American Options under Proportional Transaction Costs: Seller's Price Algorithm, Hedging Strategy and Optimal Stopping

Alet Roux and Tomasz Zastawniak

Department of Mathematics, University of York
Heslington, York YO10 5DD, United Kingdom
ar521@york.ac.uk, tz506@york.ac.uk

Abstract American options are priced and hedged in a general discrete market in the presence of arbitrary proportional transaction costs inherent in trading the underlying asset, modelled as bid-ask spreads. Pricing, hedging and optimal stopping algorithms are established for a short position (seller's position) in an American option with an arbitrary payoff settled by physical delivery. The seller's price representation as the expectation of the stopped payoff under an approximate martingale measure is also considered. The algorithms cover and extend the various special cases considered in the literature to-date. Any specific restrictions that were imposed on the form of the payoff, the magnitude of transaction costs or the discrete market model itself are relaxed. The pricing algorithm under transaction costs can be viewed as a natural generalisation of the iterative Snell envelope construction.

1 Introduction

This paper studies no-arbitrage pricing and hedging of the seller's position in American options when trading in the underlying asset is subject to proportional transaction costs. The results apply to options with arbitrary payoffs in any discrete market model and proportional transaction costs of any magnitude. We shall be concerned with computing the seller's price of an American option, also known as the upper hedging price or the ask price of the option. Apart from pricing, we put forward algorithms for computing the least expensive strategy superhedging the option seller's position, the optimal stopping time - generally a mixed (randomised) stopping time rather than an ordinary one - and the expectation representation for the seller's price.

The first to examine American options under proportional transaction costs in a similar setting and level of generality to the present paper were Chalasani and Jha [CJ01]. They established expectation representations for options with cash settlement, subject to the simplifying assumption that transaction costs apply at any time, except at any particular stopping time chosen by the buyer to exercise the option. An important feature that emerged in Chalasani and Jha's representation for the option seller's price was the role played by mixed stopping times (also known as randomised stopping times) in place of pure (ordinary) stopping times. Chalasani and Jha also pointed out the non-trivial nature of computing the option prices in their representations and the need to develop algorithms approximating these prices. Our algorithm solves this problem by providing the exact values of the prices.

Bouchard and Temam [BT05] established a dual representation for the set of initial endowments allowing to superhedge an American option in a discrete time market model with proportional transaction costs in the setting of Kabanov, Rásonyi and Stricker [KRS03] and Schachermayer [Sch04]. In particular, they reproduced Chalasani and Jha's [CJ01] expectation representation of the seller's price.

Papers concerned with various special cases involving the hedging prices of American options under proportional transaction costs include Kociński [Koc99], [Koc01], who studied sufficient conditions for the existence of perfectly replicating strategies for American options, Perrakis and Lefoll [PL00], [PL04], who investigated American calls and puts in the binomial model, and Tokarz and Zastawniak [TZ05], who worked with general American payoffs in the binomial model under small proportional transaction costs.

Another group of papers, using preference-based or risk minimisation approaches rather than superhedging for American options under proportional transaction costs, includes Davis and Zariphopoulou [DZ95], Mercurio and Vorst [MV97], Constantinides and Zariphopoulou [CZ01], and Constantinides and Perrakis [CP04]. The work by Levental and Skorohod [LS97], and Jakubenas, Levental and Ryznar [JLR03] shows that superhedging in continuous time leads to unrealistic results for American options under proportional transaction costs, thus providing motivation for the need to explore discrete time approaches.

The present paper complements and extends the results obtained by Chalasani and Jha [CJ01] and Bouchard and Temam [BT05] by providing efficient pricing, hedging and optimal stopping algorithms for arbitrary American options under proportional transaction costs in a general discrete setting. It also extends the work on hedging prices by the other authors listed above, removing any restrictions imposed in the various special cases that have been considered. As a by-product, we establish by a very different method the same expectation representation for the seller's price of an American option involving mixed (randomised) stopping times as in [CJ01] or [BT05]. Our constructions provide a geometric insight into the origin of mixed stopping times at each time step. A 'clinical' example is provided to further elucidate this very important and interesting aspect of pricing and hedging American options under transaction costs. Numerical examples show the flexibility and efficiency of the pricing algorithm in a realistic market model approximation.

Our seller's price algorithm under transaction costs bears a close resemblance to the Snell envelope construction in the standard friction-free setting. As is well known, the Snell envelope Z_t can be constructed by backward induction using the recursive relationships

$$Z_t = \max\{V_t, Y_t\}, \tag{1.1}$$

$$V_t = \mathbb{E}(Z_{t+1}|\mathcal{F}_t), \tag{1.2}$$

where V_t is the continuation value and Y_t is the payoff at time t. The analogy with the pricing algorithm under transaction costs is self-evident: Z_t, Y_t and V_t are replaced by polyhedral concave functions denoted by the same symbols in Algorithm 3.1, the maximum in (1.1) corresponds to taking the concave cap of V_t and Y_t, whereas the expectation in (1.2) is replaced by the restriction to the bid-ask spread interval of the concave cap of the functions Z_{t+1} over all successor nodes. Because of this correspondence, Algorithm 3.1 can be regarded as a natural extension of the Snell envelope construction.

Similar results, algorithms and representations have been established for European options in the general discrete setting by Roux, Tokarz and Zastawniak [RTZ06], and for American options under small transaction costs in the binomial tree model by Tokarz and Zastawniak [TZ05]. The case of American options in an arbitrary discrete market model subject to arbitrary transaction costs turns out to be significantly more challenging due to the appearance of mixed stopping times. Nevertheless, the underlying idea is similar to that in the European options paper [RTZ06].

Consider the concave function $x \mapsto Z_t^x$ such that

$$Z_t^x = \inf(\alpha_t + x\beta_t), \tag{1.3}$$

the infimum being taken over all portfolios (α_t, β_t) of cash (or bonds) and stock held at time t that allow to superhedge a given American option at any time from t to the expiry time. In the discrete model the infimum is attained whenever it is finite, and each Z_t is a polyhedral concave function. At time 0, in order to be able to hedge his or her position in the American option, a seller holding no initial position in stock needs to receive at least a cash amount

$$\alpha_0 = \max_{x \in \mathbb{R}} Z_0^x.$$

This value is the seller's price of the option. In effect, to find the seller's price, one needs to compute the function Z_0, which amounts to solving a large convex optimisation problem over the set of all superhedging self-financing strategies. In Algorithm 3.1 this is achieved by means of a dynamic programming type backward iterative procedure for computing Z_t, given Z_{t+1} and the option payoff at time t.

Similarly as for European options [RTZ06], this pricing algorithm offers considerable efficiency. In the case of path-independent stock prices and American options with path-independent payoff, the polyhedral concave functions Z_t are also path-independent. Moreover, in recombinant trees the number of extreme points of each of these functions grows at most polynomially with the number of time steps to expiry. As a result, the complexity of the pricing algorithm also grows at most polynomially.

The position of the seller versus the buyer of an American option is asymmetric. In the presence of transaction costs, in the buyer's case one is no longer dealing with a convex optimisation problem, see Remark 3.1. The algorithms put forward in this paper, which depend on convex duality between the set of portfolios (α_t, β_t) in (1.3) and the concave functions Z_t, do not readily extend to computing the buyer's price or hedging strategy. In a forthcoming paper we shall present a solution for the buyer's case, circumventing this difficulty by way of certain new ideas involving duality without convexity. In particular, the Z's are no longer concave functions in this new approach, but become self-intersecting polygonal lines with monotone gradient. A precursor of this solution under small transaction costs can be found in Tokarz and Zastawniak [TZ05].

The contents of this paper are organised as follows. In Section 2 we fix the notation, specify the market model with transaction costs, and present the necessary information on mixed stopping times, approximate martingale probabilities and concave functions. Section 3 is the main part of the paper. Following some definitions, three algorithms are presented here, namely the pricing, optimal hedging and optimal stopping algorithms for the seller of an American option in the presence of proportional transaction costs. The last algorithm also produces the optimal approximate martingale measure featuring in the expectation representation of the seller's price. This is followed by Theorem 3.2, which proves the correctness of the algorithms by establishing a number of representations for the seller's price. An illustrative example showing all three algorithms in action, simple enough to be re-computed by hand, concludes this section. In Section 4 we produce numerical examples with a more realistic flavour. Finally, Section 5 serves as an appendix containing some technical results.

2 Preliminaries

We shall use similar notational conventions and model assumptions as in Roux, Tokarz and Zastawniak [RTZ06].

Let Ω be a finite probability space equipped with the sigma-field 2^Ω consisting of all subsets of Ω and a probability measure Q on $\mathcal{F}$ such that $Q\{\omega\} > 0$ for each $\omega \in \Omega$. We shall consider discrete time $t = 0, 1, \ldots, T$, where T is a positive integer, and assume that a filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = 2^\Omega$ is given. For any $t = 0, 1, \ldots, T$ we denote by Ω_t the set of atoms of $\mathcal{F}_t$, and identify $\mathcal{F}_t$-measurable random variables with functions defined on Ω_t.

