Dynamic Programming Algorithms for the Ask and Bid Prices of American Options under Small Proportional Transaction Costs

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Abstract

Dynamic programming algorithms are developed for computing the ask and bid prices of American contingent claims in a binary tree setting in the presence of small proportional transaction costs, extending the recursive construction of the Snell envelope. Associated with the pricing algorithms are iterative procedures for computing optimal hedging strategies for the writer as well as for the buyer of an American option. The bid and ask prices of an American option are represented in terms of the expectation of the option payoff evaluated at an optimal stopping time with respect to an optimal martingale probability measure. As a by-product a similar dynamic programming algorithm is obtained for pricing and hedging European contingent claims in the same setting.

Key words: American options, bid-ask spread, transaction costs, dynamic programming, Snell envelope.

1 Introduction

The pricing and hedging of European options and conditions for the lack of arbitrage under proportional transactions costs have received considerable attention in recent years. In a discrete time setting questions of this kind were studied, for example, by Merton [Mer90], Dermody and Rockafellar [DR91], Boyle and Vorst [BV92], Bensaid, Lesne, Pagès and Scheinkman [BLPS92], Edirsinghe, Naik and Uppal [ENU93], Jouini and Kallal [JK95], Kusuoka [Kus95], Naik [Nai95], Shirakawa and Konno [SK95], Koehl, Pham and Touzi [KPT96], [KPT99], [KPT01], Stettner [Ste97], [Ste00], Perrakis and Lefoll [PL97], Rutkowski [Rut98], Touzi [Tou99], Jouini [Jou00], Ortu [Ort01], Palmer [Pal01a], [Pal01b], Kociński [Koc04], and others.

As compared to European options, rather less is known about the pricing and hedging of American type derivative securities under proportional transaction costs. In discrete time Mercurio and Vorst [MV97] used a local risk minimisation criterion to obtain option price bounds. Kociński [Koc99] proved the existence of a strictly replicating strategy for the seller of an American contingent claim in the binary tree model and gave sufficient conditions for the existence of an optimal replicating strategy. In [Koc01] he obtained a lower bound for the seller's price in a general discrete time model, and expressed the seller's price of an attainable American option as the cost of a replicating strategy under small transaction costs. Perrakis and Lefoll [PL00], [PL04] presented a procedure for computing the ask and bid prices and optimal super-hedging strategies for American calls on stocks paying known dividends and for American puts in the binary tree model, subject to certain restrictions on the model parameters. Chalasani and Jha [CJ01] obtained general representations involving so-called randomised stopping times for the ask and bid prices of American contingent claims.

The study of pricing and hedging of contingent claims in discrete time models with transaction costs appears particularly significant because of a number of results according to which the optimal super-hedge in continuous time is the trivial buy-to-hold strategy, which is hardly acceptable in practice; see Soner, Shreve and Cvitanić [SSC95], Cvitanić, Pham and Touzi [CPT99], and Levental and Skorohod [LS97] for European options, and Levental and Skorohod [LS97] and Jakubenas, Levental and Ryznar [JLR03] for American options. Other ways around this difficulty, not pursued here, include the expected utility maximisation approach as, for example, in Hodges and Neuberger [HN89], Davis, Panas, Zariphopoulou [DPZ93], Davis, Zariphopoulou [DZ95], Constantinides and Zariphopoulou [CZ01], and Constantinides and Perrakis [CP02], [CP04], or imperfect hedging with rebalancing the portfolio at discrete times only, as in Leyland [Lel85], Hoggard, Whalley and Wilmott [HWW94], Kabanov and Safarian [KS97], and others.

In the present paper dynamic programming algorithms are developed for the pricing and hedging of American contingent claims in a binary tree setting in the presence of small proportional costs on transactions in the underlying security (or, equivalently, small bid-ask spreads). The pricing algorithms, which turn out somewhat different for the ask and bid option prices, can be viewed as extensions of the recursive construction of the Snell envelope in the friction free case. Moreover, representations of the ask price (seller's price, also known as the upper hedging price) and the bid price (buyer's price, or lower hedging price) of an American contingent claim Y as, respectively,

$$\pi^a(Y) = \max_{\tau} \max_{\mathbb{P}} \mathbb{E}(Y_\tau), \quad \pi^b(y) = \max_{\tau} \min_{\mathbb{P}} \mathbb{E}(Y_\tau) \tag{1.1}$$

over stopping times τ and (suitably defined as in Jouini and Kallal [JK95]) martingale probabilities $\mathbb{P}$, with $\mathbb{E}$ being the expectation under $\mathbb{P}$, are established.

As a by-product we also construct a dynamic programming type algorithm for pricing and hedging European options in the same setting, and reprove, by a new method based on the algorithm, the known representations

$$\pi^a(Y) = \max_{\mathbb{P}} \mathbb{E}(X), \quad \pi^b(y) = \min_{\mathbb{P}} \mathbb{E}(X)$$

for the ask and bid prices of a European contingent claim X in our setting.

The representations (1.1) of the ask and bid prices, while resembling those in Chalasani and Jha [CJ01], differ significantly in that standard stopping times rather than randomised ones are used. This is contrary to a suggestion in [CJ01] that randomised stopping times are required to represent American option prices under proportional transaction costs.

Representations (1.1) are also very similar to those for the ask and bid American option prices in incomplete models (with no transactions costs or any other kind of friction) to be found in Harrison and Kreps [HK79] in continuous time and in Föllmer and Schied [FS02] in discrete time. However, in contrast to a friction free but possibly incomplete market, where

$$\max_{\tau} \min_{\mathbb{P}} \mathbb{E}(Y_\tau) = \min_{\mathbb{P}} \max_{\tau} \mathbb{E}(Y_\tau) \tag{1.2}$$

(see Theorem 6.41 in [FS02]), if transaction costs are present, then the minimax property (1.2) may be violated. The bid price of an American option is then given by the left-hand side of (1.2), but not necessarily by the right-hand side.

If there are no transaction costs, then our algorithms for computing the ask and bid prices of an American option with payoff process Y reduce to the standard construction of the Snell envelope Z of Y by backward recursion. In the presence of transaction costs Algorithm 4.1 for the ask price still resembles the Snell envelope construction, but an interesting novel feature appears: In the presence of transaction costs it becomes necessary to keep track of two quantities Z_t^a and Z_t^b at each node of the tree at any time t prior to option expiry, instead of the single quantity $Z_t = \max\{Y_t, \mathbb{E}_t(Z_{t+1})\}$ defining the Snell envelope. Algorithm 4.2 for the bid price is slightly more complicated and the analogy with the Snell envelope a little harder to see. Once again, there are two quantities U_t^a and U_t^b to keep track of, which can be seen as analogues of the value of continuation $\mathbb{E}_t(Z_{t+1})$ in the standard Snell envelope construction. (Here $\mathbb{E}_t$ denotes the conditional risk neutral expectation at time t.)

