0500010

UNIVERSITY OF YORK

BA, BSc and MMath Examinations 2006
MATHEMATICS
Analysis I

Time Allowed: $1\frac{1}{2}$ hours.

Answer all questions in Part A (questions 1–5) and two questions from Part B (questions 6–8).
Part A carries 40 marks; questions in part B carry 25 marks each.
Standard calculators will be provided.
The marking scheme shown on each question is indicative only.

Part A. Answer all questions in this part of the paper

1 (of 8). Suppose S is a non-empty subset of $\mathbb{R}$. Define the terms *supremum* and *infimum*, as applied to S. Under what circumstances do the supremum and infimum exist?

For each of the following sets, write down the supremum and infimum, if they exist. *You are not required to justify your answers*.

$$S_1 = \{x \in \mathbb{R} : (x-1)^2 < 3\}; \qquad S_2 = \{x \in \mathbb{R} : x^3 \leq 2\}.$$

[8]

2 (of 8). State the definition of *convergence* of a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ to a limit $a \in \mathbb{R}$, and use this definition to show that

$$\frac{2n}{n+3} \to 2 \text{ as } n \to \infty.$$

[10]

3 (of 8). Determine the limits as $n \to \infty$ of the following sequences. *Make free use of standard combination rules and standard limits, briefly indicating which results you are using; "ε" arguments are* ***not*** *required.*

$$a_n = \frac{2-n+n^2}{n^2+3}; \qquad b_n = \frac{3^n - n}{4^n + \sqrt{2}}; \qquad c_n = \frac{n! + n}{(n+1)^n}.$$

[6]

4 (of 8). Determine which of the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{x^n}$ ($x \neq 0$ fixed);

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n+1)}$.

[6]

5 (of 8). What is meant by the *radius of convergence* of a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

(including the cases 0 and ∞)? Find the radius of convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^{n/2}}$. **[10]**

Part B. Answer two questions from this part of the paper

6 (of 8). Suppose S is a bounded, non-empty subset of $\mathbb{R}$. Define the terms *upper bound* and *lower bound*, as applied to S. **[4]**

Let
$$-S = \{-x : x \in S\}.$$
Show that b is an upper bound for S if and only if $-b$ is a lower bound for $-S$, and that $\inf(-S) = -\sup(S)$. Deduce that $\sup(-S) = -\inf(S)$. **[14]**

State without proof the *Archimedean property* (or *axiom of Archimedes*) and use it to show that
$$\inf\{1/n : n \in \mathbb{N}\} = 0.$$
What is the supremum of this set? Justify your answer. **[7]**

7 (of 8). *Throughout this question, make free use of the standard combination rules and the basic fact that* $1/n \to 0$ *as* $n \to \infty$.

State without proof the *Sandwich Theorem*. **[3]**

Show by induction that $2^n > n$ for all $n \in \mathbb{N}$, and deduce that $1/2^n \to 0$ as $n \to \infty$. **[6]**

By expanding the numerator and denominator into n factors, show that if $n \geq 3$ then
$$\frac{2^n}{n!} \leq \frac{4}{n}.$$
Deduce that $2^n/n! \to 0$ as $n \to \infty$. **[8]**

Apply a similar technique to show that $(n!)^2/(2n)! \to 0$ as $n \to \infty$. **[8]**

8 (of 8). *Throughout this question, make free use of any standard facts about convergence of geometric series.*

State without proof the *comparison test*. **[4]**

Suppose $(a_n)_{n\in\mathbb{N}}$ is a sequence such that for every $x \in \mathbb{R}$ with $|x| < 1$, the sequence $(a_n x^n)_{n\in\mathbb{N}}$ is bounded; say $|a_n x^n| \leq C_x$ for all $n \in \mathbb{N}$.

Suppose $0 \leq x < 1$. By considering the factorisation and estimate

$$|a_n x^n| = |a_n x^{n/2}| x^{n/2} \leq C_{\sqrt{x}} x^{n/2},$$

or otherwise, show that the series

$$\sum_{n=1}^{\infty} a_n x^n$$

is absolutely convergent. **[10]**

Show that the same is true if $-1 < x < 0$. Express these results in terms of the radius of convergence of a power series. **[11]**

 End of examination.

