Some unexpected rank 1 asymptotics

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Asymptotic equivalence in normed spaces

Suppose \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) are two sequences in a normed space. Say \(a_n \sim b_n\) as \(n \to \infty\) if

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\frac{\|a_n - b_n\|}{\|a_n\|} \to 0
\]
as \(n \to \infty\).

Suppose \(T\) is a bounded operator on a Banach space \(X\). Under what circumstances is there a sequence \((S_n)_{n \in \mathbb{N}}\) of rank 1 operators such that \(T^n \sim S_n\) as \(n \to \infty\)? If such a sequence exists, how can it be described?
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Some expected rank 1 asymptotics — the power method

Suppose a bounded operator $T$ on a Banach space $X$ has an algebraically simple eigenvalue $\lambda$ with $|\lambda| = r(T)$ and a radius $\rho < r(T)$ such that $\sigma(T) \setminus \{\lambda\} \subseteq D(0, \rho)$.

- The complementary spectral projections associated with $\lambda$ and $\sigma(T) \setminus \{\lambda\}$ are $P_x = \phi(x)u$ (rank 1) and $Q_x = x - P_x$ where $Tu = \lambda u$, $T^*\phi = \lambda \phi$ and $\phi(u) = 1$.

- Since $T, P, Q$ commute, $P$ projects onto the eigenspace of $\lambda$ and $PQ = QP = 0$, we have

$$T^n = (P+Q)T^n(P+Q) = PT^nP + QT^nQ = \lambda^n P + (QTQ)^n$$

- $\|\lambda^n P\| = |\lambda|^n\|u\|\|\phi\|$ and $\sigma(QTQ) \subseteq D(0, \rho)$ so $\|(QTQ)^n\| = o(\rho^n) = o(|\lambda|^n)$ as $n \to \infty$.

- Conclusion: $T^n \sim \lambda^n P$ (rank 1) as $n \to \infty$. 
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What other operators have this property?

- If $T$ has a more complicated peripheral spectrum then $T^n$ will typically not be asymptotically of rank 1.
- e.g. a non-nilpotent $N \times N$ complex matrix is asymptotically of rank 1 iff among the eigenvalues of maximal magnitude there is exactly one Jordan block of maximal size.
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- What about quasinilpotent (spectrum $= \{0\}$) operators?
Suppose $k \in L^1(0, 1)$ and $p \in [0, \infty]$. Then

$$(V_k f)(t) = \int_0^t k(t - s)f(s) \, ds$$

defines a bounded map on $L^p(0, 1)$ with $\|V_k\| \leq \|k\|_1 = \int_0^1 |k|$. This is the Volterra convolution operator with kernel $k$; $V_k$ is compact and quasinilpotent. If $g, k \in L^1(0, 1)$ then

$$V_k V_g = V_g V_k = V_{k \ast g}$$

where

$$(k \ast g)(t) = \int_0^t k(t - s)g(s) \, ds$$

In particular,

$$V_k^n = V_{k \ast n}$$

where

$$k \ast n = k \ast k \ast \cdots \ast k$$
Riemann-Liouville fractional integration operators (I)

For $\alpha > 0$ and $\mu \in \mathbb{R}$ define $V_{(\mu)}^\alpha \in B(L^p(0, 1))$ by

$$(V_{(\mu)}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \exp(\mu(t - s))(t - s)^{\alpha - 1} f(s) \, ds$$

For fixed $\mu$, these form a semigroup: $V_{(\mu)}^{\alpha + \beta} = V_{(\mu)}^\alpha V_{(\mu)}^\beta$.

$V_{(0)}^\alpha := V_{(0)}^\alpha$ are the Riemann-Liouville fractional integration operators; $V := V_{(0)}^1$ is indefinite integration:

$$(V f)(t) = \int_0^t f(s) \, ds$$

What happens as $\alpha \to \infty$?
Riemann-Liouville fractional integration operators (II)


► When $\alpha$ is large, $t^{\alpha-1}$ can be well approximated on $[0, 1]$ by $\exp((\alpha - 1)(t - 1))$, so $V^{\alpha}_{(\mu)}$ can be well approximated by

$$
(T_{\mu,\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \exp(\mu(t-s)) \exp((\alpha-1)(t-s-1)) f(s) \, ds
$$

► When $\alpha$ is large, $\exp((\alpha - 1)(t - s - 1))$ is much smaller for $s > t$ than it is for $s < t$, so $T_{\mu,\alpha}$ can be well approximated by $S_{\mu,\alpha}$ where