The paper is organised as follows: In Section 2 we specify the model with proportional transaction costs, discuss the no-arbitrage conditions, recall some basic definitions, notation, and facts, and state the small transaction costs assumption. Algorithm 3.1 for European options is considered in Section 3, not just as a simple special case, but also because the results are needed later in Lemma 4.7. Section 4 contains the main results of the paper, namely Algorithms 4.1 and 4.2 for computing the ask and bid prices of an American contingent claim, along with Theorems 4.4 and 4.8, which establish the correctness of the algorithms and provide various representations for the ask and bid prices. Finally, following some concluding remarks in Section 5, we provide a couple of auxiliary technical propositions in the Appendix.

2 Small Proportional Transaction Costs

2.1 Model Specifications

We adopt the binary tree model with trading times $t = 0, \dots, T$ for some fixed positive integer T. The corresponding probability space Ω consists of sequences $\omega^1\omega^2 \cdots \omega^T$ with $\omega^1, \dots, \omega^T \in \{\mathrm{u}, \mathrm{d}\}$, where u and d stand for *up* and *down*. We take $\mathcal{F}$ to be the σ-field consisting of all subsets of Ω, and $\mathbb{Q}$ to be a probability measure on $\mathcal{F}$ such that $\mathbb{Q}\{\omega\} > 0$ for each $\omega \in \Omega$.

A *node* $\omega_t = \omega^1\omega^2 \cdots \omega^t$ of the tree at time $t = 0, \dots, T$, with $\omega^1, \dots, \omega^t \in \{\mathrm{u}, \mathrm{d}\}$, will be identified with the event $\{\eta \in \Omega \mid \eta^1 = \omega^1, \dots, \eta^t = \omega^t\}$. In particular, ω_0 will be identified with Ω. The family of all nodes ω_t at time t will be denoted by Ω_t. We take a filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$, where $\mathcal{F}_t$ is the σ-field generated by the family Ω_t for each $t = 0, \dots, T$. We shall often identify $\mathcal{F}_t$-measurable random variables on Ω with random variables on Ω_t.

The market model will consist of a risky and a risk-free security, a *stock* and a *bond*. Trading in stock is subject to proportional transaction costs. A share can be bought for the *ask* price S^a_t or sold for the *bid* price S^b_t at any time $t = 0, \dots, T$, both price processes S^a and S^b being adapted to the filtration. For any $t = 0, \dots, T-1$ and any node $\omega_t \in \Omega_t$ the corresponding single-step subtree of stock prices can be depicted as

$$
\begin{array}{ccc}
 & & \begin{array}{c} S^a_{t+1}(\omega_t\mathrm{u}) \\ S^b_{t+1}(\omega_t\mathrm{u}) \end{array} \\
\begin{array}{c} S^a_t(\omega_t) \\ S^b_t(\omega_t) \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\
 & & \begin{array}{c} S^a_{t+1}(\omega_t\mathrm{d}) \\ S^b_{t+1}(\omega_t\mathrm{d}) \end{array}
\end{array}
$$

Throughout this paper we shall work under the following assumption of *small transaction costs*, which simply means that the bid-ask spread intervals at each node do not overlap in any single-step tree fragment as above.

Assumption (small transaction costs) *For each* $t = 0, \dots, T-1$ *and each* $\omega_t \in \Omega_t$

$$S^b_{t+1}(\omega_t\mathrm{d}) \le S^a_{t+1}(\omega_t\mathrm{d}) < S^b_t(\omega_t) \le S^a_t(\omega_t) < S^b_{t+1}(\omega_t\mathrm{u}) \le S^a_{t+1}(\omega_t\mathrm{u}). \quad (2.3)$$

Without loss of generality we shall assume the bond to be a risk-free security with zero interest rate, the bond price being 1 for all $t = 0, \dots, T$. Equivalently, all prices can be regarded as discounted prices.

Jouini and Kallal [JK95] have studied general conditions for the lack of arbitrage in a model with proportional transaction costs. It follows directly from their result, Theorem 2.1 below, that the small transaction costs assumption admits no arbitrage opportunities in the market model.

2.2 Self-Financing Strategies, Arbitrage Opportunities, Martingale Measures

By a *self-financing strategy* we shall understand a pair (α, β) of predictable processes α_t, β_t for $t = 0, \ldots, T$ describing the positions in cask and stock such that $\beta_0 = 0$ and

$$\alpha_{t+1} - \alpha_t \leq -(\beta_{t+1} - \beta_t)^+ S_t^a + (\beta_{t+1} - \beta_t)^- S_t^b \tag{2.4}$$

for each $t = 0, \ldots, T-1$. The set of all self-financing strategies will be denoted by $\Phi(S^a, S^b)$.

The time $t = 0, \ldots, T$ liquidation value of a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ will be defined as

$$\vartheta_t(\alpha, \beta) = \alpha_t + \beta_t^+ S_t^b - \beta_t^- S_t^a.$$

We can also refer to $\vartheta_T(\alpha, \beta)$ as the terminal value of the strategy.

By an *arbitrage opportunity* we understand a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ with $\alpha_0 \leq 0$ and terminal value $\vartheta_T(\alpha, \beta) \geq 0$ such that $\mathbb{Q}\{\vartheta_T(\alpha, \beta) > 0\} > 0$.

Following Jouini and Kallal [JK95], we say that a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ is a *martingale measure* if there is a martingale S under $\mathbb{P}$ such that $S_t^b \leq S_t \leq S_t^a$ for each $t = 0, \ldots, T$. The set of such martingale measures $\mathbb{P}$ will be denoted by $\mathcal{P}$.

The following result, obtained by Jouini and Kallal [JK95], who used a slightly different notion of arbitrage, referred to as 'free lunch' in their work, is also valid under the above definition of an arbitrage opportunity in the present setting, see Tokarz [Tok04].

Theorem 2.1 (Jouini and Kallal [JK95]) *There is no arbitrage opportunity if and only if $\mathcal{P}$ is non-empty.*

In particular, it follows immediately that the small transaction costs assumption (2.3) admits no arbitrage opportunity.

We conclude this section with a simple property of self-financing strategies, which will prove useful later on. By $\mathcal{T}$ we denote the set of all stopping times τ such that $0 \leq \tau \leq T$.