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1. The supremum of S is the least upper bound of S, if S is bounded above. The infimum is the greatest lower bound of S, if S is bounded below. 4 Marks

$$\sup(S_1) = 1 + \sqrt{3}; \qquad \inf(S_1) = 1 - \sqrt{3}$$
$$\sup(S_2) = 2^{1/3}; \qquad \inf(S_2) \text{ does not exist}$$
4 Marks

Total: 8 Marks

2. a_n converges to a as $n \to \infty$ if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that if $n > N_\varepsilon$ then $|a_n - a| < \varepsilon$. 4 Marks
If $a_n = 2n/(n+3)$ and $a = 1$ then we calculate:

$$\begin{aligned} |a_n - a| < \varepsilon &\iff |2n/(n+3) - 2| < \varepsilon \\ &\iff |-6/(n+3)| < \varepsilon \\ &\iff 6/(n+3) < \varepsilon \\ &\iff n > 6/\varepsilon - 3 \end{aligned}$$

Given $\varepsilon > 0$, let N_ε be some natural number greater than $6/\varepsilon - 3$. If $n > N_\varepsilon$ then $n > 6/\varepsilon - 3$ so by the above calculation (all steps being "if and only if") it follows that $|a_n - a| < \varepsilon$, showing that $a_n \to a$ as $n \to \infty$. 6 Marks

Total: 10 Marks

3.

$$a_n = \frac{2 - n + n^2}{n^2 + 3} = \frac{2/n^2 - 1/n + 1}{1 + 3/n^2} \to \frac{0 - 0 + 1}{1 + 0} = 1$$

using standard combination rules and $1/n, 1/n^2 \to 0$.

$$b_n = \frac{3^n - n}{4^n + \sqrt{2}} = \frac{(3/4)^n - n/4^n}{1 + \sqrt{2}/4^n} \to \frac{0 - 0}{1 + 0} = 0$$

using combination rules, $(3/4)^n, (1/4)^n \to 0$ since $|3/4| < 1, |1/4| < 1$ and "exponential beats power".

$$c_n = \frac{n! + n}{(n+1)^n} = \frac{n!}{(n+1)^n} + \frac{n}{(n+1)^n} \to 0 + 0 = 0$$

using combination rules and "n^n beats factorial". [6 Marks]

[Total: 6 Marks]

4. (a) Ratio test:

$$\left|\frac{(n+1)!}{x^{n+1}}\frac{x^n}{n!}\right| = \frac{n+1}{|x|}$$

This tends to $+\infty$ for any $x \neq 0$. so the series diverges. [3 Marks]

(b) $1/\log(n+1)$ decreases and tends to zero, so by Leibniz's alternating series test, the series converges. [3 Marks]

[Total: 6 Marks]

5. The radius of convergence of the given power series is 0 if it converges only when $x = x_0$, or ∞ if it converges for all $x \in \mathbb{R}$. Otherwise, it is the number $R > 0$ with the property that the series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$. [4 Marks]

In both cases we use the ratio test.

(a)

$$\left|\frac{(x+2)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(x+2)^n}\right| = |x+2|\sqrt{n/(n+1)} \to |x+2|$$

This show that we have convergence if $|x+2| < 1$ and divergence if $|x+2| > 1$; that is, $R = 1$. [3 Marks]

(b)

$$\left|\frac{(x+1)^{n+1}}{(n+1)^{(n+1)/2}}\frac{n^{n/2}}{(x+1)^n}\right| = \frac{|x+1|}{\sqrt{n+1}}\left(\frac{n}{n+1}\right)^{n/2} \to 0$$

using the fact that $n/(n+1) < 1$. Since the limits is < 1 for all x, we have convergence for all x; that is, $R = \infty$. [3 Marks]

[Total: 10 Marks]

Remarks. *Everything in Section A is supposed to be straightforward and, if not identical to examples they've already seen, certainly similar. Full credit, of course, for equally correct solutions, e.g. root test in very last part.*

6. An upper bound of S is a number $b \in \mathbb{R}$ such that for all $x \in S$, $x \leq b$. A lower bound is a number $a \in \mathbb{R}$ such that for all $x \in S$, $a \leq x$. (4 Marks)

b is an upper bound for $S \iff$ for all $x \in S$, $x \leq b \iff$ for all $x \in S$, $-x \geq -b \iff$ for all $y \in -S$, $-b \leq y \iff -b$ is a lower bound for $-S$. (4 Marks)

Since $\sup(S)$ is an upper bound for S, $-\sup(S)$ is a lower bound for $-S$. If $a > -\sup(S)$ then $-a < \sup(S)$, so $-a$ is not an upper bound for S; it follows that a is not a lower bound for $-S$. This shows that $-\sup(S)$ is the greatest lower bound of $-S$; that is, $\inf(-S) = -\sup(S)$. (6 Marks)

To see that $\sup(-S) = -\inf(S)$, it is possible to use a similar argument; but it is easier to observe that $-(-S) = S$ so, appying the above result to $(-S)$, we have $\inf(S) = \inf(-(-S)) = -\sup(-S)$, or equivalently $\sup(-S) = -\inf(S)$. (4 Marks)