The filtration can be identified with a *tree*, the atoms of $\mathcal{F}_t$ corresponding to the *nodes* of the tree at time t. For nodes $\mu \in \Omega_t$ and $\nu \in \Omega_{t+1}$ such that $\nu \subset \mu$ we say that ν is a *successor node* of μ, and we denote by

$$\operatorname{succ} \mu = \{\nu \in \Omega_{t+1} \,|\, \nu \subset \mu\}$$

the set of successor nodes of $\mu \in \Omega_t$.

If P is a probability measure on $\mathcal{F}$, it can be identified with the family of probability measures P_t on $\mathcal{F}_t$ for $t = 0, 1, \ldots, T$ such that $P_t(\mu) = P(\mu)$ for any $\mu \in \Omega_t$.

2.1 Market Model

The market model consists of a risk-free bond, a stock, and an American option written on the stock. Trading in stock is subject to proportional transaction costs: At any time $t = 0, \ldots, T$ shares can be bought at the *ask price* S^{a}_t or sold at the *bid price* S^{b}_t. The processes S^{a} and S^{b} are adapted to the filtration and satisfy $S^{\mathrm{a}}_t \geq S^{\mathrm{b}}_t > 0$ for each t. Without loss of generality, we assume that the bond price is 1 for all $t = 0, 1, \ldots, T$, that is, the interest rate is nil. A trader's position in bonds can therefore be identified with cash holdings, and all prices with discounted prices.

The *liquidation value* of a portfolio (γ, δ) of cash (or bonds) and stock at time t is given by

$$\vartheta_t(\gamma, \delta) = \gamma + \delta^+ S^{\mathrm{b}}_t - \delta^- S^{\mathrm{a}}_t.$$

This can be obtained by selling stock for S^{b}_t per share to close a long position $\delta \geq 0$, or buying stock for S^{a}_t per share to close a short position $\delta < 0$. A *self-financing strategy* is a pair (α, β) of predictable processes α_t, β_t representing positions in cash (or bonds) and stock at $t = 0, \ldots, T$ such that $\beta_0 = 0$ and

$$\vartheta_t(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) \geq 0 \tag{2.1}$$

for each $t = 0, \ldots, T-1$. The set of such strategies will be denoted by $\Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$. An *arbitrage opportunity* is a self-financing strategy $(\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$ such that

$$\alpha_0 \leq 0, \quad \vartheta_T(\alpha_T, \beta_T) \geq 0, \quad Q\{\vartheta_T(\alpha_T, \beta_T) > 0\} > 0.$$

The following result, originally established by Jouini and Kallal [JK95] and Ortu [Ort01] in slightly different models, also holds in the case in hand, as was demonstrated by Tokarz [Tok04].

Theorem 2.1 *There is no arbitrage opportunity if and only if there exist a probability measure P on Ω equivalent to Q and a martingale S under P such that $S^{\mathrm{b}}_t \leq S_t \leq S^{\mathrm{a}}_t$ for each $t = 0, \ldots, T$.*

Form now on we shall assume that the market model admits no arbitrage opportunities.

2.2 Pure and Mixed Stopping Times

We denote by $\mathcal{T}$ the set of stopping times with values in $\{0, 1, \ldots, T\}$. Recall that a stopping time $\tau \in \mathcal{T}$ is a random variable such that for each $t = 0, 1, \ldots, T$

$$\{\tau = t\} \in \mathcal{F}_t.$$

We shall sometimes refer to such τ's as *pure stopping times* to distinguish them from mixed stopping times, defined below.

By a *mixed stopping time* (also called a *randomised stopping time* as in, for example, Chow, Robins and Siegmund [CRS71], Baxter and Chacon [BC77] or

Chalasani and Jha [CJ01]) we understand any non-negative adapted process χ such that

$$\sum_{t=0}^{T} \chi_t = 1,$$

and denote the set of mixed stopping times by $\mathcal{X}$.

Each pure stopping time τ can be identified with a mixed stopping time χ^τ such that for any $t = 0, 1, \dots, T$

$$\chi_t^\tau = 1_{\{\tau = t\}}.$$

The set $\mathcal{X}$ of mixed stopping times is convex, with χ^τ for $\tau \in \mathcal{T}$ as its extreme points. Each $\chi \in \mathcal{X}$ can therefore be expressed as

$$\chi = \sum_{\tau \in \mathcal{T}} \sigma_\tau \chi^\tau, \tag{2.2}$$

where $\sigma_\tau \geq 0$ for each $\tau \in \mathcal{T}$ and $\sum_{\tau \in \mathcal{T}} \sigma_\tau = 1$.

For any adapted process Z and any mixed stopping time χ the *time-χ value of Z* is defined as the random variable

$$Z_\chi = \sum_{t=0}^{T} \chi_t Z_t.$$

If τ is a pure stopping time, then Z_{χ^τ} is the familiar random variable

$$Z_{\chi^\tau} = \sum_{t=0}^{T} 1_{\{\tau = t\}} Z_t = Z_\tau.$$

Moreover, if χ is a convex combination (2.2) of pure stopping times, then

$$Z_\chi = \sum_{t=0}^{T} \sum_{\tau \in \mathcal{T}} \sigma_\tau \chi_t^\tau Z_t = \sum_{\tau \in \mathcal{T}} \sigma_\tau Z_\tau.$$

With every mixed stopping time $\chi \in \mathcal{X}$ we associate a predictable non-increasing process χ^* such that for each $t = 0, 1, \dots, T$

$$\chi_t^* = \sum_{s=t}^{T} \chi_s,$$

and for any adapted process Z we put

$$Z_t^{\chi^*} = \sum_{s=t}^{T} \chi_s Z_s.$$

Moreover, it will prove convenient to define

$$\chi_{T+1}^* = 0, \quad Z_{T+1}^{\chi^*} = 0.$$

2.3 Approximate Martingale Probabilities

The family of pairs (P, S) consisting of a probability measure P on Ω equivalent to Q and a martingale S under P such that for each $t = 0, 1, \dots, T$

$$S_t^{\mathrm{b}} \leq S_t \leq S_t^{\mathrm{a}},$$

which feature in Theorem 2.1, will be denoted by $\mathcal{P}$. Probability measures P of this kind will be called *martingale probabilities*. This family of pairs (P, S) can be used to represent the prices of European options under transaction costs, see Jouini and Kallal [JK95] or Roux, Tokarz and Zastawniak [RTZ06]. To represent the prices of American options we need certain larger families than $\mathcal{P}$.

For any mixed stopping time χ we denote by $\mathcal{P}(\chi)$ the family of pairs (P, S) consisting of a probability measure P on Ω equivalent to Q and an adapted process S such that for each $t = 0, 1, \dots, T$

$$\begin{gathered} S_t^{\mathrm{b}} \leq S_t \leq S_t^{\mathrm{a}}, \\ \chi_{t+1}^* S_t^{\mathrm{b}} \leq \mathbb{E}_P(S_{t+1}^{\chi^*} | \mathcal{F}_t) \leq \chi_{t+1}^* S_t^{\mathrm{a}}, \end{gathered} \tag{2.3}$$

where $\mathbb{E}_P$ denotes the expectation under P. We call such a P a χ-*approximate equivalent martingale probability*, and say that S is a χ-*approximate martingale* under P. If the assumption that P should be equivalent to Q is dropped, the corresponding family of pairs (P, S) will be denoted by $\bar{\mathcal{P}}(\chi)$, and P will be called a χ-*approximate martingale probability*. This notation and terminology is similar to that in Chalasani and Jha [CJ01].

Lemma 2.2 *For each mixed stopping time* $\chi \in \mathcal{X}$

$$\mathcal{P} \subset \mathcal{P}(\chi) \subset \bar{\mathcal{P}}(\chi).$$

Proof The second inclusion is obvious. To prove the first one, take any $(P, S) \in \mathcal{P}$ and any $\chi \in \mathcal{X}$. Because $S_t^{\mathrm{b}} \leq S_t \leq S_t^{\mathrm{a}}$, it is sufficient to show that for each $t = 0, 1, \dots, T$

$$\chi_{t+1}^* S_t = \mathbb{E}_P(S_{t+1}^{\chi^*} | \mathcal{F}_t). \tag{2.4}$$

We proceed by backward induction. For $t = T$ both sides of (2.4) are equal to zero. Suppose that (2.4) holds for some $t = 1, \dots, T$. Then, since χ^* is a predictable process and S is a martingale under P,

$$\begin{aligned} \mathbb{E}_P(S_t^{\chi^*} | \mathcal{F}_{t-1}) &= \mathbb{E}_P(\chi_t S_t + S_{t+1}^{\chi^*} | \mathcal{F}_{t-1}) = \mathbb{E}_P(\chi_t S_t + \mathbb{E}_P(S_{t+1}^{\chi^*} | \mathcal{F}_t) | \mathcal{F}_{t-1}) \\ &= \mathbb{E}_P(\chi_t S_t + \chi_{t+1}^* S_t | \mathcal{F}_{t-1}) = \mathbb{E}_P(\chi_t^* S_t | \mathcal{F}_{t-1}) = \chi_t^* \mathbb{E}_P(S_t | \mathcal{F}_{t-1}) \\ &= \chi_t^* S_{t-1}, \end{aligned}$$

which completes the proof. ■

It follows from Lemma 2.2 and Theorem 2.1 that the families $\mathcal{P}$ and $\mathcal{P}(\chi)$, $\bar{\mathcal{P}}(\chi)$ for any $\chi \in \mathcal{X}$ are non-empty in an arbitrage-free market model.