Lemma 2.2 *If $(\alpha, \beta) \in \Phi(S^a, S^b)$, $\mathbb{P} \in \mathcal{P}$, S is a martingale under $\mathbb{P}$ such that $S^b \leq S \leq S^a$, and $\mathbb{E}$ denotes the expectation under $\mathbb{P}$, then*

(a) *$\alpha + \beta S$ is a supermartingale under $\mathbb{P}$;*

(b) *$\mathbb{E}(\vartheta_\tau(\alpha, \beta)) \leq \alpha_0$ for each stopping time $\tau \in \mathcal{T}$.*

Proof **(a)** The self-financing condition (2.4) together with $S_t^b \leq S_t \leq S_t^a$ give

$$\alpha_{t+1} - \alpha_t \leq -(\beta_{t+1} - \beta_t)^+ S_t^a + (\beta_{t+1} - \beta_t)^- S_t^b \leq -(\beta_{t+1} - \beta_t) S_t.$$

As a result, for each $t = 0, \ldots, T-1$

$$\mathbb{E}(\alpha_{t+1} + \beta_{t+1} S_{t+1} | \mathcal{F}_t) = \alpha_{t+1} + \beta_{t+1} S_t \leq \alpha_t + \beta_t S_t.$$

(b) Since $S^b_\tau \leq S_\tau \leq S^a_\tau$,

$$\vartheta_\tau(\alpha, \beta) = \alpha_\tau + \beta^+_\tau S^b_\tau - \beta^-_\tau S^a_\tau \leq \alpha_\tau + \beta_\tau S_\tau.$$

Because $\alpha + \beta S$ is a supermartingale under $\mathbb{P}$ and $\beta_0 = 0$ it follows that

$$\mathbb{E}(\vartheta_\tau(\alpha, \beta)) \leq \mathbb{E}(\alpha_\tau + \beta_\tau S_\tau) \leq \alpha_0,$$

as required. ■

3 European Options

We begin with the special case of European contingent claims, for which representations of the ask and bid prices in terms of optimal super- and sub-hedging strategies as well as in terms of expectations of the payoff under optimal martingale probability measures (but not in terms of a dynamic programming algorithm) have already been studied by Jouini and Kallal [JK95]. The constructions and results will then be extended to obtain new representations and algorithms for American option ask and bid prices in the presence of small proportional transaction costs. The results of the present section will also be needed later in the proof of Lemma 4.7.

A European contingent claim can be characterised by a random variable X, the *payoff* at expiry time T. The time 0 *ask* and *bid prices* of such an option can be defined, respectively, as

$$\pi^a(X) = \min\{\alpha_0 \,|\, (\alpha, \beta) \in \Phi(S^a, S^b),\ \vartheta_T(\alpha, \beta) \geq X\}, \tag{3.5}$$

$$\pi^b(X) = \max\{-\alpha_0 \,|\, (\alpha, \beta) \in \Phi(S^a, S^b),\ -\vartheta_T(\alpha, \beta) \leq X\}. \tag{3.6}$$

The minimum and maximum are attained because the corresponding sets are closed and, respectively, bounded below and above in the discrete setting. Thus, the ask price $\pi^a(X)$ is the lowest price that the option writer should demand to be able to hedge the position without any risk of loss. The bid price $\pi^b(X)$ is the highest amount that a buyer can raise to pay for the option such that his or her position can be hedged without running any risk of loss. Clearly,

$$\pi^b(X) = -\pi^a(-X). \tag{3.7}$$

Jouini and Kallal [JK95] obtained the following representation for the bid-ask spread of European option prices:

$$[\pi^b(X), \pi^a(X)] = \overline{\{\mathbb{E}(X) \,|\, \mathbb{P} \in \mathcal{P}\}},$$

where $\mathbb{E}$ is the expectation under $\mathbb{P}$ and where $\overline{A}$ denotes the closure of a set $A \subset \mathbb{R}$. In particular, it follows that

$$\pi^a(X) = \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(X), \tag{3.8}$$

$$\pi^b(X) = \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(X). \tag{3.9}$$

Here the maximum and minimum are attained because $\mathcal{P}$ is a compact set in the discrete setting under the small transaction costs assumption (2.3), and $\mathbb{P} \mapsto \mathbb{E}(X)$ is a continuous function.

3.1 Algorithm

Neither the representations (3.5), (3.6) nor (3.8), (3.9) are particularly effective for computing the ask or bid prices of a European option. They involve the solution of complex multidimensional optimisation problems. (Note that (3.8), (3.9) are not even linear optimisation problems because $\mathcal{P}$ is not convex, in general.) Here we put forward a dynamic programming type recursive algorithm to compute these prices quickly and efficiently. This will then be extended to the case of American options.

The following assertion, which will be used in the proof of the correctness of the algorithm, follows directly from the results in Jouini and Kallal [JK95]. We state and prove it here nevertheless so as to keep this paper self-contained.

Lemma 3.1 *For any European option X*

$$\pi^b(X) \le \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(X) \le \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(X) \le \pi^a(X).$$

Proof The middle inequality is obvious.

Take a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\pi^a(X) = \alpha_0$ and $X \le \vartheta_T(\alpha, \beta)$. By Lemma 2.2, for each $\mathbb{P} \in \mathcal{P}$

$$\mathbb{E}(X) \le \mathbb{E}(\vartheta_T(\alpha, \beta)) \le \alpha_0 = \pi^a(X),$$

which proves the last inequality.

Now take $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\pi^b(X) = -\alpha_0$ and $-\vartheta_T(\alpha, \beta) \le X$. Thus, again by Lemma 2.2, for each $\mathbb{P} \in \mathcal{P}$

$$\mathbb{E}(X) \ge -\mathbb{E}(\vartheta_T(\alpha, \beta)) \ge -\alpha_0 = \pi^b(X).$$

This proves the first inequality. ■

Next, let us introduce some notation, which will also be used in subsequent sections on American options. For any $u, v, w \in \{a, b\}$, any $t = 0, \ldots, T-1$ and $\omega_t \in \Omega_t$, and any $\mathcal{F}_{t+1}$-measurable $\mathbb{R}^2$-valued random variables $\mathbf{G} = (G^a, G^b)$ and $\mathbf{H} = (H^a, H^b)$ we put

$$\mathbb{E}_t^{uvw}(\mathbf{G}; \mathbf{H}|\omega_t) = p_t^{uvw}(\omega_t) G^v(\omega_t \mathrm{u}) + (1 - p_t^{uvw}(\omega_t)) H^w(\omega_t \mathrm{d}),$$

$$p_t^{uvw}(\omega_t) = \frac{S_t^u(\omega_t) - S_{t+1}^w(\omega_t \mathrm{d})}{S_{t+1}^v(\omega_t \mathrm{u}) - S_{t+1}^w(\omega_t \mathrm{d})}.$$

We shall write $\mathbb{E}_t^{uvw}(\mathbf{G};\mathbf{H})$ to denote the $\mathcal{F}_t$-measurable random variable $\omega_t \mapsto \mathbb{E}_t^{uvw}(\mathbf{G};\mathbf{H}|\omega_t)$.

This notation is slightly more complicated than necessary in the case of European options, for which we shall always take $\mathbf{G} = \mathbf{H}$. However it will become necessary to allow $\mathbf{G} \neq \mathbf{H}$ when discussing the bid price of an American option later on.

Algorithm 3.1 (ask price of European option) Given a European option with payoff X and expiry time T, an $\mathbb{R}^2$-valued process $\mathbf{Z} = (Z^a, Z^b)$ is constructed by backward induction as follows:

1. Put
$$Z_T^a = Z_T^b = X;$$

2. For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ take
$$Z_{t-1}^u = \max_{v,w \in \{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t). \tag{3.10}$$

Claim 3.1 The ask price of the European option is given by
$$\pi^a(X) = \max\{Z_0^a, Z_0^b\}.$$

The claim will be verified in what follows, see Theorem 3.4.

