A note on Melnikov-Petrachenko option pricing in binomial market with transaction costs

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March 3, 2005

Abstract
In the paper by Melnikov and Petrachenko published in Finance and Stochastics 9 (2005), 141–149, a procedure is put forward for pricing and replicating an arbitrary European contingent claim in the binomial market with transaction costs. We present an example to show that the option price arrived at by this replication procedure can lead to arbitrage. This is related to the fact that under transaction costs a superreplicating strategy may cost less to set up than a strictly replicating one.

Key words: transaction costs, arbitrage, option pricing, replication, superreplication.

JEL Classification: G11, G13

Mathematics Subject Classification (2000): 62P05

1 Introduction
We shall use the same notation and model setup as in Melnikov and Petrachenko’s paper [MP05].

The Theorem on page 146 in [MP05] asserts that the unique fair price \( C_N(f_N(\cdot)) \) of any given contingent claim \( f_N(\omega) = (f_1^N(\omega), f_2^N(\omega)) \) exercised by the delivery of a portfolio of \( f_1^N(\omega) \) bonds and \( f_2^N(\omega) \) shares is given by formula (9) in [MP05], which reads (using the notation in [MP05])

\[
C_N(f_N) = \beta_1 \circ (B_0^{dep} | B_0^{cred}) + \gamma_1 \circ (S_0^{ask} | S_0^{bid}).
\]
The formula as printed in [MP05] contains a typesetting error: the bid and ask stock prices $S_{\text{bid}}$ and $S_{\text{ask}}$ have been interchanged, which is corrected in the formula above. Having made the correction, we check the amended formula (9) against the derivation and against the numerical examples in [MP05] to ensure that this is what was originally intended. We then conclude that the amended formula (9) can still produce an option price that leads to arbitrage in some cases. This is shown by means of an example involving a single-step binomial tree with bid-ask spreads satisfying the assumptions in [MP05] and a call option with cash settlement.

The problem is related to the fact, first indicated by Dermody and Rockafellar [DR91] and Bensaid, Lesne, Pages and Scheinkman [BLPS92], that in the presence of transaction costs a superreplicating strategy may be less expensive to set up than a replicating one.

Incidentally, formula (9) as printed in [MP05], that is, without the correction above, can also lead to arbitrage, as can be shown by the same example.

### 2 Checking and re-deriving the option pricing formula in [MP05]

Let us re-derive (9) by following the original argument, starting with formula (8) on pages 144 and 145 in [MP05]. According to this argument, given a contingent claim $f_N = (f^{1}_N, f^{2}_N)$, the investor must find a self-financing strategy $(\beta_n, \gamma_n)_{n=0,1,...,N+1}$ replicating the claim, $(\beta_{N+1}, \gamma_{N+1}) = (f^{1}_N, f^{2}_N)$, starting from a portfolio $(\beta_0, 0)$ with no stock, that is, such that $\gamma_0 = 0$. The initial capital in the portfolio $(\beta_0, 0)$ becomes the price of the contingent claim. The self-financing condition (6) in [MP05] must be satisfied at time 0,

$$ \beta_1 \circ (B^{\text{dep}}_0 | B^{\text{cred}}_0) - \beta_0 \circ (B^{\text{dep}}_0 | B^{\text{cred}}_0) + \gamma_1 \circ (S^{\text{ask}}_0 | S^{\text{bid}}_0) = 0. $$

This gives the option price

$$ C_N(f_N) = \beta_0 \circ (B^{\text{dep}}_0 | B^{\text{cred}}_0) $$

$$ = \beta_1 \circ (B^{\text{dep}}_0 | B^{\text{cred}}_0) + \gamma_1 \circ (S^{\text{ask}}_0 | S^{\text{bid}}_0), $$

as in the amended version of (9) above. In addition, Melnikov and Petrachenko assume that there is a single initial bond price $B_0 = B^{\text{dep}}_0 = B^{\text{cred}}_0$. Under this assumption, the option price becomes