2.4 Concave Functions

In this section we collect some information and fix notation concerned with concave functions. More details can be found in Rockafellar [Roc97].

A *proper concave function* is any function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that for any $x \leq y \leq z$

$$(z - x)f(y) \geq (z - y)f(x) + (y - x)f(z)$$

and $f(x) > -\infty$ for some $x \in \mathbb{R}$. The *effective domain* of such a function f is defined as

$$\operatorname{dom} f = \{x \in \mathbb{R} \,|\, f(x) > -\infty\}.$$

Moreover, f is a *polyhedral proper concave function* if $\operatorname{dom} f$ is closed and there exist real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that for each $x \in \operatorname{dom} f$

$$f(x) = \min_{i=1,\dots,n} (a_i x + b_i).$$

Definition 2.1 The *concave cap* $\operatorname{cap}\{f_1, \dots, f_n\}$ of functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is defined as the smallest concave function h such that

$$h \geq \max\{f_1, \dots, f_n\}.$$

Lemma 2.3 *Suppose that $f_1, \dots, f_n$ are polyhedral proper concave functions on $\mathbb{R}$ with bounded effective domains. Then $\operatorname{cap}\{f_1, \dots, f_n\}$ is also a polyhedral proper concave function with bounded effective domain, such that for any $x \in \operatorname{dom}(\operatorname{cap}\{f_1, \dots, f_n\})$*

$$\operatorname{cap}\{f_1, \dots, f_n\}(x) = \max \sum_{i=1}^{n} \lambda_i f_i(x_i), \tag{2.5}$$

where the maximum is taken over all $\lambda_1, \dots, \lambda_n \geq 0$ and $x_1 \in \operatorname{dom} f_1, \dots, x_n \in \operatorname{dom} f_n$ satisfying

$$\sum_{i=1}^{n} \lambda_i = 1, \quad \sum_{i=1}^{n} \lambda_i x_i = x.$$

For an outline of the proof of this lemma, see Roux, Tokarz and Zastawniak [RTZ06]. For details we refer to Rockafellar [Roc97].

Remark 2.1 Carathéodory's theorem allows a stronger assertion, namely that the maximum in (2.5) is attained for some $\lambda_1, \dots, \lambda_n$ and $x_1, \dots, x_n$ as above such that all except at most two of the λ_i's are zero.

For any proper concave function f and any $x \in \operatorname{dom} f$ we denote by $D^- f(x)$ and $D^+ f(x)$ the *left and right derivatives* of f at x, adopting the convention that

$$\begin{aligned} D^- f(x) &= +\infty \quad \text{if } f(y) = -\infty \text{ for all } y < x, \\ D^+ f(x) &= -\infty \quad \text{if } f(y) = -\infty \text{ for all } y > x. \end{aligned}$$

Because f is concave, $D^- f(x) \geq D^+ f(x)$ for each $x \in \operatorname{dom} f$.

3 American Options under Transaction Costs: Seller's Case

We consider an American option with an adapted payoff process (ξ, ζ) and expiry time T. The option is exercised by the delivery of a portfolio (ξ_τ, ζ_τ) of cash (or bonds) and stock at a stopping time $\tau \in \mathcal{T}$ chosen by the buyer.

An option seller's superhedging strategy $(\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$ is characterised by the property that a seller who adopts this strategy will be left with a solvent portfolio $(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau)$, that is, a portfolio such that

$$\vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \geq 0$$

upon delivering the payoff (ξ_τ, ζ_τ) at any stopping time $\tau \in \mathcal{T}$ that the buyer may select. The smallest possible initial value α_0 among all strategies (α, β) of this kind is called the *seller's price* (*ask price*, *upper hedging price*) of the option and denoted by $\pi^{\mathrm{a}}(\xi, \zeta)$. The precise definition is

$$\pi^{\mathrm{a}}(\xi, \zeta) = \min\{\alpha_0 \mid (\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}}), \forall \tau \in \mathcal{T} : \vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \geq 0\}.$$

In a discrete arbitrage-free market model the minimum is attained for some strategy $(\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$. Such a strategy will be called a *seller's optimal superhedging strategy*.

In the following sections we present three constructions: Algorithm 3.1 for computing the seller's price $\pi^a(\xi, \zeta)$, Algorithm 3.2 for a seller's optimal superhedging strategy $(\hat{\alpha}, \hat{\beta})$, and Algorithm 3.3 to find an optimal mixed stopping time $\hat{\chi}$ and risk-neutral representation of the sellers price of the form $\pi^{\mathrm{a}}(\xi, \zeta) = \mathbb{E}_{\hat{P}}((\xi + \hat{S}\zeta)_{\hat{\chi}})$, where $(\hat{P}, \hat{S}) \in \bar{\mathcal{P}}(\hat{\chi})$. In Theorem 3.2 we shall establish a number of representations for $\pi^{\mathrm{a}}(\xi, \zeta)$ and prove the correctness of the algorithms.

Remark 3.1 In this paper we are concerned only with the seller's price $\pi^{\mathrm{a}}(\xi, \zeta)$ and seller's optimal superhedging strategy for an American option (ξ, ζ) under transaction costs. One could also consider an option buyer's *superhedging strategy* $(\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$ such that a buyer who follows it will be left with a solvent portfolio $(\alpha_\tau + \xi_\tau, \beta_\tau + \zeta_\tau)$, that is, a portfolio such that

$$\vartheta_\tau(\alpha_\tau + \xi_\tau, \beta_\tau + \zeta_\tau) \geq 0$$

upon receiving the payoff (ξ_τ, ζ_τ) at some stopping time $\tau \in \mathcal{T}$ chosen by him/herself. This leads to the definition of the *buyer's price* (*bid price*, *lower hedging price*)

$$\pi^{\mathrm{b}}(\xi, \zeta) = \max\{-\alpha_0 \mid (\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}}), \exists \tau \in \mathcal{T} : \vartheta_\tau(\alpha_\tau + \xi_\tau, \beta_\tau + \zeta_\tau) \geq 0\},$$

and a *buyer's optimal superhedging strategy* (α, β), that is, one for which the maximum is attained. The two prices $\pi^{\mathrm{a}}(\xi, \zeta)$ and $\pi^{\mathrm{b}}(\xi, \zeta)$ are the upper and, respectively, lower bounds of the no-arbitrage interval of option prices.

The algorithms presented below do not readily extend to the buyer's case. This is because the seller's algorithms amount, essentially, to solving a convex linear optimisation problem, whereas this is no longer so for the buyer. In particular, while the set of seller's superhedging strategies is convex, the set of buyer's superhedging strategies is not, in general. The buyer's case requires an essentially different approach and will be studied in forthcoming work.

3.1 Pricing

Algorithm 3.1 Given an American option with payoff process (ξ,ζ) and expiry time T, for each $t=0,1,\ldots,T$ define

$$Y_t^x = \begin{cases} \xi_t + x\zeta_t & \text{if } S_t^{\mathrm{b}} \leq x \leq S_t^{\mathrm{a}}, \\ -\infty & \text{otherwise,} \end{cases}$$

and construct by backward induction three adapted processes V, $\tilde{V}$, Z with values among polyhedral proper concave functions with bounded effective domain:

- Put
  $$Z_T = V_T = \tilde{V}_T = Y_T.$$
  This defines functions $x \mapsto Z_T^x(\mu)$ and $x \mapsto V_T^x(\mu)$ at each node $\mu \in \Omega_T$.
- For any $t = 0,1,\ldots,T-1$ put
  $$\tilde{V}_t(\mu) = \operatorname{cap}\{Z_{t+1}(\nu) \,|\, \nu \in \Omega_{t+1} \text{ is a successor node of } \mu\}$$
  at each node $\mu \in \Omega_t$, and
  $$Z_t = \operatorname{cap}\{V_t, Y_t\},$$
  where
  $$V_t^x = \begin{cases} \tilde{V}_t^x & \text{if } S_t^{\mathrm{b}} \leq x \leq S_t^{\mathrm{a}}, \\ -\infty & \text{otherwise.} \end{cases}$$

The algorithm returns
$$\max_{x\in\mathbb{R}} Z_0^x,$$
shown in Theorem 3.2 to be the ask price $\pi^{\mathrm{a}}(\xi,\zeta)$ of the American option.

All functions Z_t, V_t and $\tilde{V}_t$ for $t=0,1,\ldots,T$ have non-empty effective domains. This is because, by Theorem 2.1, in an arbitrage-free model there exists a measure P equivalent to Q and a martingale S under P such that $S_t^{\mathrm{b}} \leq S_t \leq S_t^{\mathrm{a}}$ for each $t=0,1,\ldots,T$. For any such P and S we have $Z_t^{S_t} > -\infty$, $V_t^{S_t} > -\infty$ and $\tilde{V}_t^{S_t} > -\infty$ for each $t=0,1,\ldots,T$.