Remark 3.1 (bid price of European option) Due to (3.7) Algorithm 3.1 can also be used to compute the bid price $\pi^b(X)$ of a European option.

The proof of correctness of Algorithm 3.1 will involve a number objects that need to be constructed. We begin by constructing processes $\hat{S}, \hat{Z}$ such that:

1. For some $u \in \{a, b\}$
$$\hat{S}_0 = S_0^u,$$
$$\hat{Z}_0 = Z_0^u,$$
and
$$Z_0^u = \max\{Z_0^a, Z_0^b\}. \tag{3.11}$$

2. For each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$ there are $v, w \in \{a, b\}$ such that
$$\hat{S}_{t+1}(\omega_t\mathrm{u}) = S_{t+1}^v(\omega_t\mathrm{u}), \quad \hat{S}_{t+1}(\omega_t\mathrm{d}) = S_{t+1}^w(\omega_t\mathrm{d}),$$
$$\hat{Z}_{t+1}(\omega_t\mathrm{u}) = Z_{t+1}^v(\omega_t\mathrm{u}), \quad \hat{Z}_{t+1}(\omega_t\mathrm{d}) = Z_{t+1}^w(\omega_t\mathrm{d}),$$
and
$$Z_t^u(\omega_t) = \mathbb{E}_t^{uvw}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\omega_t) \tag{3.12}$$
for each $u \in \{a, b\}$. Such v, w exist by Proposition 6.1.

Remark 3.2 The processes $\hat{S}, \hat{Z}$ may not be unique. The lack of uniqueness may arise whenever there is more than one pair $v, w \in \{a, b\}$ satisfying (3.12), or there is more then one $u \in \{a, b\}$ such that (3.11) holds. In such cases one can choose any $\hat{S}, \hat{Z}$ satisfying the conditions above.

Let $\hat{\mathbb{P}}$ be the probability measure turning $\hat{S}$ into a martingale. It is well defined and equivalent to $\mathbb{Q}$ because, by the small transaction costs assumption (2.3), $\hat{S}_{t+1}(\omega_t \mathrm{u}) > \hat{S}_t(\omega_t) > \hat{S}_{t+1}(\omega_t \mathrm{d})$ for each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$. We denote by $\hat{\mathbb{E}}$ the expectation under $\hat{\mathbb{P}}$.

Lemma 3.2 **(a)** *$\hat{Z}$ is a martingale under $\hat{\mathbb{P}}$;*

(b) $\hat{\mathbb{E}}(X) = \hat{Z}_0$.

Proof **(a)** Take any $t = 0, \ldots, T-1$ and any $\omega_t \in \Omega_t$. By the construction of $\hat{S}, \hat{Z}$ there are $u, v, w \in \{a, b\}$ such that

$$\begin{aligned} &\hat{S}_t(\omega_t) = S_t^u(\omega_t), \quad \hat{S}_{t+1}(\omega_t \mathrm{u}) = S_{t+1}^v(\omega_t \mathrm{u}), \quad \hat{S}_{t+1}(\omega_t \mathrm{d}) = S_{t+1}^w(\omega_t \mathrm{d}), \\ &\hat{Z}_t(\omega_t) = Z_t^u(\omega_t), \quad \hat{Z}_{t+1}(\omega_t \mathrm{u}) = Z_{t+1}^v(\omega_t \mathrm{u}), \quad \hat{Z}_{t+1}(\omega_t \mathrm{d}) = Z_{t+1}^w(\omega_t \mathrm{d}), \end{aligned}$$

and (3.12) holds. Thus

$$\hat{Z}_t(\omega_t) = Z_t^u(\omega_t) = \mathbb{E}_t^{uvw}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\omega_t) = \hat{\mathbb{E}}(\hat{Z}_{t+1}|\omega_t).$$

(b) Since $Z_T^a = Z_T^b = X$, it follows that $\hat{Z}_T = X$. Because $\hat{Z}$ is a martingale under $\hat{\mathbb{P}}$ we obtain $\hat{\mathbb{E}}(X) = \hat{\mathbb{E}}(\hat{Z}_T) = \hat{Z}_0$. ■

Next, let $\hat{\alpha}, \hat{\beta}$ be predictable processes such that $\hat{\beta}_0 = 0$ and for each $t = 0, \ldots, T$

$$\hat{\alpha}_t + \hat{\beta}_t \hat{S}_t = \hat{Z}_t. \tag{3.13}$$

Since $\hat{S}$ and $\hat{Z}$ are martingales under $\hat{\mathbb{P}}$, it follows that for each $t = 0, \ldots, T-1$

$$\begin{aligned} \hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{S}_t &= \hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{\mathbb{E}}(\hat{S}_{t+1}|\mathcal{F}_t) \\ &= \hat{\mathbb{E}}(\hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{S}_{t+1}|\mathcal{F}_t) = \hat{\mathbb{E}}(\hat{Z}_{t+1}|\mathcal{F}_t) = \hat{Z}_t. \end{aligned} \tag{3.14}$$

Observe that $(\hat{\alpha}, \hat{\beta})$ is the self-financing strategy hedging the European option X in the standard (friction free) model with stock price process $\hat{S}$. Namely $\hat{\alpha}_t + \hat{\beta}_t \hat{S}_t = \hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{S}_t$ for all $t = 0, \ldots, T-1$, and $\hat{\alpha}_T + \hat{\beta}_T \hat{S}_T = X$. Nevertheless, it does not immediately follow that $(\hat{\alpha}, \hat{\beta})$ is also a hedging strategy for X in the model with transaction costs. This will be proved in Lemma 3.3.

Lemma 3.3 **(a)** $(\hat{\alpha}, \hat{\beta}) \in \Phi(S^a, S^b)$.

(b) $\vartheta_T(\hat{\alpha}, \hat{\beta}) = X$.

Proof (a) By (3.13) and (3.14) the self-financing condition (2.4) can be written as

$$(\hat\beta_{t+1} - \hat\beta_t)^+ (S_t^a - \hat S_t) + (\hat\beta_{t+1} - \hat\beta_t)^- (\hat S_t - S_t^b) \le 0$$

or, equivalently, as

$$(\hat\beta_{t+1} - \hat\beta_t)(S_t^r - \hat S_t) \le 0 \quad \forall r \in \{a, b\}. \tag{3.15}$$

We shall prove that $(\hat\alpha, \hat\beta)$ satisfies (3.15) for each $t = 0, \dots, T-1$.

