Some unexpected rank 1 asymptotics

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Asymptotic equivalence in normed spaces

Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences in a normed space. Say $a_n \sim b_n$ as $n \to \infty$ if

$$\frac{\|a_n - b_n\|}{\|a_n\|} \to 0$$

as $n \to \infty$.

Suppose T is a bounded operator on a Banach space X. Under what circumstances is there a sequence $(S_n)_{n\in\mathbb{N}}$ of rank 1 operators such that $T^n \sim S_n$ as $n \to \infty$? If such a sequence exists, how can it be described?

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Some expected rank 1 asymptotics — the power method

Suppose a bounded operator T on a Banach space X has an algebraically simple eigenvalue λ with $|\lambda| = r(T)$ and a radius $\rho < r(T)$ such that $\sigma(T) \setminus \{\lambda\} \subseteq D(0, \rho)$.

- The complementary spectral projections associated with λ and $\sigma(T) \setminus \{\lambda\}$ are $Px = \phi(x)u$ (rank 1) and $Qx = x - Px$ where $Tu = \lambda u$, $T^*\phi = \lambda\phi$ and $\phi(u) = 1$
- Since T, P, Q commute, P projects onto the eigenspace of λ and $PQ = QP = 0$, we have

$$T^n = (P+Q)T^n(P+Q) = PT^nP + QT^nQ = \lambda^n P + (QTQ)^n$$

- $\|\lambda^n P\| = |\lambda|^n \|u\|\|\phi\|$ and $\sigma(QTQ) \subseteq D(0, \rho)$ so $\|(QTQ)^n\| = o(\rho^n) = o(|\lambda|^n)$ as $n \to \infty$.
- Conclusion: $T^n \sim \lambda^n P$ (rank 1) as $n \to \infty$.

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What other operators have this property?

- If T has a more complicated peripheral spectrum then T^n will typically not be asymptotically of rank 1.
- e.g. a non-nilpotent $N \times N$ complex matrix is asymptotically of rank 1 iff among the eigenvalues of maximal magnitude there is exactly one Jordan block of maximal size.
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Volterra Convolution Operators

Suppose $k \in L^1(0,1)$ and $p \in [0,\infty]$. Then

$$(V_k f)(t) = \int_0^t k(t-s) f(s)\,\mathrm{d}s$$

defines a bounded map on $L^p(0,1)$ with $\|V_k\| \leq \|k\|_1 = \int_0^1 |k|$. This is the *Volterra convolution operator with kernel* k; V_k is compact and quasinilpotent. If $g, k \in L^1(0,1)$ then

$$V_k V_g = V_g V_k = V_{k*g}$$

where

$$(k*g)(t) = \int_0^t k(t-s) g(s)\,\mathrm{d}s$$

In particular,

$$V_k^n = V_{k^{*n}}$$

where

$$k^{*n} = k * k * \cdots * k$$

Riemann-Liouville fractional integration operators (I)

For $\alpha > 0$ and $\mu \in \mathbb{R}$ define $V^{\alpha}_{(\mu)} \in B(L^p(0,1))$ by

$$(V^{\alpha}_{(\mu)} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \exp(\mu(t-s))(t-s)^{\alpha-1} f(s)\,\mathrm{d}s$$

For fixed μ, these form a semigroup: $V^{\alpha+\beta}_{(\mu)} = V^{\alpha}_{(\mu)} V^{\beta}_{(\mu)}$.
$V^{\alpha} := V^{\alpha}_{(0)}$ are the *Riemann-Liouville fractional integration operators*; $V := V^{1}_{(0)}$ is indefinite integration:

$$(Vf)(t) = \int_0^t f(s)\,\mathrm{d}s$$

What happens as $\alpha \to \infty$?

Riemann-Liouville fractional integration operators (II)

Outline argument (Eveson 2003, 2005, cf. Thorpe 1998, Kershaw 1999):

- When α is large, $t^{\alpha-1}$ can be well approximated on $[0,1]$ by $\exp((\alpha-1)(t-1))$, so $V^{\alpha}_{(\mu)}$ can be well approximated by

$$(T_{\mu,\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \exp(\mu(t-s))\exp((\alpha-1)(t-s-1))f(s)\,\mathrm{d}s$$

- When α is large, $\exp((\alpha-1)(t-s-1))$ is much smaller for $s > t$ than it is for $s < t$, so $T_{\mu,\alpha}$ can be well approximated by $S_{\mu,\alpha}$ where

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Riemann-Liouville fractional integration operators (III)

This leads to various asymptotically equal sequences; e.g. the Riemann-Liouville operators

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s$$

are asymptotically equal to the rank 1 operators given by

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,\mathrm{d}s$$

Can now read off information from the simpler asymptotic form, e.g. on $L^p(0,1)$

$$\|V^{\alpha}_{(\mu)}\| \sim \frac{C_p \mathrm{e}^{\mu}}{\Gamma(\alpha+1)}$$

where $C_p = p^{-1/p} q^{-1/q}$ $(p^{-1}+q^{-1}=1)$, $C_1 = C_\infty = 1$.