3.2 Hedging

Lemma 3.1 *Let $t = 0, 1, \dots, T-1$ and let γ, δ be $\mathcal{F}_t$-measurable random variables such that*

$$\gamma + x\delta \geq Z_t^x \tag{3.1}$$

for each $x \in \mathbb{R}$. Then there are $\mathcal{F}_t$-measurable random variables ρ, σ such that

$$\rho + x\sigma \geq Z_{t+1}^x \tag{3.2}$$

for each $x \in \mathbb{R}$, and

$$\vartheta_t(\gamma - \rho, \delta - \sigma) \geq 0. \tag{3.3}$$

Proof By the construction of Z_t in Algorithm 3.1,

$$\gamma + x\delta \geq Z_t^x \geq V_t^x$$

for each $x \in \mathbb{R}$. From now on the proof is exactly the same as that of Lemma 4.1 in Roux, Tokarz and Zastawniak [RTZ06], and is presented here for completeness. The only change required involves replacing Z_t and $\tilde{Z}_t$ in [RTZ06] by V_t and, respectively, $\tilde{V}_t$. Let y be an $\mathcal{F}_t$-measurable random variable such that

$$V_t^y - y\delta = \max\{V_t^x - x\delta \,|\, x \in \mathbb{R}\}. \tag{3.4}$$

Because V_t is a polyhedral proper concave function with bounded effective domain, the maximum is attained. In particular, it follows that

$$y \in [S_t^{\mathrm{b}}, S_t^{\mathrm{a}}], \quad \tilde{V}_t^y = V_t^y > -\infty.$$

If we can find σ and ρ such that (3.3) holds and

$$D^-\tilde{V}_t^y \geq \sigma \geq D^+\tilde{V}_t^y, \quad \rho + y\sigma = \tilde{V}_t^y, \tag{3.5}$$

then it will follow by the concavity and the construction of $\tilde{V}_t$ that for any $x \in \mathbb{R}$

$$\rho + x\sigma \geq \tilde{V}_t^x \geq Z_{t+1}^x,$$

proving (3.2).

To see that we can indeed find such σ and ρ, we consider the following four cases, covering all possibilities:

1. $S_t^{\mathrm{b}} < y < S_t^{\mathrm{a}}$. Then $D^-V_t^y = D^-\tilde{V}_t^y$, $D^+V_t^y = D^+\tilde{V}_t^y$, and (3.4) implies that $D^-V_t^y \geq \delta \geq D^+V_t^y$. We put

   $$\sigma = \delta, \quad \rho = V_t^y - y\sigma.$$

   As a result, (3.5) holds. Observe that

   $$\begin{aligned} \rho + y\sigma &= V_t^y \leq \gamma + y\delta, \\ \sigma &= \delta, \end{aligned}$$

   which implies (3.3).

2. $S^{\mathrm{b}}_t = y < S^{\mathrm{a}}_t$. Then $D^+V^y_t = D^+\tilde{V}^y_t$ and (3.4) implies that $\delta \geq D^+V^y_t$. We put

$$\sigma = \min\{\delta, D^-\tilde{V}^y_t\}, \quad \rho = V^y_t - y\sigma.$$

As a result, (3.5) holds. Moreover,

$$\begin{aligned} \rho + y\sigma = V^y_t &\leq \gamma + y\delta, \\ \sigma &\leq \delta. \end{aligned}$$

Since $y = S^{\mathrm{b}}_t$, this implies (3.3).

3. $S^{\mathrm{b}}_t < y = S^{\mathrm{a}}_t$. In this case we put

$$\sigma = \max\{\delta, D^+\tilde{V}^y_t\}, \quad \rho = V^y_t - y\sigma,$$

and follow a similar argument as in case 2.

4. $S^{\mathrm{b}}_t = y = S^{\mathrm{a}}_t$. We take any finite σ such that

$$D^-\tilde{V}^y_t \geq \sigma \geq D^+\tilde{V}^y_t$$

and put

$$\rho = V^y_t - y\sigma.$$

Clearly, (3.5) holds. It also follows that

$$\rho + y\sigma = V^y_t \leq \gamma + y\delta,$$

and since $S^{\mathrm{b}}_t = y = S^{\mathrm{a}}_t$, (3.3) is satisfied.

This completes the proof of Lemma 3.1. ■

The lemma can be used to construct a seller's superhedging strategy. Observe that the proof of Lemma 4.1 in [RTZ06] can provide concrete formulae for ρ and σ, leading to the following algorithm.

Algorithm 3.2 We construct a strategy $(\hat{\alpha}, \hat{\beta})$ by induction as follows:

- Put

$$\hat{\alpha}_0 = \max_{x \in \mathbb{R}} Z^x_0, \quad \hat{\beta}_0 = 0.$$

  Then $\hat{\alpha}_0, \hat{\beta}_0$ are $\mathcal{F}_0$-measurable, and for each $x \in \mathbb{R}$

$$\hat{\alpha}_0 + x\hat{\beta}_0 \geq Z^x_0.$$

- For any $t = 1, \ldots, T$, if $\mathcal{F}_{t-1}$-measurable random variables $\hat{\alpha}_{t-1}, \hat{\beta}_{t-1}$ such that for each $x \in \mathbb{R}$

$$\hat{\alpha}_{t-1} + x\hat{\beta}_{t-1} \geq Z^x_{t-1}$$

  have already been constructed, then Lemma 3.1 provides $\mathcal{F}_{t-1}$-measurable random variables $\hat{\alpha}_t, \hat{\beta}_t$ such that

$$\vartheta_t(\hat{\alpha}_{t-1} - \hat{\alpha}_t, \hat{\beta}_{t-1} - \hat{\beta}_t) \geq 0$$

  and for each $x \in \mathbb{R}$

$$\hat{\alpha}_t + x\hat{\beta}_t \geq Z^x_t.$$

Since $\hat{\alpha}_t, \hat{\beta}_t$ are $\mathcal{F}_{t-1}$-measurable, they are also $\mathcal{F}_t$-measurable, which makes it possible to iterate the last step.

As a result, we obtain a self-financing strategy $(\hat{\alpha}, \hat{\beta}) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$ such that for each $\tau \in \mathcal{T}$ and for each $x \in [S^{\mathrm{b}}_\tau, S^{\mathrm{a}}_\tau]$

$$\hat{\alpha}_\tau + x\hat{\beta}_\tau \geq Z^x_\tau \geq Y^x_\tau = \xi_\tau + x\zeta_\tau,$$

implying that for each $\tau \in \mathcal{T}$

$$\vartheta_\tau(\hat{\alpha}_\tau - \xi_\tau, \hat{\beta}_\tau - \zeta_\tau) \geq 0.$$

This means that $(\hat{\alpha}, \hat{\beta})$ is a seller's superhedging strategy, so that

$$\pi^{\mathrm{a}}(\xi, \zeta) \leq \hat{\alpha}_0 = \max_{x \in \mathbb{R}} Z^x_0. \tag{3.6}$$

In Theorem 3.2 we shall prove that $(\hat{\alpha}, \hat{\beta})$ is in fact a seller's optimal superhedging strategy, that is, $\pi^{\mathrm{a}}(\xi, \zeta) = \hat{\alpha}_0$.

3.3 Stopping Time and Risk-Neutral Expectation

We construct a mixed stopping time $\hat{\chi}$, a probability measure $\hat{P}$ and an adapted process $\hat{S}$ such that $(\hat{P}, \hat{S}) \in \bar{\mathcal{P}}(\hat{\chi})$ and

$$\max_{x \in \mathbb{R}} Z^x_0 = \mathbb{E}_{\hat{P}}((\xi + \hat{S}\zeta)_{\hat{\chi}}).$$

In Theorem 3.2 we show that $\pi^{\mathrm{a}}(\xi, \zeta) = \mathbb{E}_{\hat{P}}((\xi + \hat{S}\zeta)_{\hat{\chi}})$.

Algorithm 3.3 Construct by induction a mixed stopping time $\hat{\chi}$, a probability measure $\hat{P}$, an adapted process $\hat{S}$ and auxiliary adapted processes $\hat{U}, \hat{Z}, \hat{X}, \hat{V}, \hat{Y}$ as follows:

- For $t = 0$ there is a $\hat{U}_0 \in \operatorname{dom} Z_0$ such that

  $$Z^{\hat{U}_0}_0 = \max_{x \in \mathbb{R}} Z^x_0.$$

  By Lemma 2.3, since $Z_0 = \operatorname{cap}\{V_0, Y_0\}$, there exist $\hat{X}_0 \in \operatorname{dom} V_0$, $\hat{S}_0 \in \operatorname{dom} Y_0$ and $\lambda \in [0, 1]$ such that

  $$\hat{U}_0 = (1 - \lambda)\hat{X}_0 + \lambda\hat{S}_0,$$
  $$Z^{\hat{U}_0}_0 = (1 - \lambda)V^{\hat{X}_0}_0 + \lambda Y^{\hat{S}_0}_0.$$

  Put

  $$\hat{Z}_0 = Z^{\hat{U}_0}_0, \quad \hat{V}_0 = V^{\hat{X}_0}_0, \quad \hat{Y}_0 = Y^{\hat{S}_0}_0.$$