First we shall verify (3.15) for $t = 0$. Consider the tree fragment

$$\begin{array}{ccc} & & \omega_0 \mathrm{u} \\ & \nearrow & \\ \omega_0 & & \\ & \searrow & \\ & & \omega_0 \mathrm{d} \end{array}$$

For brevity, we shall omit ω_0 in the expressions below. By the construction of $\hat S, \hat Z$ there are $u, v, w \in \{a, b\}$ such that at the respective nodes

$$\begin{array}{ccc} & & \hat S_1(\mathrm{u}) = S_1^v(\mathrm{u}) \\ & & \hat Z_1(\mathrm{u}) = Z_1^v(\mathrm{u}) \\ \hat S_0 = S_0^u & \nearrow & \\ \hat Z_0 = Z_0^u & \searrow & \\ & & \hat S_1(\mathrm{d}) = S_1^w(\mathrm{d}) \\ & & \hat Z_1(\mathrm{d}) = Z_1^w(\mathrm{d}) \end{array}$$

and

$$Z_0^u = \mathbb{E}_0^{uvw}(\mathbf{Z}_1; \mathbf{Z}_1).$$

Observe that

$$\hat\beta_1 = \frac{\hat Z_1(\mathrm{u}) - \hat Z_1(\mathrm{d})}{\hat S_1(\mathrm{u}) - \hat S_1(\mathrm{d})} = \frac{Z_1^v(\mathrm{u}) - Z_1^w(\mathrm{d})}{S_1^v(\mathrm{u}) - S_1^w(\mathrm{d})}.$$

By the construction of $\hat Z_0$ we know that $\hat Z_0 = Z_0^u \ge Z_0^r$ for each $r \in \{a, b\}$, and by the construction of Z_0^r in Algorithm 3.1, $Z_0^r \ge \mathbb{E}_0^{rvw}(\mathbf{Z}_1; \mathbf{Z}_1)$. This gives

$$\mathbb{E}_0^{uvw}(\mathbf{Z}_1; \mathbf{Z}_1) \ge \mathbb{E}_0^{rvw}(\mathbf{Z}_1; \mathbf{Z}_1),$$

which can be transformed into

$$\hat\beta_1 (S_0^r - S_0^u) \le 0$$

for each $r \in \{a, b\}$. Because $\hat\beta_0 = 0$ this implies (3.15) for $t = 0$, as required.

Next, we need to demonstrate that $(\hat\alpha, \hat\beta)$ satisfies (3.15) for each $t = 1, \dots, T-1$. Let us take any node $\omega_{t-1} \in \Omega_{t-1}$ and consider the tree fragment

$$\begin{array}{ccccc} & & & & \omega_{t-1}\mathrm{uu} \\ & & & \nearrow & \\ & & \omega_{t-1}\mathrm{u} & & \\ & & & \searrow & \\ & \nearrow & & & \omega_{t-1}\mathrm{ud} \\ \omega_{t-1} & & & & \\ & \searrow & & & \\ & & \omega_{t-1}\mathrm{d} & & \end{array}$$

We shall verify (3.15) at node $\omega_{t-1}\mathrm{u}$. For brevity, ω_{t-1} will be omitted in the expressions below. By the construction of $\hat{S}, \hat{Z}$ there are $u, v, w, g, h \in \{a, b\}$ such that at the respective nodes

$$
\begin{array}{ccccc}
 & & & & \begin{array}{c}\hat{S}_{t+1}(\mathrm{uu}) = S^g_{t+1}(\mathrm{uu})\\ \hat{Z}_{t+1}(\mathrm{uu}) = Z^g_{t+1}(\mathrm{uu})\end{array}\\
 & & \begin{array}{c}\hat{S}_t(\mathrm{u}) = S^v_t(\mathrm{u})\\ \hat{Z}_t(\mathrm{u}) = Z^v_t(\mathrm{u})\end{array} & \begin{array}{c}\nearrow\\ \searrow\end{array} & \\
\begin{array}{c}\hat{S}_{t-1} = S^u_{t-1}\\ \hat{Z}_{t-1} = Z^u_{t-1}\end{array} & \begin{array}{c}\nearrow\\ \searrow\end{array} & & & \begin{array}{c}\hat{S}_{t+1}(\mathrm{ud}) = S^h_{t+1}(\mathrm{ud})\\ \hat{Z}_{t+1}(\mathrm{ud}) = Z^h_{t+1}(\mathrm{ud})\end{array}\\
 & & \begin{array}{c}\hat{S}_t(\mathrm{d}) = S^w_t(\mathrm{d})\\ \hat{Z}_t(\mathrm{d}) = Z^w_t(\mathrm{d})\end{array} & &
\end{array}
$$

and

$$Z^u_{t-1} = \mathbb{E}^{uvw}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t), \quad Z^v_t(\mathrm{u}) = \mathbb{E}^{vgh}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u}).$$

Observe that

$$\begin{aligned}
\hat{\beta}_t &= \frac{\hat{Z}_t(\mathrm{u}) - \hat{Z}_t(\mathrm{d})}{\hat{S}_t(\mathrm{u}) - \hat{S}_t(\mathrm{d})} = \frac{Z^v_t(\mathrm{u}) - Z^w_t(\mathrm{d})}{S^v_t(\mathrm{u}) - S^w_t(\mathrm{d})},\\
\hat{\beta}_{t+1}(\mathrm{u}) &= \frac{\hat{Z}_{t+1}(\mathrm{uu}) - \hat{Z}_{t+1}(\mathrm{ud})}{\hat{S}_{t+1}(\mathrm{uu}) - \hat{S}_{t+1}(\mathrm{ud})} = \frac{Z^g_{t+1}(\mathrm{uu}) - Z^h_{t+1}(\mathrm{ud})}{S^g_{t+1}(\mathrm{uu}) - S^h_{t+1}(\mathrm{ud})}.
\end{aligned}$$

By the construction of Z^u_{t-1} in Algorithm 3.1, for any $r \in \{a, b\}$

$$\mathbb{E}^{uvw}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}^{urw}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t),$$

which can be transformed into

$$\hat{\beta}_t\,(S^r_t(\mathrm{u}) - S^v_t(\mathrm{u})) \geq Z^r_t(\mathrm{u}) - Z^v_t(\mathrm{u}).$$

Furthermore, for any $r \in \{a, b\}$

$$Z^r_t(\mathrm{u}) \geq \mathbb{E}^{rgh}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u})$$

by the construction of $Z^r_t(\mathrm{u})$ in Algorithm 3.1. As a result,

$$\begin{aligned}
Z^r_t(\mathrm{u}) - Z^v_t(\mathrm{u}) &\geq \mathbb{E}^{rgh}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u}) - \mathbb{E}^{vgh}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u})\\
&= \hat{\beta}_{t+1}(\mathrm{u})\,(S^r_t(\mathrm{u}) - S^v_t(\mathrm{u})).
\end{aligned}$$

It follows that

$$(\hat{\beta}_{t+1}(\mathrm{u}) - \hat{\beta}_t)(S^r_t(\mathrm{u}) - S^v_t(\mathrm{u})) \leq 0$$

for any $r \in \{a, b\}$, that is, (3.15) holds at node $\omega_{t-1}\mathrm{u}$, as required. The argument to verify (3.15) at node $\omega_{t-1}\mathrm{d}$ is similar.

The above proves (3.15) and therefore (2.4) at all nodes for all times $t = 0, \ldots, T-1$.