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Schatten Norms of Riemann-Liouville Operators

(Eveson 2003) We have

$$\frac{\|V^\alpha - S_\alpha\|}{\|S_\alpha\|} \to 0 \text{ as } \alpha \to \infty$$

where $\|\cdot\|$ is the operator norm on $L^p(0,1)$. In the case $p = 2$, there are other norms of interest, in particular *Schatten norms*.

- The operator norm: covered by the L^p analysis.
- The Hilbert-Schmidt norm: not so hard.
- The trace norm: not so easy.

Since every Schatten norm is bounded below by the operator norm and above by the trace norm, the equivalence holds in every Schatten norm.

In every normalised Schatten norm (i.e. one in which rank-1 orthogonal projections have norm 1), we have

$$\|V^\alpha\| \sim \frac{1}{2\Gamma(\alpha+1)}$$

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More general Volterra convolution operators (I)

(Eveson 2005) Suppose $k \in L^1(0,1)$ and define $V_k \in B(L^p(0,1))$ by

$$(V_k f)(t) = \int_0^t k(t-s) f(s)\,\mathrm{d}s$$

Suppose the exponential function $\mathrm{e}^{\mu t}$ meets k tangentially at 0, i.e. k is differentiable from the right at 0, $k(0) = 1$ and $k'(0+) = \mu$. Then $V_k^n \sim V_{(\mu)}^n \sim$ rank-1.

Idea of proof: write

$$k(t) = \mathrm{e}^{\mu t} + h(t); \qquad V_k = V_{(\mu)} + V_h; \qquad V_k^n = (V_{(\mu)} + V_h)^n$$

where $h(t) = o(t)$ $(t \to 0)$. Expand the RHS using the binomial theorem, use the fact that $\|V_{(\mu)}^n\| \asymp 1/n!$ but $\|V_h^n\| = o(1/(2n)!)$ $(n \to \infty)$.

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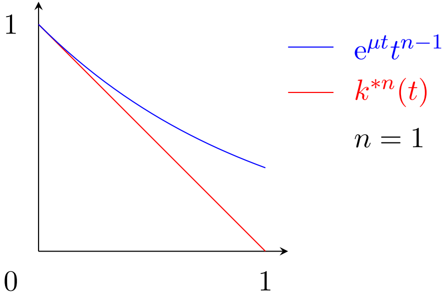
More general Volterra convolution operators (II)

For example, let $k(t) = 1 - t$ so e^{-t} meets $k(t)$ tangentially at $t = 0$, and let k^{*n} be the nth convolution power of k, i.e. the kernel of V_k^n.



More generally: if $k(t) = t^{\alpha-1} g(t)$ where $g(0) \neq 0$ and $g'(0+)$ exists then

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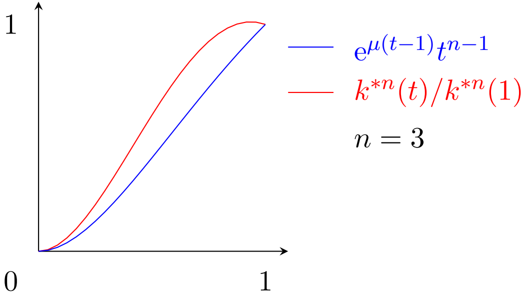

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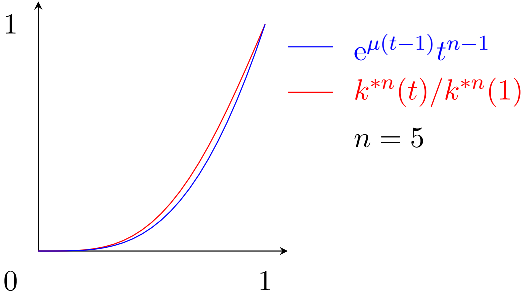

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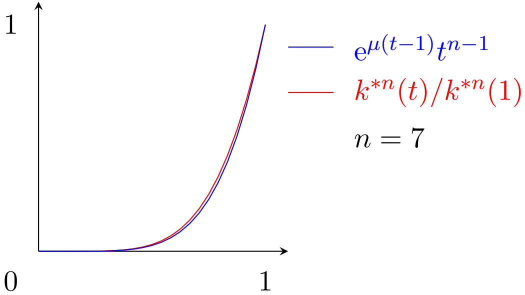

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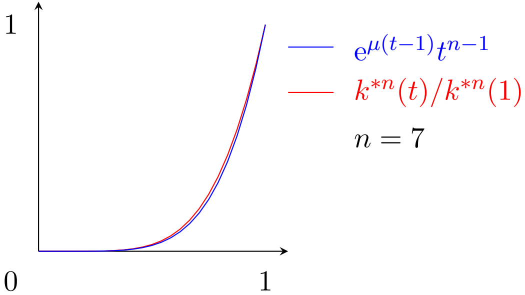

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The polylaplacian in one dimension: Thorpe BCs

Consider the Riemann-Liouville operator

$$(V^n f)(t) = \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)\, \mathrm{d}s$$

for $n \in \mathbb{N}$. Thorpe (1998) observed that $(V^n)^*(V^n)$ on $L^2(0,1)$ is the solution operator for the BVP

$$(-1)^n g^{(2n)} = f;$$

$$g^{(j)}(0) = 0 \quad (0 \le j \le n-1); \qquad g^{(j)}(1) = 0 \quad (n \le j \le 2n-1).$$

Since V^n is asymptotically of rank 1, the same is true for $(V^n)^*(V^n)$. What about other boundary conditions?