  Also put

  $$\hat{\chi}_0 = \lambda$$

  and

  $$\hat{P}_0 = 1.$$

- For any $t = 1, \dots, T$ suppose that $\hat{\chi}_s, \hat{P}_s, , \hat{S}_s$ and $\hat{U}_s, \hat{Z}_s, \hat{X}_s, \hat{V}_s, \hat{Y}_s$ have already been constructed for each $s = 0, 1, \dots, t-1$. Take any node $\mu \in \Omega_{t-1}$. By Lemma 2.3, since $\hat{X}_{t-1}(\mu) \in \operatorname{dom} V_{t-1}(\mu) \subset \operatorname{dom} \tilde{V}_{t-1}(\mu)$ and $\tilde{V}_{t-1}(\mu) = \operatorname{cap}\{Z_t(\nu) \,|\, \nu \in \operatorname{succ} \mu\}$, it follows that

$$\hat{X}_{t-1}(\mu) = \sum_{\nu \in \operatorname{succ} \mu} p_\nu \hat{U}_t(\nu),$$
$$\hat{V}_{t-1}(\mu) = V_{t-1}^{\hat{X}_{t-1}(\mu)}(\mu) = \sum_{\nu \in \operatorname{succ} \mu} p_\nu Z_t^{\hat{U}_t(\nu)}(\nu)$$

for some $p_\nu \geq 0$ and $\hat{U}_t(\nu) \in \operatorname{dom} Z_t(\nu)$, where $\nu \in \operatorname{succ} \mu$, such that

$$1 = \sum_{\nu \in \operatorname{succ} \mu} p_\nu.$$

Consider two cases:

  - If $t < T$, for each $\nu \in \operatorname{succ} \mu$ use Lemma 2.3 again to deduce from $Z_t(\nu) = \operatorname{cap}\{V_t(\nu), Y_t(\nu)\}$ and $\hat{U}_t(\nu) \in \operatorname{dom} Z_t(\nu)$ that there exist $\hat{X}_t(\nu) \in \operatorname{dom} V_t(\nu)$, $\hat{S}_t(\nu) \in \operatorname{dom} Y_t(\nu)$ and $\lambda_\nu \in [0, 1]$ such that

$$\hat{U}_t(\nu) = (1 - \lambda_\nu)\hat{X}_t(\nu) + \lambda_\nu \hat{S}_t(\nu),$$
$$Z_t^{\hat{U}_t(\nu)}(\nu) = (1 - \lambda_\nu) V_t^{\hat{X}_t(\nu)}(\nu) + \lambda_\nu Y_t^{\hat{S}_t(\nu)}(\nu).$$

  - If $t = T$, then for each $\nu \in \operatorname{succ} \mu$ put

$$\hat{X}_T(\nu) = \hat{S}_T(\nu) = \hat{U}_t(\nu), \quad \lambda_\nu = 1.$$

Next put

$$\hat{Z}_t(\nu) = Z_t^{\hat{U}_t(\nu)}(\nu), \quad \hat{V}_t(\nu) = V_t^{\hat{X}_t(\nu)}(\nu), \quad \hat{Y}_t(\nu) = Y_t^{\hat{S}_t(\nu)}(\nu).$$

Also put

$$\hat{\chi}_t(\nu) = \lambda_\nu \left(1 - \sum_{s=0}^{t-1} \hat{\chi}_s(\nu)\right)$$

and

$$\hat{P}_t(\nu) = p_\nu \hat{P}_{t-1}(\mu),$$

concluding the inductive step.

The objects constructed in Algorithm 3.3 are by no means unique, and we can choose any $\hat{\chi}, \hat{P}, \hat{S}, \hat{U}, \hat{Z}, \hat{X}, \hat{V}, \hat{Y}$ satisfying the above conditions.

It follows from the construction that

$$\hat{\chi}_t^* \hat{U}_t = \hat{\chi}_{t+1}^* \hat{X}_t + \hat{\chi}_t \hat{S}_t, \tag{3.7}$$
$$\hat{\chi}_t^* \hat{Z}_t = \hat{\chi}_{t+1}^* \hat{V}_t + \hat{\chi}_t \hat{Y}_t \tag{3.8}$$

for each $t = 0, 1, \ldots, T$, and

$$\hat{X}_{t-1} = \mathbb{E}_{\hat{P}}(\hat{U}_t|\mathcal{F}_{t-1}), \tag{3.9}$$

$$\hat{V}_{t-1} = \mathbb{E}_{\hat{P}}(\hat{Z}_t|\mathcal{F}_{t-1}), \tag{3.10}$$

for each $t = 1, \ldots, T$.

The last two equalities, in turn, imply that

$$\hat{\chi}^*_{t+1}\hat{X}_t = \mathbb{E}_{\hat{P}}(\hat{S}^{\hat{\chi}^*}_{t+1}|\mathcal{F}_t), \tag{3.11}$$

$$\hat{\chi}^*_{t+1}\hat{V}_t = \mathbb{E}_{\hat{P}}(\hat{Y}^{\hat{\chi}^*}_{t+1}|\mathcal{F}_t) \tag{3.12}$$

for each $t = 0, 1, \ldots, T$. We give a proof of (3.11) by backward induction. That of (3.12) is similar and will be omitted. For $t = T$ both sides of (3.11) are equal to zero. Suppose that (3.11) holds for some $t = 1, \ldots, T$. Then by (3.7) and (3.9)

$$\begin{aligned}
\hat{\chi}^*_t\hat{X}_{t-1} &= \mathbb{E}_{\hat{P}}(\hat{\chi}^*_t\hat{U}_t|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_{\hat{P}}(\hat{\chi}^*_{t+1}\hat{X}_t + \hat{\chi}_t\hat{S}_t|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_{\hat{P}}(\mathbb{E}_{\hat{P}}(\hat{S}^{\hat{\chi}^*}_{t+1}|\mathcal{F}_t) + \hat{\chi}_t\hat{S}_t|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_{\hat{P}}(\hat{S}^{\hat{\chi}^*}_{t+1} + \hat{\chi}_t\hat{S}_t|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_{\hat{P}}(\hat{S}^{\hat{\chi}^*}_t|\mathcal{F}_{t-1}),
\end{aligned}$$

completing the proof of (3.11).

Combining (3.11) with the fact that $\hat{S}_t \in \operatorname{dom} Y_t = [S^{\mathrm{b}}_t, S^{\mathrm{a}}_t]$ and $\hat{X}_t \in \operatorname{dom} V_t \subset [S^{\mathrm{b}}_t, S^{\mathrm{a}}_t]$, we obtain

$$S^{\mathrm{b}}_t \leq \hat{S}_t \leq S^{\mathrm{a}}_t,$$
$$\hat{\chi}^*_{t+1}S^{\mathrm{b}}_t \leq \hat{\chi}^*_{t+1}\hat{X}_t = \mathbb{E}_{\hat{P}}(\hat{S}^{\hat{\chi}^*}_{t+1}|\mathcal{F}_t) \leq \hat{\chi}^*_{t+1}S^{\mathrm{a}}_t$$

for each $t = 0, 1, \ldots, T$, concluding that

$$(\hat{P}, \hat{S}) \in \bar{\mathcal{P}}(\hat{\chi}). \tag{3.13}$$

Moreover, by (3.8) and (3.12),

$$\begin{aligned}
\max_{x\in\mathbb{R}} Z^x_0 &= \hat{Z}_0 = \chi^*_0\hat{Z}_0 = \hat{\chi}^*_1\hat{V}_0 + \hat{\chi}_0\hat{Y}_0 = \mathbb{E}_{\hat{P}}(\hat{Y}^{\hat{\chi}^*}_1) + \hat{\chi}_0\hat{Y}_0 \\
&= \mathbb{E}_{\hat{P}}(\hat{Y}^{\hat{\chi}^*}_1 + \hat{\chi}_0\hat{Y}_0) = \mathbb{E}_{\hat{P}}(\hat{Y}^{\hat{\chi}^*}_0) = \mathbb{E}_{\hat{P}}(\hat{Y}_{\hat{\chi}}) = \mathbb{E}_{\hat{P}}((\xi + \hat{S}\zeta)_{\hat{\chi}}).
\end{aligned} \tag{3.14}$$

Remark 3.2 By Remark 2.1 the above construction can be carried out in such a way that $\hat{P}$ is concentrated on a binomial subtree.

3.4 Seller's Price Representations

Theorem 3.2 *The seller's price of an American option with payoff process* (ξ, ζ) *and exercise time* T *can be represented as*

$$\begin{aligned}\pi^{\mathrm{a}}(\xi,\zeta) &= \hat{\alpha}_0 = \max_{x\in\mathbb{R}} Z_0^x = \mathbb{E}_{\hat{P}}((\xi+\hat{S}\zeta)_{\hat{\chi}}) \\ &= \max_{\chi\in\mathcal{X}} \max_{(P,S)\in\bar{\mathcal{P}}(\chi)} \mathbb{E}_P((\xi+S\zeta)_\chi) = \max_{\chi\in\mathcal{X}} \sup_{(P,S)\in\mathcal{P}(\chi)} \mathbb{E}_P((\xi+S\zeta)_\chi),\end{aligned}$$

where $\hat{\alpha}_0$ *is the initial value of the seller's hedging strategy* $(\hat{\alpha}, \hat{\beta})$ *constructed in Algorithm* 3.2*,* Z_0 *is the polyhedral proper concave function in Algorithm* 3.1*,* $\hat{\chi}$ *is the mixed stopping time,* $\hat{P}$ *the probability measure and* $\hat{S}$ *the adapted process in Algorithm* 3.3*, the sets* $\mathcal{P}(\chi)$ *and* $\bar{\mathcal{P}}(\chi)$ *are defined in Section* 2.3*, and* $\mathbb{E}_P$ *denotes the expectation under probability measure* P*.*