(b) Since $\hat{\alpha}_T + \hat{\beta}_T \hat{S}_T = \hat{Z}_T = X$, the condition $\vartheta_T(\hat{\alpha}, \hat{\beta}) = X$ can be written as

$$\hat{\beta}_T^+(\hat{S}_T - S_T^b) + \hat{\beta}_T^-(S_T^a - \hat{S}_T) = 0,$$

or, equivalently, as

$$\hat{\beta}_T(S_T^r - \hat{S}_T) \geq 0 \quad \forall r \in \{a, b\}. \tag{3.16}$$

Take any $\omega_{T-1} \in \Omega_{T-1}$ and consider the tree fragment

$$\begin{array}{ccc} & & \omega_{T-1}\mathrm{u} \\ & \nearrow & \\ \omega_{T-1} & & \\ & \searrow & \\ & & \omega_{T-1}\mathrm{d} \end{array}$$

We shall verify (3.16) at node $\omega_{T-1}\mathrm{u}$. For brevity, ω_{T-1} will be omitted in the expressions below. By the construction of $\hat{S}, \hat{Z}$ there are $u, v, w \in \{a, b\}$ such that at the respective nodes

$$\begin{array}{ccc} & & \begin{array}{c} \hat{S}_T(\mathrm{u}) = S_T^v(\mathrm{u}) \\ \hat{Z}_T(\mathrm{u}) = Z_T^v(\mathrm{u}) \end{array} \\ \begin{array}{c} \hat{S}_{T-1} = S_{T-1}^u \\ \hat{Z}_{T-1} = Z_{T-1}^u \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & \begin{array}{c} \hat{S}_T(\mathrm{d}) = S_T^w(\mathrm{d}) \\ \hat{Z}_T(\mathrm{d}) = Z_T^w(\mathrm{d}) \end{array} \end{array}$$

and

$$Z_{T-1}^u = \mathbb{E}_{T-1}^{uvw}(\mathbf{Z}_T; \mathbf{Z}_T).$$

Observe that

$$\hat{\beta}_T = \frac{\hat{Z}_T(\mathrm{u}) - \hat{Z}_T(\mathrm{d})}{\hat{S}_T(\mathrm{u}) - \hat{S}_T(\mathrm{d})} = \frac{Z_T^v(\mathrm{u}) - Z_T^w(\mathrm{d})}{S_T^v(\mathrm{u}) - S_T^w(\mathrm{d})}.$$

By the construction of Z_{T-1}^u in Algorithm 3.1, for any $r \in \{a, b\}$

$$\mathbb{E}_{T-1}^{uvw}(\mathbf{Z}_T; \mathbf{Z}_T) \geq \mathbb{E}_{T-1}^{urw}(\mathbf{Z}_T; \mathbf{Z}_T),$$

which can be transformed into

$$\hat{\beta}_T \left(S_T^r(\mathrm{u}) - S_T^v(\mathrm{u}) \right) \geq 0$$

using the fact that $Z_T^r(\mathrm{u}) = Z_T^v(\mathrm{u}) = X(\mathrm{u})$. This implies (3.16) and therefore $\vartheta_T(\hat{\alpha}, \hat{\beta}) = X$ at node $\omega_{T-1}\mathrm{u}$, as required. The argument to verify $\vartheta_T(\hat{\alpha}, \hat{\beta}) = X$ at node $\omega_{T-1}\mathrm{d}$ is similar. This, then, covers every node at time T. ■

Theorem 3.4 *In a model with proportional transaction costs subject to the small transaction costs assumption* (2.3) *the ask price $\pi^a(X)$ of a European option X can be represented as*

$$\pi^a(X) = \hat{\alpha}_0 = \hat{Z}_0 = \max\{Z_0^a, Z_0^b\} = \hat{\mathbb{E}}(X) = \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}(X).$$

In particular, this implies the correctness of Algorithm 3.1 for computing $\pi^a(X)$ asserted in Claim 3.1.

Proof Observe that $\pi^a(X) \leq \hat{\alpha}_0$ by the definition (3.5) of $\pi^a(X)$ and by Lemma 3.3. Furthermore, $\hat{\alpha}_0 = \hat{Z}_0$ by the construction of $\hat{\alpha}_0$, $\hat{Z}_0 = \hat{\mathbb{E}}(X)$ by Lemma 3.2, $\hat{\mathbb{E}}(X) \leq \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(X)$ because $\hat{\mathbb{P}} \in \mathcal{P}$, and $\max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(X) \leq \pi^a(X)$ by Lemma 3.1. Finally, $\hat{Z}_0 = \max\{Z_0^a, Z_0^b\}$ by the construction of $\hat{Z}$, completing the proof. ■

4 American Options

An American contingent claim can be characterised by an adapted process Y, where Y_t is the payoff at time $t = 0, \ldots, T$.

The time 0 *ask* and *bid prices* of an American option with payoff process Y are defined as

$$\pi^a(Y) = \min\{\alpha_0 \mid (\alpha, \beta) \in \Phi(S^a, S^b),\ \forall\, \tau \in \mathcal{T} : \vartheta_\tau(\alpha, \beta) \geq Y_\tau\}, \tag{4.17}$$

$$\pi^b(Y) = \max\{-\alpha_0 \mid (\alpha, \beta) \in \Phi(S^a, S^b),\ \exists\, \tau \in \mathcal{T} : -\vartheta_\tau(\alpha, \beta) \leq Y_\tau\}, \tag{4.18}$$

where $\mathcal{T}$ is the set of all stopping times τ such that $0 \leq \tau \leq T$. The minimum and maximum are attained because the corresponding sets are closed and, respectively, bounded below and above in the discrete setting. Similarly as for European options, the ask price $\pi^a(Y)$ is the lowest price the writer of the option should demand, and the bid price $\pi^b(Y)$ is the highest amount an option buyer could raise to pay for the option so as to be able to hedge their respective positions without any risk of loss.

We shall obtain dynamic programming type algorithms for computing the American option ask and bid prices $\pi^a(Y)$ and $\pi^b(Y)$ in a market with small transaction costs, that is, subject to assumption (2.3). The algorithms can be viewed as extensions of the Snell envelope construction from the case without friction. We shall also construct hedging strategies for both the writer and the buyer of an American option under small transaction costs. Moreover, representations for the American option ask and bid prices in terms of optimal martingale measures and stopping times will be established.

We begin with the following assertion, extending Lemma 3.1 to American options.

Lemma 4.1 *For any American option Y*

$$\pi^b(Y) \leq \max_{\tau\in\mathcal{T}} \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(Y_\tau) \leq \max_{\tau\in\mathcal{T}} \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(Y_\tau) \leq \pi^a(Y).$$

Proof The middle inequality is obvious.

Take a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\pi^a(Y) = \alpha_0$ and $Y_\tau \leq \vartheta_\tau(\alpha, \beta)$ for all $\tau \in \mathcal{T}$. By Lemma 2.2, for each $\tau \in \mathcal{T}$ and each $\mathbb{P} \in \mathcal{P}$

$$\mathbb{E}(Y_\tau) \leq \mathbb{E}(\vartheta_\tau(\alpha, \beta)) \leq \alpha_0 = \pi^a(Y),$$

which proves the last inequality.