Archimedes axiom states that the natural numbers are not bounded above; that is, for any $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > x$. Clearly 0 is a lower bound of the given set. If a is another lower bound, then $a \leq 1/n$ for all $n \in \mathbb{N}$. In case $a > 0$ this implies $n < 1/a$ for all $n \in \mathbb{N}$, in contradiction to Archimedes. We therefore have $a \leq 0$, so 0 is the greatest lower bound. (5 Marks)

Since $1 \leq n$ for all $n \in \mathbb{N}$, $1/n \leq 1$ for all $n \in \mathbb{N}$. This shows that 1 is an upper bound of the given set. But $1 = 1/1$ is also an element of the set, so it is the maximum and, in particular, the supremum. (2 Marks)

Remarks. *Candidates typically find formal calculations with bounds difficult, so I've tried to make this question reasonably straightforward. The calculation* $\inf(S) = -\sup(-S)$ *is in the notes, in the context if proving that infima exist. The infimum of* $\{1/n : n \in \mathbb{N}\}$

is also in the notes, as part of a list of equivalent formulation of the Archimedean proper, but the proof isn't as direct as this.

Total: 25 Marks

7. The Sandwich Theorem states that if $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are real sequences such that $a_n \leq b_n \leq c_n$ for all n and a_n and c_n both converge to the same limit a as $n \to \infty$, then $b_n \to a$ as $n \to \infty$.

3 Marks

Certainly $2^1 > 1$. If, for some n, $2^n > n$, then

$$2^{n+1} = 2.2^n > 2n = n + n \geq n + 1.$$

3 Marks

It follows that $2^n >$ for all n. We now have

$$0 < \frac{1}{2^n} < \frac{1}{n}$$

so $1/2^n \to 0$ as $n \to \infty$ by the sandwich theorem.

3 Marks

We have

$$\frac{2^n}{n!} = \frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\cdots\frac{2}{n}$$

The product of the first two terms is 2. The next $n-3$ terms are all less than 1. The last term is $2/n$. This gives

$$0 < \frac{2^n}{n!} \leq 2\frac{2}{n} = \frac{4}{n}$$

for $n \geq 3$.

5 Marks

The RHS tends to 0, so $2^n/n! \to 0$ as $n \to \infty$ by the sandwich theorem (the cases $n = 1, 2$ not making any difference to the limit).

3 Marks

Similarly, the first n terms in the expansion $(2n)! = 1.2\ldots(2n)$ cancel with $n!$ to give

$$\frac{(n!)^2}{(2n)!} = \frac{1}{n+1}\frac{2}{n+2}\cdots\frac{n}{2n}$$

and each term on the RHS is $\leq 1/2$. This gives

$$0 < \frac{(n!)^2}{(2n)!} \leq \frac{1}{2^n}$$

5 Marks

The RHS tends to 0 by an earlier result so $(n!)^2/(2n)! \to 0$ as $n \to \infty$ by the sandwich theorem. [3 Marks]

Remarks. *Inequalities like this, and the consequences of the sandwich theorem, are familiar, though mostly in more general contexts such as $x^n/n! \to 0$ for any x. The last inequality was on a problem sheet.*

[Total: 25 Marks]

8. Suppose $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are series such that $|p_n| \le |q_n|$ for all n and $\sum_{n=1}^{\infty} q_n$ is absolutely convergent. Then $\sum_{n=1}^{\infty} p_n$ is absolutely convergent. [4 Marks]

The suggested factorisation

$$a_n x^n = a_n x^{n/2} x^{n/2} = a_n(\sqrt{x})^n(\sqrt{x})^n$$

yields

$$|a_n x^n| \le C_{\sqrt{x}}(\sqrt{x})^n$$

Since $0 < x < 1$, we also have $0 < \sqrt{x} < 1$ so the geometric sum $\sum_{n=1}^{\infty}(\sqrt{x})^n$ converges. By the comparison test, $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent. [10 Marks]

In the case $-1 < x < 0$, we can instead use the factorisation

$$a_n x^n = a_n(-1)^n|x|^{n/2}|x|^{n/2} = a_n(-1)^n(\sqrt{|x|})^n(\sqrt{|x|})^n$$

which leads to

$$|a_n x^n| \le C_{\sqrt{|x|}}(\sqrt{|x|})^n$$

The comparison test now works in the same way to show that $\sum_{n=1}^{\infty} a_n x^n$ is absolutely convergent, since $0 < \sqrt{|x|} < 1$. [8 Marks]

This hows that the power series $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R \ge 1$. [3 Marks]

Remarks. *Apart from the comparison test and the convergence of the geometric series, this is unseen. It's quick and easy for the good students.*

Total: 25 Marks