$$ C_N(f_N) = \beta_0 B_0 $$

$$ = \beta_1 B_0 + \gamma_1 \circ (S^{\text{ask}}_0 | S^{\text{bid}}_0). $$

The last formula can be checked against the two-step numerical example in Sect. 5 in [MP05]. Substituting into this formula the values of $\beta_1$ and $\gamma_1$ in Table 1 in [MP05] (to be precise, the strategies need to be re-computed to a greater number of decimal places than stated in the table to ensure accuracy...
of option prices to at least three decimal places) and the values \( S_0^{\text{ask}} = 105, \)
\( S_0^{\text{bid}} = 100, B_0 = 100 \) assumed in Sect. 5, we obtain

\[
C_2(f_2) = 7.602, \quad C_2(\hat{f}_2) = 7.312, \quad C_2(\tilde{f}_2) = 1.587,
\]
that is, precisely the same numerical results as in [MP05] for the three options with payoffs \( f_2, \hat{f}_2, \tilde{f}_2 \) considered there.

This, then, means that the amended version of the option pricing formula (9) stated above must be what was originally intended in [MP05].

3 Example of arbitrage in the option pricing formula in [MP05]

Having corrected the typesetting error in the option pricing formula (9), we shall now consider the following example.

Example  Take a single-step binomial model with ask and bid stock prices

\[
\begin{align*}
S_0^{\text{ask}} &= 5 \\
S_0^{\text{bid}} &= 1 \\
S_1^{\text{ask}} &= 3 \\
S_1^{\text{bid}} &= 2
\end{align*}
\]

and bond prices

\[
B_0 = B_0^{\text{dep}} = B_0^{\text{cred}} = B_1^{\text{dep}} = B_1^{\text{cred}} = 1,
\]

which satisfies Melnikov and Petrachenko’s assumptions; see (1) and (2) in [MP05]. Also consider a call option \( f_1 = (f_1^1, f_1^2) \) with cash settlement given by the pay-offs

\[
\begin{align*}
f_1(1) &= (f_1^1(1), f_1^2(1)) = (2, 0), \\
f_1(0) &= (f_1^1(0), f_1^2(0)) = (0, 0).
\end{align*}
\]

The replicating strategy computed by the method in [MP05] is

\[
(\beta_2, \gamma_2) = (2, 0) \\
(\beta_1, \gamma_1) = (-2, 1)
\]

and the amended formula (9) gives the option price

\[
C_1(f_1) = \beta_1 \diamond (B_0^{\text{dep}} | B_0^{\text{cred}}) + \gamma_1 \diamond (S_0^{\text{ask}} | S_0^{\text{bid}}) = -2 \times 1 + 1 \times 5 = 3.
\]

However, this price leads to arbitrage. Namely, selling the option at price 3, one could then keep a portfolio of 2 bonds and no stock to superreplicate the option, which leaves an arbitrage profit of 1. There is even more arbitrage profit to be made at the down node at time 1.
Remark  The above example provides yet another argument why the option pricing formula (9) as printed in [MP05] has the \( S_0^{\text{ask}}, S_0^{\text{bid}} \) prices typed in the wrong places. If used in the above example, it would price the option as

\[
\beta_1 \circ (B_0^{\text{dep}} | B_0^{\text{cred}}) + \gamma_1 \circ (S_0^{\text{bid}} | S_0^{\text{ask}}) = -2 \times 1 + 1 \times 1 = -1.
\]

Since the option has non-negative payoffs, this negative price would lead to arbitrage for the buyer. Indeed it can readily be verified, for example, by following the algorithm in [Tok04], that the no-arbitrage option price interval in the above example is from 0 to 2.

We conclude that the quantity \( C_N(f_N) \) in [MP05] cannot be regarded as the fair price for an arbitrary option because it can lead to arbitrage in some cases.

It may be instructive to compare the result in [MP05] to other approaches involving replication rather than superreplication of options under transaction costs, for example, [BV92], [Ste97], [Rut98], [KPT99], [Ste00], [Pal01b], [Pal01a] or [Koc04]. These papers introduce various assumptions and/or consider specific options for which the cost of setting up a replicating strategy happens to coincide with the least expensive superreplicating strategy. This is so, for example, for European calls with physical delivery, as in [BV92]. However, in [MP05] no assumptions of this kind are imposed and options with arbitrary payoffs are admitted including, in particular, calls with cash settlement, which can lead to arbitrage as in our example. If restricted to call options with physical delivery, the pricing formula in [MP05] is correct. It is also correct if restricted to small transaction costs in the sense defined in [Tok04], [TZ04] and [Zas04].

References