Proof The equalities follow immediately by (3.6), (3.13), (3.14), and Lemmas 5.1 and 5.2. ∎

3.5 Example

Example 3.1 Consider a two-period binary tree model and an American option with the following ask and bid stock prices $S^{\mathrm{a}}, S^{\mathrm{b}}$ and payoff process (ξ, ζ):

$$\begin{array}{llcllcll}
 & & & & & & S_1^{\mathrm{a}} = 15 & \xi_2 = 9 \\
 & & & S_1^{\mathrm{a}} = 14 & \xi_1 = 3 & \nearrow & S_1^{\mathrm{b}} = 15 & \zeta_2 = 0 \\
 & & & S_1^{\mathrm{b}} = 8 & \zeta_1 = 0 & \searrow & & \\
S_0^{\mathrm{a}} = 10 & \xi_0 = 0 & \nearrow & & & & S_1^{\mathrm{a}} = 12 & \xi_2 = 0 \\
S_0^{\mathrm{b}} = 10 & \zeta_0 = 0 & \searrow & & & & S_1^{\mathrm{b}} = 12 & \zeta_2 = 0 \\
 & & & S_1^{\mathrm{a}} = 6 & \xi_1 = 0 & \nearrow & & \\
 & & & S_1^{\mathrm{b}} = 6 & \zeta_1 = 0 & \searrow & & \\
 & & & & & & S_1^{\mathrm{a}} = 4 & \xi_2 = 0 \\
 & & & & & & S_1^{\mathrm{b}} = 4 & \zeta_2 = 0
\end{array}$$

The risk-free rate is equal to zero (all bond prices are 1). The nodes in the tree at time 1 will be referred to as u and d, and those at time 2 as uu, ud, du and dd. The ask and bid stock prices as well as the payoffs are taken to be the same at nodes ud and du (they are path-independent). The option is settled in cash, that is, $\zeta \equiv 0$.

In Figure 1 we present the construction in Algorithm 3.1 for two of the nodes, namely u and the root node, which are the interesting ones in this example; the construction at any of the remaining nodes is straightforward. Looking at function Z_0 (which takes only one finite value), we find the seller's price of the option to be

$$\pi^{\mathrm{a}}(\xi,\zeta) = \max_{x\in\mathbb{R}} Z_0^x = Z_0^{10} = 4.$$

Figure 1 also shows the values of the processes $\hat{U}, \hat{Z}, \hat{X}, \hat{V}, \hat{S}, \hat{Y}$ from Algorithm 3.3 at the root node and at node u. An optimal mixed stopping time $\hat{\chi}$ and pair $(\hat{P}, \hat{S}) \in \bar{\mathcal{P}}(\hat{\chi})$ satisfying

$$\pi^{\mathrm{a}}(\xi,\zeta) = \mathbb{E}_{\hat{P}}((\xi+\hat{S}\zeta)_{\hat{\chi}}) = 4$$

can be computed using Algorithm 3.3:

$$
\begin{array}{ccccc}
 & & & & \begin{array}{l}\hat{\chi}_2(\mathrm{uu}) = \frac{1}{3}\\ \hat{P}_2(\mathrm{uu}) = \frac{2}{3}\\ \hat{S}_2(\mathrm{uu}) = 15\end{array}\\
 & & \begin{array}{l}\hat{\chi}_1(\mathrm{u}) = \frac{2}{3}\\ \hat{P}_1(\mathrm{u}) = 1\\ \hat{S}_1(\mathrm{u}) = 8\end{array} & \begin{array}{c}\nearrow\\ \searrow\end{array} & \\
 & \nearrow & & & \begin{array}{l}\hat{\chi}_2(\mathrm{ud}) = \frac{1}{3}\\ \hat{P}_2(\mathrm{ud}) = \frac{1}{3}\\ \hat{S}_2(\mathrm{ud}) = 12\end{array}\\
\begin{array}{l}\hat{\chi}_0 = 0\\ \hat{P}_0 = 1\\ \hat{S}_0 = 10\end{array} & & & & \\
 & \searrow & & & \begin{array}{l}\hat{\chi}_2(\mathrm{du}) = 1\\ \hat{P}_2(\mathrm{du}) = 0\\ \hat{S}_2(\mathrm{du}) = 12\end{array}\\
 & & \begin{array}{l}\hat{\chi}_1(\mathrm{u}) = 0\\ \hat{P}_1(\mathrm{d}) = 0\\ \hat{S}_1(\mathrm{d}) = 6\end{array} & \begin{array}{c}\nearrow\\ \searrow\end{array} & \\
 & & & & \begin{array}{l}\hat{\chi}_2(\mathrm{dd}) = 1\\ \hat{P}_2(\mathrm{dd}) = 0\\ \hat{S}_2(\mathrm{dd}) = 4\end{array}
\end{array}
$$

It is interesting to trace the mixed stopping time values $\hat{\chi}_0 = 0$ and $\hat{\chi}_1(\mathrm{u}) = \frac{2}{3}$ to the diagrams in Figure 1, where we can see that

$$
\begin{aligned}
Z_0^{10} &= 1V_0^{10} + 0Y_0^{10},\\
Z_1^{10}(\mathrm{u}) &= \frac{1}{3}V_1^{14}(\mathrm{u}) + \frac{2}{3}Y_1^{8}(\mathrm{u}).
\end{aligned}
$$

In Figure 1 we can also see how the probabilities $\hat{P}_1(\mathrm{u}) = 1$ and $\hat{P}_2(\mathrm{uu}) = \frac{2}{3}$, $\hat{P}_2(\mathrm{ud}) = \frac{1}{3}$ arise. They follow from

$$
\begin{aligned}
V_0^{10} &= 1Z_1^{10}(\mathrm{u}) + 0Z_1^{6}(\mathrm{d}),\\
V_1^{14}(\mathrm{u}) &= \frac{2}{3}Z_2^{15}(\mathrm{uu}) + \frac{1}{3}Z_2^{12}(\mathrm{ud}).
\end{aligned}
$$

An optimal superhedging strategy can be found by means of Algorithm 3.2:

$$
\begin{gathered}
(\hat{\alpha}_0, \hat{\beta}_0) = (4, 0),\\
(\hat{\alpha}_1, \hat{\beta}_1) = (-1, \tfrac{1}{2}),\\
(\hat{\alpha}_2(\mathrm{u}), \hat{\beta}_2(\mathrm{u})) = (-36, 3), \quad (\hat{\alpha}_2(\mathrm{d}), \hat{\beta}_2(\mathrm{d})) = (2, 0).
\end{gathered}
$$

Last but not least, the example serves to show that mixed stopping times play an essential role in representing the seller's price, and cannot be replaced by pure stopping times: The highest possible value of $\mathbb{E}_P(\xi_\tau + S_\tau\zeta_\tau)$ that can be obtained for any pure stopping time $\tau \in \mathcal{T}$ and any $(P, S) \in \bar{\mathcal{P}}(\chi^\tau)$ is 3, which falls short of $\pi^{\mathrm{a}}(\xi, \zeta) = 4$.

4 Numerical Results

The results of this paper may be used to price American options with various payoffs under transaction costs. We present two numerical examples. The first

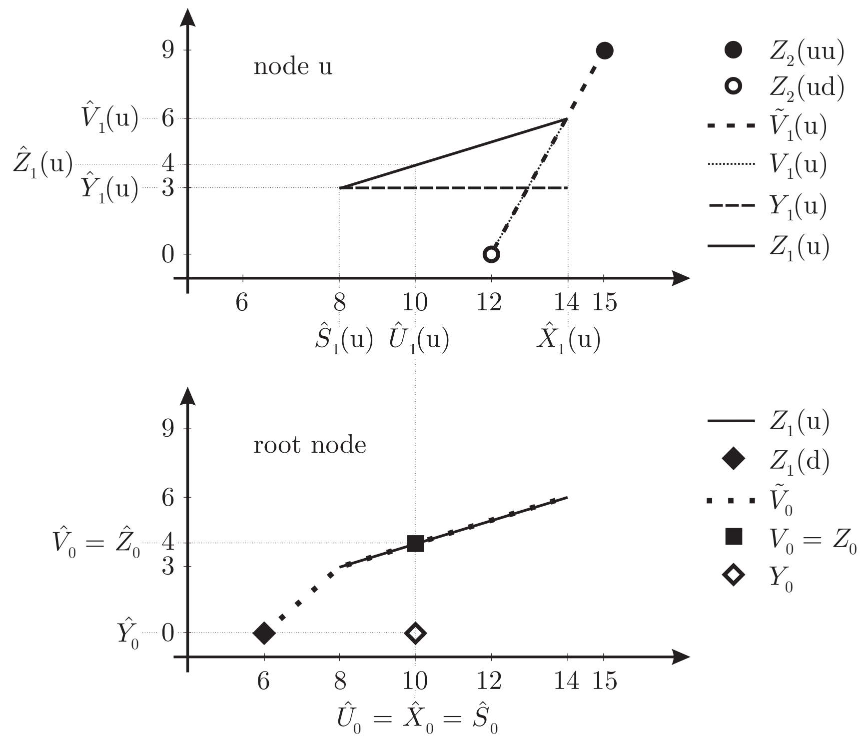

Figure 1: Seller's algorithm at node u and the root node in Example 3.1

is based on a binomial tree model, and the second involves a trinomial tree. The binomial tree example overlaps with earlier numerical work by Perrakis and Lefoll [PL04], Table 1, which we extend to more time steps.

