# A counter-example to an option pricing formula under transaction costs

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**Abstract**

In the paper by Melnikov and Petrachenko 'On option pricing in binomial market with transaction costs,' *Finance Stoch.* 9 (2005), 141–149, a procedure is put forward for pricing and replicating an arbitrary European contingent claim in the binomial model with bid-ask spreads. We present a counter-example to show that the option pricing formula stated in that paper can in fact lead to arbitrage. This is related to the fact that under transaction costs a superreplicating strategy may be less expensive to set up than a strictly replicating one.

*Key words:* transaction costs, arbitrage, option pricing, replication, superreplication.

*JEL Classification:* G11, G13

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## 1 Introduction

We use the same notation and model setup as in Melnikov and Petrachenko's paper [MP05].

The Theorem on page 146 in [MP05] asserts that the fair price $\mathbb{C}_N(f_N)$ of any given contingent claim $f_N = \left(f_N^1, f_N^2\right)$ exercised at time $N$ by the delivery of a portfolio of $f_N^1$ bonds and $f_N^2$ shares is equal to the initial cost of setting up a self-financing strategy that replicates the portfolio exactly, given by formula (9) in [MP05].

We present a counter-example to show that this option pricing formula can in fact lead to arbitrage in some cases. The problem is related to the fact, first indicated by Dermody and Rockafellar [DR91] and Bensaid, Lesne, Pagès and Scheinkman [BLPS92], that in the presence of transaction costs a super-replicating strategy may have a lower initial value than a strictly replicating one.

Before proceeding to the counter-example, we recall the main features of the market model in [MP05]. It is based on a binomial tree with $N$ time steps, the nodes at time $n = 0, 1, \dots, N$ being identified with sequences $\delta_1, \dots, \delta_n$ of 1's and 0's, corresponding to up and down stock price movements. The standard probability space and filtration $\mathcal{F}_n$ for the binomial tree model is used. Bond prices $B_n^{\mathrm{cred}}, B_n^{\mathrm{dep}}$ for short and long positions in the money market and bid-ask spreads $S_n^{\mathrm{bid}}, S_n^{\mathrm{ask}}$ for stock prices are given at each node. It is assumed that the $B_n^{\mathrm{cred}}, B_n^{\mathrm{dep}}$ are $\mathcal{F}_{n-1}$-measurable, the $S_n^{\mathrm{bid}}, S_n^{\mathrm{ask}}$ are $\mathcal{F}_n$-measurable,

$$0 < S_n^{\mathrm{bid}} \le S_n^{\mathrm{ask}}, \quad 0 < B_n^{\mathrm{dep}} \le B_n^{\mathrm{cred}},$$

and for each node $\delta_1, \dots, \delta_{n-1}$ the bid and ask stock prices at the subsequent up and down nodes satisfy

$$S_n^{\mathrm{bid}}(\delta_1, \dots, \delta_n, 1) > S_n^{\mathrm{ask}}(\delta_1, \dots, \delta_n, 0).$$

In this model, for any European option to be exercised at time $N$ by the delivery of a portfolio $f_N = (f_N^1, f_N^2)$, a self-financing strategy $(\beta_n, \gamma_n)$, $n = 1, \dots, N+1$ is constructed that replicates the option, that is, satisfies

$$\beta_{N+1} = f_N^1, \quad \gamma_{N+1} = f_N^2.$$

The initial cost of setting up the replicating strategy, given by formula (9) in [MP05], is claimed to provide the fair price $\mathbb{C}_N(f_N(\cdot))$ of the option. We note that formula (9) as printed in [MP05] contains a typesetting error: the bid and ask stock prices $S_0^{\mathrm{bid}}$ and $S_0^{\mathrm{ask}}$ have been interchanged. The formula derived in [MP05] should read

$$\mathbb{C}_N(f_N) = \beta_1 \diamond (B_0^{\mathrm{dep}} | B_0^{\mathrm{cred}}) + \gamma_1 \diamond (S_0^{\mathrm{ask}} | S_0^{\mathrm{bid}}), \tag{9'}$$

where, $\xi \diamond (\theta_1|\theta_2) = \xi\theta_1$ if $\xi \ge 0$, and $\xi \diamond (\theta_1|\theta_2) = \xi\theta_2$ if $\xi < 0$. The two-step numerical example in Sect. 5 of [MP05] has indeed been computed using (9') rather than (9).

# 2 Counter-example showing arbitrage in the option pricing formula

Take a single-step binomial model with ask and bid stock prices

$$\begin{array}{ccc} & & S_1^{\mathrm{ask}} = 6 \\ & & S_1^{\mathrm{bid}} = 4 \\ S_0^{\mathrm{ask}} = 5 & & \\ S_0^{\mathrm{bid}} = 1 & & \\ & & S_1^{\mathrm{ask}} = 3 \\ & & S_1^{\mathrm{bid}} = 2 \end{array}$$

and bond prices

$$B_0^{\text{dep}} = B_0^{\text{cred}} = B_1^{\text{dep}} = B_1^{\text{cred}} = 1,$$

which satisfies the assumptions in [MP05]. Also consider a call option $f_1 = (f_1^1, f_1^2)$ with cash settlement given by the payoffs

$$\begin{aligned} f_1(1) &= (f_1^1(1), f_1^2(1)) = (2,0), \\ f_1(0) &= (f_1^1(0), f_1^2(0)) = (0,0). \end{aligned}$$

The replicating strategy computed by the method in [MP05] is

$$(\beta_1,\gamma_1) = (-2,1) \quad \begin{array}{l} (\beta_2,\gamma_2) = (2,0) \\ \\ (\beta_2,\gamma_2) = (0,0) \end{array}$$

and (9') gives the option price

$$\mathbb{C}_1(f_1) = \beta_1 \diamond (B_0^{\text{dep}} | B_0^{\text{cred}}) + \gamma_1 \diamond (S_0^{\text{ask}} | S_0^{\text{bid}}) = -2 \times 1 + 1 \times 5 = 3.$$

However, this price leads to arbitrage. Namely, selling the option at price 3, one could then keep a portfolio of 2 bonds and no stock to superreplicate the option, which leaves an arbitrage profit of 1. There is even more arbitrage profit to be made at the down node at time 1.

We conclude that $\mathbb{C}_N(f_N)$ cannot be regarded as the fair price for an arbitrary option because it can lead to arbitrage in some cases.

We remark that trading in stock would always lead to a loss in this counter-example, so no trading would in fact occur. It is, nevertheless, possible to produce a counter-example with two or more time steps that is free of this somewhat artificial feature.

# 3 Final remarks

It may be instructive to compare the approach in [MP05] to other papers involving replication rather than superreplication of options under transaction costs, for example, [BV92], [Ste97], [Rut98], [KPT99], [Ste00], [Pal01b], [Pal01a] or [Koc04]. These papers introduce various assumptions and/or consider specific options such that the cost of setting up a replicating strategy happens to coincide with the least expensive superreplicating strategy. This is so, for example, for European calls with physical delivery, as in [BV92]. However, in [MP05] no assumptions of this kind are imposed and options with arbitrary payoffs are admitted including, in particular, calls with cash settlement, which can lead to arbitrage as in the counter-example above. If restricted to call options with physical delivery, pricing formula (9') is free of arbitrage. It is also free of arbitrage if restricted to small transaction costs in the sense defined in [Tok04], [TZ04] and [Zas04].

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