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▷ $S_{\mu,\alpha}$ has rank 1.
This leads to various asymptotically equal sequences; e.g. the Riemann-Liouville operators

\[ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) \, ds \]

are asymptotically equal to the rank 1 operators given by

\[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s) \, ds \]

Can now read off information from the simpler asymptotic form, e.g. on \( L^p(0, 1) \)

\[ \| V^{\alpha}_{(\mu)} \| \sim \frac{C_p e^{\mu}}{\Gamma(\alpha + 1)} \]

where \( C_p = p^{-1/p} q^{-1/q} \) (\( p^{-1} + q^{-1} = 1 \)), \( C_1 = C_\infty = 1 \).
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Schatten Norms of Riemann-Liouville Operators

(Eveson 2003) We have

\[
\frac{\|V^\alpha - S_\alpha\|}{\|S_\alpha\|} \to 0 \text{ as } \alpha \to \infty
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where \(\|\cdot\|\) is the operator norm on \(L^p(0, 1)\). In the case \(p = 2\), there are other norms of interest, in particular \textit{Schatten norms}.

- The operator norm: covered by the \(L^p\) analysis.
- The Hilbert-Schmidt norm: not so hard.
- The trace norm: not so easy.

Since every Schatten norm is bounded below by the operator norm and above by the trace norm, the equivalence holds in every Schatten norm.

In every normalised Schatten norm (i.e. one in which rank-1 orthogonal projections have norm 1), we have

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Suppose the exponential function \( e^{\mu t} \) meets \( k \) tangentially at 0, i.e. \( k \) is differentiable from the right at 0, \( k(0) = 1 \) and \( k'(0+) = \mu \). Then \( V^n_k \sim V^n_\mu \sim \text{rank-1} \).

Idea of proof: write

\[
k(t) = e^{\mu t} + h(t); \quad V_k = V_\mu + V_h; \quad V^n_k = (V_\mu + V_h)^n
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where \( h(t) = o(t) \) \( (t \to 0) \). Expand the RHS using the binomial theorem, use the fact that \( \|V^n_\mu\| \asymp 1/n! \) but \( \|V^n_h\| = o(1/(2n)!) \) \( (n \to \infty) \).
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More general Volterra convolution operators (II)

For example, let \( k(t) = 1 - t \) so \( e^{-t} \) meets \( k(t) \) tangentially at \( t = 0 \), and let \( k^* \) be the \( n \)th convolution power of \( k \), i.e. the kernel of \( V^n_k \).

More generally: if \( k(t) = t^{\alpha - 1}g(t) \) where \( g(0) \neq 0 \) and \( g'(0+) \) exists then

\[
V^n_k \sim [g(0)\Gamma(\alpha)]^n V^{\alpha n}_{(g'(0)/g(0))} \sim \text{rank-1}
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$e^{\mu(t-1)}t^n-1$

$k^*n(t)/k^*n(1)$

$n = 3$

More generally: if $k(t) = t^{\alpha-1}g(t)$ where $g(0) \neq 0$ and $g'(0+)$ exists then

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\[
e^{\mu(t-1)t^{n-1}} \]

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The polylaplacian in one dimension: Thorpe BCs

Consider the Riemann-Liouville operator

$$(V^n f)(t) = \frac{1}{\Gamma(n)} \int_0^t (t - s)^{n-1} f(s) \, ds$$

for $n \in \mathbb{N}$. Thorpe (1998) observed that $(V^n)^*(V^n)$ on $L^2(0, 1)$ is the solution operator for the BVP

$$(-1)^n g^{(2n)} = f;$$

$$g^{(j)}(0) = 0 \quad (0 \leq j \leq n - 1); \quad g^{(j)}(1) = 0 \quad (n \leq j \leq 2n - 1).$$

Since $V^n$ is asymptotically of rank 1, the same is true for $(V^n)^*(V^n)$. What about other boundary conditions?
The polylaplacian in one dimension: Dirichlet BCs


Consider the BVP on $[-1, 1]$

$$(-1)^n g^{(2n)} = f; \quad g^{(j)}(\pm 1) = 0 \quad (0 \leq j \leq n - 1)$$

This has a solution operator $T_n$ on $L^2(0, 1)$, and

$$\|T_n\| = r(T_n) \sim \frac{1}{\sqrt{2}(2n)!}$$

Conjecture (Eveson / Fewster 2007):

$T_n$ is asymptotically of rank 1.
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$T_n$ is asymptotically of rank 1.
Consider the poly-Laplacian \((-\Delta)^m\) on the unit ball \(B_n\) in \(\mathbb{R}^n\), with Dirichlet boundary conditions. Green’s function (Boggio 1905) is

\[
G_{m,n}(x, y) = k_{m,n} |x - y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{(v^2 - 1)^{m-1}}{v^{n-1}} dv
\]

where

\[
[x, y] = (|x|^2|y|^2 - 2x.y + 1)^{1/2}
\]

\[
k_{m,n} = \frac{\Gamma(1 + n/2)}{n\pi^{n/2}4^{m-1}[(m - 1)!]^2} > 0.
\]

Assume \(m > n/2\), so \(G_{m,n}\) is continuous. Solution operator is

\[
(T_{m,n}f)(x) = \int_{B_n} G_{m,n}(x, y)f(y) dy
\]

(compact, symmetric, positive definite, non-negative kernel).