Now let us take a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\pi^b(Y) = -\alpha_0$ and $-\vartheta_\tau(\alpha, \beta) \leq Y_\tau$ for some stopping time $\tau \in \mathcal{T}$. Then, again by Lemma 2.2,

$$\mathbb{E}(Y_\tau) \geq -\mathbb{E}(\vartheta_\tau(\alpha, \beta)) \geq -\alpha_0 = \pi^b(Y)$$

for each $\mathbb{P} \in \mathcal{P}$. This proves the first inequality. ■

4.1 Ask Price Algorithm

We propose the following algorithm for computing the ask price $\pi^a(Y)$ of an American option Y under the small transaction costs assumption (2.3).

Algorithm 4.1 (ask price of American option) Given an American option with payoff process Y and expiry time T, construct an $\mathbb{R}^2$-valued process $\mathbf{Z} = (Z^a, Z^b)$ by backward induction as follows:

1. Put
$$Z_T^a = Z_T^b = Y_T;$$

2. For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ take
$$Z_{t-1}^u = \max\{Y_{t-1}, \max_{v,w \in \{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t)\}.$$

Claim 4.1 The ask price of the American option is given by

$$\pi^a(Y) = \max\{Z_0^a, Z_0^b\}.$$

The claim will be verified in Theorem 4.4.

For each $u \in \{a, b\}$ and $t = 0, \ldots, T$ we put

$$V_t^u = \begin{cases} \max_{v,w \in \{a,b\}} \mathbb{E}_t^{uvw}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}) & \text{if } t < T, \\ Y_T & \text{if } t = T, \end{cases} \tag{4.19}$$

so that $Z_t^u = \max\{Y_t, V_t^u\}$, and we construct processes $\hat{S}, \hat{V}, \hat{Z}$ such that:

1. For some $u \in \{a, b\}$
$$\hat{S}_0 = S_0^u,$$
$$\hat{V}_0 = V_0^u,$$
$$\hat{Z}_0 = Z_0^u,$$

   and
$$V_0^u = \max\{V_0^a, V_0^b\}. \tag{4.20}$$

2. For each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$ there are $v, w \in \{a, b\}$ such that

$$\begin{array}{ll} \hat{S}_{t+1}(\omega_t \mathrm{u}) = S^v_{t+1}(\omega_t \mathrm{u}), & \hat{S}_{t+1}(\omega_t \mathrm{d}) = S^w_{t+1}(\omega_t \mathrm{d}), \\ \hat{V}_{t+1}(\omega_t \mathrm{u}) = V^v_{t+1}(\omega_t \mathrm{u}), & \hat{V}_{t+1}(\omega_t \mathrm{d}) = V^w_{t+1}(\omega_t \mathrm{d}), \\ \hat{Z}_{t+1}(\omega_t \mathrm{u}) = Z^v_{t+1}(\omega_t \mathrm{u}), & \hat{Z}_{t+1}(\omega_t \mathrm{d}) = Z^w_{t+1}(\omega_t \mathrm{d}), \end{array}$$

   and

$$V^u_t(\omega_t) = \mathbb{E}^{uvw}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1} | \omega_t) \tag{4.21}$$

   for each $u \in \{a, b\}$. Such v, w exist by Proposition 6.1.

Remark 4.1 The processes $\hat{S}, \hat{V}, \hat{Z}$ may not be unique. The lack of uniqueness may arise whenever there is more than one pair $v, w \in \{a, b\}$ satisfying (4.21), or there is more then one $u \in \{a, b\}$ such that (4.20) holds. In such cases one can choose any $\hat{S}, \hat{V}, \hat{Z}$ satisfying the conditions above.

Let $\hat{\mathbb{P}}$ be the probability measure turning $\hat{S}$ into a martingale. It is well defined and equivalent to $\mathbb{Q}$ because, by the small transaction costs assumption (2.3), $\hat{S}_{t+1}(\omega_t \mathrm{u}) > \hat{S}_t(\omega_t) > \hat{S}_{t+1}(\omega_t \mathrm{d})$ for each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$. We denote by $\hat{\mathbb{E}}$ the expectation under $\hat{\mathbb{P}}$. We also define a stopping time $\hat{\tau} \in \mathcal{T}$ by

$$\hat{\tau} = \min\{t \,|\, \hat{Z}_t = Y_t\}.$$

Lemma 4.2 **(a)** *$\hat{V}_t = \hat{\mathbb{E}}(\hat{Z}_{t+1} | \mathcal{F}_t)$ for each $t = 0, \ldots, T-1$;*

(b) *$\hat{Z}$ is the Snell envelope of Y under $\hat{\mathbb{P}}$;*

(c) *$\hat{\mathbb{E}}(Y_{\hat{\tau}}) = \hat{Z}_0$.*

Proof **(a)** By the construction of $\hat{S}, \hat{V}, \hat{Z}$, for any $t = 0, \ldots, T-1$ and any $\omega_t \in \Omega_t$ there are $u, v, w \in \{a, b\}$ such that

$$\begin{array}{lll} \hat{S}_t(\omega_t) = S^u_t(\omega_t), & \hat{S}_{t+1}(\omega_t \mathrm{u}) = S^v_{t+1}(\omega_t \mathrm{u}), & \hat{S}_{t+1}(\omega_t \mathrm{d}) = S^w_{t+1}(\omega_t \mathrm{d}), \\ \hat{V}_t(\omega_t) = V^u_t(\omega_t), & \hat{V}_{t+1}(\omega_t \mathrm{u}) = V^v_{t+1}(\omega_t \mathrm{u}), & \hat{V}_{t+1}(\omega_t \mathrm{d}) = V^w_{t+1}(\omega_t \mathrm{d}), \\ \hat{Z}_t(\omega_t) = Z^u_t(\omega_t), & \hat{Z}_{t+1}(\omega_t \mathrm{u}) = Z^v_{t+1}(\omega_t \mathrm{u}), & \hat{Z}_{t+1}(\omega_t \mathrm{d}) = Z^w_{t+1}(\omega_t \mathrm{d}), \end{array}$$

and (4.21) holds. Thus

$$\hat{V}_t(\omega_t) = V^u_t(\omega_t) = \mathbb{E}^{uvw}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1} | \omega_t) = \hat{\mathbb{E}}(\hat{Z}_{t+1} | \omega_t).$$

(b) Because $Z^a_T = Z^b_T = Y_T$, we have

$$\hat{Z}_T = Y_T.$$

Since $Z^u_t = \max\{Y_t, V^u_t\}$ for each u, it follows from (a) that for each $t = 0, \ldots, T-1$

$$\hat{Z}_t = \max\{Y_t, \hat{V}_t\} = \max\{Y_t, \hat{\mathbb{E}}(\hat{Z}_{t+1} | \mathcal{F}_t)\}.$$

(c) Because $\hat{Z}$ is the Snell envelope of Y, we know that the stopped process $\hat{Z}_{\hat{\tau} \wedge t}$ is a martingale under $\hat{\mathbb{P}}$. Since $\hat{Z}_{\hat{\tau}} = Y_{\hat{\tau}}$, it follows that $\hat{\mathbb{E}}(Y_{\hat{\tau}}) = \hat{\mathbb{E}}(\hat{Z}_{\hat{\tau}}) = \hat{Z}_0$. ■