In friction-free models, where cash and shares are freely exchangeable, it is self-evident that exercising an American option is of benefit to its owner whenever the cash equivalent payoff of the option is non-negative. In the presence of transaction costs, the case is no longer as clear-cut, as the desirability of the payoff (and hence the exercise decision) may also depend on the current position in stock and bonds of the owner at the time that the payoff becomes available. Motivated by the work of Perrakis and Lefoll [PL04], we alleviate this problem by awarding the buyer the right to not exercise the option at all, formally by adding an extra time step $T + 1$ to the model and setting the option payoff at that time to be zero.

Example 4.1 Consider a binomial tree model. The stock price process S satisfies

$$S_t = \varepsilon_t S_{t-1}$$

for $t = 1, \ldots, T$, where $S_0 = 100$ is given, and where $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of independent identically distributed random variables that take only two values

$$e^{-\sigma\sqrt{h}}, e^{\sigma\sqrt{h}},$$

both with positive probability. Here $\sigma = 0.2$ is the stock volatility, and $h = \frac{1}{4T}$, where $4T$ is the number of time steps per annum. We assume given a continuously compounded interest rate 10% and a transaction cost rate $k \in [0,1)$ so that, for $t = 1, \dots, T$, the bid and ask stock prices are

$$S_t^{\mathrm{b}} = (1-k)S_t, \quad S_t^{\mathrm{a}} = (1+k)S_t.$$

To be consistent with [PL04] we also assume that there are no transaction costs at time 0, i.e.

$$S_t^{\mathrm{b}} = S_0^{\mathrm{a}} = S_0.$$

Table 1 contains the ask prices of an American put option with strike price K exercised at time $Th = 0.25$ by the delivery of a portfolio $(K, -1)$ of cash and stock at any exercise time of the buyer's choosing. For $T = 20$ and $T = 40$, our results corresponds exactly to those of Perrakis and Lefoll [PL04], Table 1.

Table 1 Seller's prices of American put options in the binomial model

Strike	Number of time steps (T)			
(K)	20	40	52^a	250^a
85	0.295	0.374	0.424^a	0.904^a
90	0.849	1.009	1.088^a	1.835^a
95	1.954	2.218	2.320^a	3.310^a
100	3.867	4.155	4.286^a	5.413^a
105	6.689	6.964	7.078^a	8.178^a
110	10.416	10.601	10.698^a	11.585^a
115	15.000	15.028	15.063^a	15.580^a

[a]Not considered in Perrakis and Lefoll [PL04].

Example 4.2 Consider now a trinomial tree model. We assume that the stock price process S satisfies

$$S_t = \varepsilon_t S_{t-1}$$

for $t = 1, \dots, T$, where $S_0 = 100$ is given, and where $\varepsilon_1, \varepsilon_2, \dots$ is a sequence of independent identically distributed random variables taking only the values

$$e^{-\sigma\sqrt{h}}, 1, e^{\sigma\sqrt{h}},$$

all with positive probability. Here $\sigma = 0.2$ is the stock volatility, and $h = \frac{1}{T}$, where T is the number of time steps per annum. Also given is a continuously compounded interest rate of 10% and a transaction cost rate $k \in [0,1)$. For $t = 0, \dots, T$, the bid and ask stock prices are

$$S_t^{\mathrm{b}} = (1-k)S_t, \quad S_t^{\mathrm{a}} = (1+k)S_t.$$

In Table 2 we give the ask prices of a call option and a bull spread with one year to expiry. The call option with strike price 100 may be exercised by acceptance of the portfolio $(-100, 1)$ by the buyer at any time. The bull spread is a basket with cash settlement consisting of one long call with strike price 95 and one short call with strike price 105, i.e. the payoff of the spread is

$$(S_t - 95)1_{\{S_t>95\}} - (S_t - 105)1_{\{S_t>105\}}$$

in cash at any time $t = 0, \ldots, T$.

Table 2 Seller's prices of American options in the trinomial model

Transaction cost rate (k)	Number of time steps (T) 12	24	52	250
	Call option			
0%	13.098	13.183	13.230	13.261
1%	15.036	15.489	16.130	18.195
2%	16.822	17.557	18.630	21.100
3%	18.503	19.472	20.893	25.280
	Bull spread			
0%	8.896	8.881	9.215	9.450
1%	9.502	9.375	9.651	9.699
2%	9.824	9.592	9.847	9.895
3%	9.917	9.786	9.962	9.984

5 Appendix: Technical Results

Lemma 5.1 *For an American option with payoff process* (ξ, ζ)

$$\sup_{\chi\in\mathcal{X}} \sup_{(P,S)\in\bar{\mathcal{P}}(\chi)} \mathbb{E}_P((\xi + S\zeta)_\chi) \le \pi^{\mathrm{a}}(\xi, \zeta).$$

Proof We need to show that

$$\mathbb{E}_P((\xi + S\zeta)_\chi) \le \alpha_0$$

for any $\chi \in \mathcal{X}$, any $(P, S) \in \bar{\mathcal{P}}(\chi)$ and any $(\alpha, \beta) \in \Phi(S^{\mathrm{a}}, S^{\mathrm{b}})$ such that for each $\tau \in \mathcal{T}$

$$\vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \ge 0. \tag{5.1}$$

The self-financing condition (2.1), which is satisfied by (α, β), together with inequalities (2.3) from the definition of $\bar{\mathcal{P}}(\chi)$ imply that

$$\chi^*_{t+1}\alpha_t + \mathbb{E}_P(S^{\chi^*}_{t+1}|\mathcal{F}_t)\beta_t \ge \chi^*_{t+1}\alpha_{t+1} + \mathbb{E}_P(S^{\chi^*}_{t+1}|\mathcal{F}_t)\beta_{t+1} \tag{5.2}$$

for each $t = 0, 1, \ldots, T$. We shall prove by backward induction that for each $t = 0, 1, \ldots, T$

$$\chi^*_{t+1}\alpha_t + \mathbb{E}_P(S^{\chi^*}_{t+1}|\mathcal{F}_t)\beta_t \ge \mathbb{E}_P((\alpha + S\beta)^{\chi^*}_{t+1}|\mathcal{F}_t). \tag{5.3}$$

Inequality (5.3) holds for $t = T$ since both sides are equal to zero. Suppose that (5.3) holds for some $t = 1, \ldots, T$. Then by (5.2)

$$\begin{aligned}
\chi_t^* \alpha_{t-1} + \mathbb{E}_P(S_t^{\chi^*}|\mathcal{F}_{t-1})\beta_{t-1} &\geq \chi_t^* \alpha_t + \mathbb{E}_P(S_t^{\chi^*}|\mathcal{F}_{t-1})\beta_t \\
&= \mathbb{E}_P(\chi_t^* \alpha_t + S_t^{\chi^*} \beta_t|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_P(\chi_t (\alpha_t + S_t\beta_t) + \chi_{t+1}^* \alpha_t + S_{t+1}^{\chi^*} \beta_t|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_P(\chi_t (\alpha_t + S_t\beta_t) + \chi_{t+1}^* \alpha_t + \mathbb{E}_P(S_{t+1}^{\chi^*}|\mathcal{F}_t)\beta_t|\mathcal{F}_{t-1}) \\
&\geq \mathbb{E}_P(\chi_t (\alpha_t + S_t\beta_t) + \mathbb{E}_P((\alpha + S\beta)_{t+1}^{\chi^*}|\mathcal{F}_t)|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_P(\chi_t (\alpha_t + S_t\beta_t) + (\alpha + S\beta)_{t+1}^{\chi^*}|\mathcal{F}_{t-1}) \\
&= \mathbb{E}_P((\alpha + S\beta)_t^{\chi^*}|\mathcal{F}_{t-1}),
\end{aligned}$$

completing the proof of (5.3). In particular, for $t = 0$ inequality (5.3) implies that $\chi_1^* \alpha_0 \geq \mathbb{E}_P((\alpha + S\beta)_1^{\chi^*})$. Since, in addition, $\chi_0\alpha_0 = \mathbb{E}_P(\chi_0(\alpha_0 + S_0\beta_0))$, it follows that

$$\begin{aligned}
\alpha_0 = \chi_0\alpha_0 + \chi_1^*\alpha_0 &\geq \mathbb{E}_P(\chi_0(\alpha_0 + S_0\beta_0) + (\alpha + S\beta)_1^{\chi^*}) \\
&= \mathbb{E}_P((\alpha + S\beta)_0^{\chi^*}) = \mathbb{E}_P((\alpha + S\beta)_\chi).
\end{aligned}$$

