting from $t_0 = 0.25$, we have $t_1 = f(t_0) \approx 0.268$, $t_2 = f(t_1) \approx 0.256$ and $t_3 = f(t_2) \approx 0.264$. $\boxed{\text{3 Marks}}$

Remarks. *Again, all familiar but an unseen example of a contraction.*

$\boxed{\text{Total: 30 Marks}}$

4. If $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, then the sequence $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. $\boxed{\text{3 Marks}}$
Suppose $x_n \to x$ as $n \to \infty$ and we have a subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, x) < \varepsilon$. Since $(n_k)_{k\in\mathbb{N}}$ is strictly increasing in $\mathbb{N}$, we have $n_k \geq k$ for all k. If follows that if $k > N$ then $n_k > N$, so $d(x_{n_k}, x) < \varepsilon$, showing that $x_{n_k} \to x$ as $k \to \infty$ $\boxed{\text{6 Marks}}$

(a) Let $x_n = n$. Then any subsequence (x_{n_k}) has $x_{n_k} > k$, so $x_{n_k} \to \infty$ as $k \to \infty$. This shows that there is no convergent subsequence. $\boxed{\text{4 Marks}}$

(b) Let $x_n = (-1)^n$. Then $x_{2n} = 1$ and $x_{2n+1} = -1$. These subsequences are trivially convergent, and the fact that they have different limits shows that $(x_n)_{n\in\mathbb{N}}$ itself is divergent. $\boxed{\text{4 Marks}}$

A metric space (X, d) is called sequentially compact if every sequence in X has a convergenbt subsequence. $\boxed{\text{3 Marks}}$

(a) Let $x_n = 1/n$. Since $x_n \to 0$ as $n \to \infty$ in $\mathbb{R}$, every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to 0 in $\mathbb{R}$. If a subsequence were to converge in $(0, 1)$, then it would also converge in $\mathbb{R}$, and would thus have two distinct limits in $\mathbb{R}$, which is impossible; it follows that no subsequence can converge in $(0, 1)$. $\boxed{\text{5 Marks}}$

(b) Let $f_n(t)$ be a continuous function on $[0, 1]$ which is zero outside the interval $I_n = [1/(n+1), 1/n]$ and has a maximum value of 1 over I_n. Then, if $m \neq n$ $|f_m - f_n| = |f_n|$ on I_n, so $d_\infty(f_m, f_n) \geq 1$. This shows that $(f_n)_{n\in\mathbb{N}}$ has no Cauchy subsequence, and hence no convergent subsequence. $\boxed{\text{5 Marks}}$

Remarks. *Nothing at all new here, but the technical details are tricky.*

$\boxed{\text{Total: 30 Marks}}$