How does this behave as \(m \to \infty\)?
Results

(Eveson 2011) Say \( T_{m,n} h_{m,n} = \lambda_{m,n} h_{m,n} \) where \( \lambda_{m,n} \) is the maximal eigenvalue of \( T_{m,n} \), \( h_{m,n} \geq 0 \) (Jentzsch), \( \|h_{m,n}\|_2 = 1 \).

For \( x, y \in B_n \) define:

\[
L_{m,n}(x) = (1 - |x|^2)^m \quad K_{m,n}(x, y) = L_{m,n}(x)L_{m,n}(y)
\]

Let \( S_{m,n} \) be the rank 1 integral operator on \( L^2(B_n) \) with kernel \( K_{m,n} \). Then as \( m \to \infty \):

(A) \( \lambda_{m,n} \sim \frac{\Gamma(n/2)}{(2\pi)^{1/2}} \frac{1}{\Gamma(2m + n/2 + 1/2)} \)

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Idea of proof

\[ G_{m,n}(x, y) = k_{m,n} |x - y|^{2m-n} \int_{[x,y]/|x-y|} (v^2 - 1)^{m-1} \frac{1}{v^{n-1}} \, dv \]

(Boggio). After change of variables and some calculation,

\[ G_{m,n}(x, y) = \frac{k_{m,n}}{2} (1 - |x|^2)^m (1 - |y|^2)^m \times \]

\[ \int_0^1 \int_0^1 \frac{w^{m-1} \, dw}{(|x - y|^2 + (1 - |x|^2)(1 - |y|^2)w)^{n/2}} \]

The shaded term is \((k_{m,n}/2)K_{m,n}(x, y)\). As \(m\) increases, its mass concentrates near \(x = y = 0\), so asymptotically the remaining term can be replaced by its value at \(x = y = 0\); this is \(\sim 1/m\).

In some sense this shows that \(G_{m,n} \sim k_{m,n}/(2m)K_{m,n}\). Now prove (A)–(E)!
Lower-order perturbations

Fix $n \in \mathbb{N}$ and a differential operator $\mathcal{A}$,

$$\mathcal{A} f = \sum_{\alpha : |\alpha| \leq d} a_\alpha D^\alpha$$

where $a_\alpha$ is a continuous function on $\overline{B_n}$. Then the solution operators for $(-\Delta)^m$ and $(-\Delta)^m + \mathcal{A}$ (with Dirichlet boundary conditions in both cases) are asymptotically equal in every Schatten norm. In particular, the results described above apply to the solution operator of $(-\Delta)^m + \mathcal{A}$.

This is because the solution operator for $(-\Delta)^m + \mathcal{A}$ is

$$T_{m,n}(I + \mathcal{A}T_{m,n})^{-1}$$

and $\mathcal{A}T_{m,n} \to 0$ as $m \to \infty$. 
Lower-order perturbations

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and \( \mathcal{A} T_{m,n} \to 0 \) as \( m \to \infty \).
A variety of linear systems have asymptotic rank-1 behaviour.

- Normalised iterates $T^n/\|T^n\|$ converge to a rank-1 spectral projection, if the peripheral spectrum of $T$ consists of a single, isolated, simple eigenvalue.

- The iterates of some quasinilpotent operators $T$ (e.g. many Volterra operators) can be asymptotically equivalent to sequences of rank-1 operators.

- Solution operators of some BVPs are asymptotically equivalent to sequences of rank-1 operators, as the order of the BVP tends to infinity, e.g. $\mathcal{A} + \Delta^m$ with Dirichlet BCs.

Instead of the case-by-case calculations currently known, is there a theoretical framework which explains why this happens?
T. Boggio, Sulle funzioni di Green d’ordine $m$, Rend. Circ. Mat. Palermo, 20 (1905)


G. A. Kalyabin, Sharp estimates for derivatives of functions in the Sobolev classes $\dot{W}_2^r(-1, 1)$, Proc. Steklov Inst. 2010