For each $\tau \in \mathcal{T}$ the superhedging condition (5.1) and $S_\tau^{\mathrm{b}} \leq S_\tau \leq S_\tau^{\mathrm{b}}$ give

$$\alpha_\tau + S_\tau\beta_\tau \geq \xi_\tau + S_\tau\zeta_\tau.$$

Representing χ as a convex combination (2.2) of pure stopping times, we therefore obtain

$$\begin{aligned}
\alpha_0 \geq \mathbb{E}_P((\alpha + S\beta)_\chi) &= \sum_{\tau\in\mathcal{T}} \sigma_\tau \mathbb{E}_P(\alpha_\tau + S_\tau\beta_\tau) \\
&\geq \sum_{\tau\in\mathcal{T}} \sigma_\tau \mathbb{E}_P(\xi_\tau + S_\tau\zeta_\tau) = \mathbb{E}_P((\xi + S\zeta)_\chi),
\end{aligned}$$

as required. ■

Lemma 5.2 *For any mixed stopping time $\chi \in \mathcal{X}$ and any American option payoff process (ξ, ζ)*

$$\max_{(P,S)\in\bar{\mathcal{P}}(\chi)} \mathbb{E}_P((\xi + S\zeta)_\chi) = \sup_{(P,S)\in\mathcal{P}(\chi)} \mathbb{E}_P((\xi + S\zeta)_\chi).$$

Proof Since $\bar{\mathcal{P}}(\chi)$ is bounded and closed, the maximum is attained. It is sufficient to show that for any $\delta > 0$ and any $(\bar{P}, \bar{S}) \in \bar{\mathcal{P}}(\chi)$ there exists a pair $(P^\delta, S^\delta) \in \mathcal{P}(\chi)$ such that

$$\left|\mathbb{E}_{\bar{P}}((\xi + \bar{S}\zeta)_\chi) - \mathbb{E}_{P^\delta}((\xi + S^\delta\zeta)_\chi)\right| < \delta. \tag{5.4}$$

Due to the lack of arbitrage, by Theorem 2.1 there exists some $(P, S) \in \mathcal{P}$. If $\mathbb{E}_P((\xi + S\zeta)_\chi) = \mathbb{E}_{\bar{P}}((\xi + \bar{S}\zeta)_\chi)$, then (5.4) is trivial. If this is not the case, take any

$$\varepsilon \in \left(0, \min\left\{1, \frac{\delta}{\left|\mathbb{E}_P\left((\xi + S\zeta)_\chi\right) - \mathbb{E}_{\bar{P}}\left((\xi + \bar{S}\zeta)_\chi\right)\right|}\right\}\right),$$

and put

$$\begin{aligned} P^\delta &= (1-\varepsilon)\bar{P} + \varepsilon P, \\ S^\delta_t &= \mathbb{E}_{P^\delta}\left((1-\varepsilon)\bar{S}_t \frac{d\bar{P}}{dP^\delta} + \varepsilon S_t \frac{dP}{dP^\delta} \,\middle|\, \mathcal{F}_t\right) \end{aligned}$$

for each $t = 0, 1, \ldots, T$. It follows that P^δ is a probability measure equivalent to Q. It also follows that

$$\begin{aligned} S^\delta_t &= \mathbb{E}_{P^\delta}\left((1-\varepsilon)\bar{S}_t \frac{d\bar{P}}{dP^\delta} + \varepsilon S_t \frac{dP}{dP^\delta} \,\middle|\, \mathcal{F}_t\right) \\ &\qquad \le S^{\mathrm{a}}_t \mathbb{E}_{P^\delta}\left((1-\varepsilon) \frac{d\bar{P}}{dP^\delta} + \varepsilon \frac{dP}{dP^\delta} \,\middle|\, \mathcal{F}_t\right) = S^{\mathrm{a}}_t \end{aligned}$$

and, in a similar way, that

$$S^{\mathrm{b}}_t \le S^\delta_t$$

for any $t = 0, 1, \ldots, T$. Next,

$$\begin{aligned} \mathbb{E}_{P^\delta}((S^\delta)^{\chi^*}_{t+1}|\mathcal{F}_t) &= (1-\varepsilon)\mathbb{E}_{\bar{P}}(\bar{S}^{\chi^*}_{t+1}|\mathcal{F}_t) + \varepsilon\mathbb{E}_P(S^{\chi^*}_{t+1}|\mathcal{F}_t) \\ &\qquad \le (1-\varepsilon)\chi^*_{t+1}S^{\mathrm{a}}_t + \varepsilon\chi^*_{t+1}S^{\mathrm{a}}_t = \chi^*_{t+1}S^{\mathrm{a}}_t \end{aligned}$$

and, similarly,

$$\chi^*_{t+1}S^{\mathrm{b}}_t \le \mathbb{E}_{P^\delta}((S^\delta)^{\chi^*}_{t+1}|\mathcal{F}_t)$$

for any $t = 0, 1, \ldots, T$. As a result, $(P^\delta, S^\delta) \in \mathcal{P}(\chi)$. Moreover,

$$\begin{aligned} \mathbb{E}_{P^\delta}((\xi + S^\delta\zeta)_\chi) &= \mathbb{E}_{P^\delta}(\xi_\chi) + \mathbb{E}_{P^\delta}((S^\delta\zeta)_\chi) \\ &= (1-\varepsilon)\mathbb{E}_{\bar{P}}(\xi_\chi) + \varepsilon\mathbb{E}_P(\xi_\chi) + (1-\varepsilon)\mathbb{E}_{\bar{P}}((\bar{S}\zeta)_\chi) + \varepsilon\mathbb{E}_P((S\zeta)_\chi) \\ &= (1-\varepsilon)\mathbb{E}_{\bar{P}}((\xi + \bar{S}\zeta)_\chi) + \varepsilon\mathbb{E}_P((\xi + S\zeta)_\chi), \end{aligned}$$

which implies that

$$\left|\mathbb{E}_{\bar{P}}((\xi + \bar{S}\zeta)_\chi) - \mathbb{E}_{P^\delta}((\xi + S^\delta\zeta)_\chi)\right| = \varepsilon\left|\mathbb{E}_P((\xi + S\zeta)_\chi) - \mathbb{E}_{\bar{P}}((\xi + \bar{S}\zeta)_\chi)\right| < \delta.$$

■

References

[BC77] S. Baxter and R. Chacon, *Compactness of stopping times*, Z. Wahrsch. verw. Gebiete **40** (1977), 169–181.

[BT05] B. Bouchard and E. Temam, *On the hedging of American options in discrete time markets with proportional transaction costs*, Electronic Journal of Probability **10** (2005), 746–760.

[CJ01] P. Chalasani and S. Jha, *Randomized stopping times and American option pricing with transaction costs*, Math. Finance **1** (2001), 33–77.

[CP04] G.M. Constantinides and S. Perrakis, *Stochastic dominance bounds on American option prices in markets with frictions*, Working paper, University of Chicago, 2004.

[CRS71] Y.S. Chow, H. Robbins, and D. Siegmund, *Great expectations: The theory of optimal stopping*, Houghton Mifflin, Boston, 1971.

[CZ01] G.M. Constantinides and T. Zariphopoulou, *Bounds on derivative prices in an intertemporal setting with proportional transaction costs and multiple securities*, Math. Finance **11** (2001), 331–346.

[DZ95] M.H.A. Davis and T. Zariphopoulou, *American options and transaction fees*, Mathematical Finance (M.H.A. Davis et al., eds.), IMA Volumes in Mathematics and Its Applications, vol. 65, Springer, New York, 1995, pp. 47–61.

[JK95] E. Jouini and H. Kallal, *Martingales and arbitrage in securities markets with transaction costs*, J. Econom. Theory **66** (1995), 178–197.

[JLR03] P. Jakubenas, S. Levental, and M. Ryznar, *The super-replication problem via probabilistic methods*, Ann. Appl. Probab. **13** (2003), 742–773.

[Koc99] M. Kociński, *Optimality of the replicating strategy for American options*, Appl. Math. (Warsaw) **26** (1999), 93–105.

[Koc01] ———, *Pricing of the American option in discrete time with proportional transaction costs*, Math. Methods Oper. Res. **53** (2001), 67–88.

[KRS03] Y. Kabanov, M. Rásonyi, and C. Stricker, *On the closedness of sums of convex cones in L^0 and the robust no-arbitrage property*, Finance Stoch. **7** (2003), 403–411.

[LS97] S. Levental and A.V. Skorohod, *On the possibility of hedging options in the presence of transaction costs*, Ann. Appl. Probab. **7** (1997), 410–443.

[MV97] F. Mercurio and T.C.F. Vorst, *Options pricing and hedging in discrete time with transaction costs*, Mathematics of Derivative Securities (M.A.H. Dempster and S.R. Pliska, eds.), Cambridge University Press, Cambridge, UK, 1997, pp. 190–215.

[Ort01] F. Ortu, *Arbitrage, linear programming and martingales in securities markets with bid-ask spreads*, Decis. Econom. Finance **24** (2001), no. 2, 79–105.

[PL00] S. Perrakis and J. Lefoll, *Option pricing and replication with transaction costs and dividends*, J. Econom. Dynam. Control **24** (2000), 1527–1561.

[PL04] ______, *The American put under transaction costs*, J. Econom. Dynam. Control **28** (2004), 915–935.

[Roc97] R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1997.

[RTZ06] A. Roux, K. Tokarz, and T Zastawniak, *European options under proportional transaction costs: An algorithmic approach*, Preprint, 2006, http://ssrn.com/abstract=897965.

[Sch04] W. Schachermayer, *The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time*, Math. Fin. **14** (2004), 19–48.

[Tok04] K. Tokarz, *European and American option pricing under proportional transaction costs*, Ph.D. thesis, University of Hull, 2004.

[TZ05] K. Tokarz and T. Zastawniak, *American contingent claims under small proportional transaction costs*, Preprint, 2005.