The polylaplacian in one dimension: Dirichlet BCs

(Eveson / Fewster 2007, Böttcher / Widom 2007, Kalyabin 2010)
Consider the BVP on $[-1, 1]$

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This has a solution operator T_n on $L^2(0, 1)$, and

$$\|T_n\| = r(T_n) \sim \frac{1}{\sqrt{2}(2n)!}$$

Conjecture (Eveson / Fewster 2007):
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The polylaplacian in n dimensions: Dirichlet BCs

Consider the polylaplacian $(-\Delta)^m$ on the unit ball B_n in $\mathbb{R}^n$, with Dirichlet boundary conditions. Green's function (Boggio 1905) is

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{(v^2-1)^{m-1}}{v^{n-1}}\,\mathrm{d}v$$

where

$$[x,y] = (|x|^2|y|^2 - 2x.y + 1)^{1/2}$$

$$k_{m,n} = \frac{\Gamma(1+n/2)}{n\pi^{n/2}4^{m-1}[(m-1)!]^2} > 0.$$

Assume $m > n/2$, so $G_{m,n}$ is continuous. Solution operator is

$$(T_{m,n}f)(x) = \int_{B_n} G_{m,n}(x,y)f(y)\,\mathrm{d}y$$

(compact, symmetric, positive definite, non-negative kernel).
How does this behave as $m \to \infty$?

Results

(Eveson 2011) Say $T_{m,n}h_{m,n} = \lambda_{m,n}h_{m,n}$ where $\lambda_{m,n}$ is the maximal eigenvalue of $T_{m,n}$, $h_{m,n} \geq 0$ (Jentzsch), $\|h_{m,n}\|_2 = 1$. For $x, y \in B_n$ define:

$$L_{m,n}(x) = (1 - |x|^2)^m \qquad K_{m,n}(x,y) = L_{m,n}(x)L_{m,n}(y)$$

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Idea of proof

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{(v^2-1)^{m-1}}{v^{n-1}} \, \mathrm{d}v$$

(Boggio). After change of variables and some calculation,

$$G_{m,n}(x,y) = \boxed{\frac{k_{m,n}}{2}(1-|x|^2)^m(1-|y|^2)^m} \times \int_0^1 \frac{w^{m-1}\,\mathrm{d}w}{(|x-y|^2+(1-|x|^2)(1-|y|^2)w)^{n/2}}$$

The shaded term is $(k_{m,n}/2)K_{m,n}(x,y)$. As m increases, its mass concentrates near $x=y=0$, so asymptotically the remaining term can be replaced by its value at $x=y=0$; this is $\sim 1/m$.
In some sense this shows that $G_{m,n} \sim k_{m,n}/(2m)K_{m,n}$. Now prove (A)–(E)!

Lower-order perturbations

Fix $n \in \mathbb{N}$ and a differential operator $\mathscr{A}$,

$$\mathscr{A} f = \sum_{\alpha : |\alpha| \leq d} a_\alpha D^\alpha$$

where a_α is a continuous function on $\overline{B_n}$. Then the solution operators for $(-\Delta)^m$ and $(-\Delta)^m + \mathscr{A}$ (with Dirichlet boundary conditions in both cases) are asymptotically equal in every Schatten norm. In particular, the results described above apply to the solution operator of $(-\Delta)^m + \mathscr{A}$.

This is because the solution operator for $(-\Delta)^m + \mathscr{A}$ is

$$T_{m,n}(I + \mathscr{A} T_{m,n})^{-1}$$

and $\mathscr{A} T_{m,n} \to 0$ as $m \to \infty$.

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Question

A variety of linear systems have asymptotic rank-1 behaviour.

- Normalised iterates $T^n/\|T^n\|$ converge to a rank-1 spectral projection, if the peripheral spectrum of T consists of a single, isolated, simple eigenvalue.
- The iterates of some quasinilpotent operators T (e.g. many Volterra operators) can be asymptotically equivalent to sequences of rank-1 operators.
- Solution operators of some BVPs are asymptotically equivalent to sequences of rank-1 operators, as the order of the BVP tends to infinity, e.g. $\mathscr{A} + \Delta^m$ with Dirichlet BCs.

Instead of the case-by-case calculations currently known, is there a theoretical framework which explains why this happens?

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