We define $\hat{\alpha}, \hat{\beta}$ to be predictable processes such that $\hat{\beta}_0 = 0$ and for each $t = 0, \ldots, T$

$$\hat{\alpha}_t + \hat{\beta}_t \hat{S}_t = \hat{Z}_t. \tag{4.22}$$

By Lemma 4.2 and since $\hat{S}$ is a martingale under $\hat{\mathbb{P}}$, it follows that for each $t = 0, \ldots, T-1$

$$\begin{aligned} \hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{S}_t &= \hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{\mathbb{E}}(\hat{S}_{t+1} | \mathcal{F}_t) \\ &= \hat{\mathbb{E}}(\hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{S}_{t+1} | \mathcal{F}_t) = \hat{\mathbb{E}}(\hat{Z}_{t+1} | \mathcal{F}_t) = \hat{V}_t. \end{aligned} \tag{4.23}$$

Clearly, $(\hat{\alpha}, \hat{\beta})$ is a hedging strategy for the American option Y in the standard (friction free) model with stock price process $\hat{S}$. That is to say, $\hat{\alpha}_t + \hat{\beta}_t \hat{S}_t \geq \hat{\alpha}_{t+1} + \hat{\beta}_{t+1} \hat{S}_t$ for each $t = 0, \ldots, T-1$, and $\hat{\alpha}_t + \hat{\beta}_t \hat{S}_t \geq Y_t$ for each $t = 0, \ldots, T$. Nevertheless, it does not immediately follow that $(\hat{\alpha}, \hat{\beta})$ is also a hedging strategy for Y in the model with transaction costs. This will be shown in Lemma 4.3.

Lemma 4.3 **(a)** $(\hat{\alpha}, \hat{\beta}) \in \Phi(S^a, S^b)$.

(b) $\vartheta_t(\hat{\alpha}, \hat{\beta}) \geq Y_t$ *for each* $t = 0, \ldots, T$.

Proof **(a)** By (4.22) and (4.23) the self-financing condition (2.4) can be written as

$$(\hat{\beta}_{t+1} - \hat{\beta}_t)^+ (S^a_t - \hat{S}_t) + (\hat{\beta}_{t+1} - \hat{\beta}_t)^- (\hat{S}_t - S^b_t) \leq \hat{Z}_t - \hat{V}_t,$$

or, equivalently, as

$$(\hat{\beta}_{t+1} - \hat{\beta}_t)(S^r_t - \hat{S}_t) \leq \hat{Z}_t - \hat{V}_t \quad \forall r \in \{a, b\}. \tag{4.24}$$

Thus, we need to demonstrate that $(\hat{\alpha}, \hat{\beta})$ satisfies (4.24) for each $t = 0, \ldots, T-1$.

First we shall verify (4.24) for $t = 0$. Consider the tree fragment:

$$\begin{array}{ccc} & & \omega_0\mathrm{u} \\ & \nearrow & \\ \omega_0 & & \\ & \searrow & \\ & & \omega_0\mathrm{d} \end{array}$$

For brevity, we shall omit ω_0 in the expressions below. By the construction of $\hat{S}, \hat{V}, \hat{Z}$ there are $u, v, w \in \{a, b\}$ such that at the respective nodes

$$\begin{array}{ccc} & & \begin{array}{l} \hat{S}_1(\mathrm{u}) = S^v_1(\mathrm{u}) \\ \hat{V}_1(\mathrm{u}) = V^v_1(\mathrm{u}) \\ \hat{Z}_1(\mathrm{u}) = Z^v_1(\mathrm{u}) \end{array} \\ & \nearrow & \\ \begin{array}{l} \hat{S}_0 = S^u_0 \\ \hat{V}_0 = V^u_0 \\ \hat{Z}_0 = Z^u_0 \end{array} & & \\ & \searrow & \\ & & \begin{array}{l} \hat{S}_1(\mathrm{d}) = S^w_1(\mathrm{d}) \\ \hat{V}_1(\mathrm{d}) = V^w_1(\mathrm{d}) \\ \hat{Z}_1(\mathrm{d}) = Z^w_1(\mathrm{d}) \end{array} \end{array}$$

and

$$V_0^u = \mathbb{E}_0^{uvw}(\mathbf{Z}_1; \mathbf{Z}_1).$$

Observe that

$$\hat{\beta}_1 = \frac{\hat{Z}_1(\mathrm{u}) - \hat{Z}_1(\mathrm{d})}{\hat{S}_1(\mathrm{u}) - \hat{S}_1(\mathrm{d})} = \frac{Z_1^v(\mathrm{u}) - Z_1^w(\mathrm{d})}{S_1^v(\mathrm{u}) - S_1^w(\mathrm{d})}.$$

By the construction of $\hat{V}_0$ and by (4.19) we know that $\hat{V}_0 = V_0^u \geq V_0^r \geq \mathbb{E}_0^{rvw}(\mathbf{Z}_1; \mathbf{Z}_1)$ for each $r \in \{a, b\}$. This gives

$$\mathbb{E}_0^{uvw}(\mathbf{Z}_1; \mathbf{Z}_1) \geq \mathbb{E}_0^{rvw}(\mathbf{Z}_1; \mathbf{Z}_1),$$

which can be transformed into

$$\hat{\beta}_1(S_0^r - S_0^u) \leq 0$$

for each $r \in \{a, b\}$. Because $\hat{\beta}_0 = 0$ and $\hat{Z}_0 \geq \hat{V}_0$, this implies (4.24) for $t = 0$, as required.

Next we shall verify (4.24) for any $t = 1, \ldots, T-1$. Let us take any node $\omega_{t-1} \in \Omega_{t-1}$ and consider the tree fragment

$$\begin{array}{ccccc}
 & & & & \omega_{t-1}\mathrm{uu} \\
 & & \omega_{t-1}\mathrm{u} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
\omega_{t-1} & \begin{array}{c}\nearrow \\ \searrow\end{array} & & & \omega_{t-1}\mathrm{ud} \\
 & & \omega_{t-1}\mathrm{d} & &
\end{array}$$

We are going to verify (4.24) at node $\omega_{t-1}\mathrm{u}$. For brevity, we shall omit ω_{t-1} in the expressions to follow. By the construction of $\hat{S}, \hat{V}, \hat{Z}$ there are $u, v, w, g, h \in \{a, b\}$ such that at the respective nodes

$$\begin{array}{ccccc}
 & & & & \begin{array}{c}\hat{S}_{t+1}(\mathrm{uu}) = S_{t+1}^g(\mathrm{uu}) \\ \hat{V}_{t+1}(\mathrm{uu}) = V_{t+1}^g(\mathrm{uu}) \\ \hat{Z}_{t+1}(\mathrm{uu}) = Z_{t+1}^g(\mathrm{uu})\end{array} \\
 & & \begin{array}{c}\hat{S}_t(\mathrm{u}) = S_t^v(\mathrm{u}) \\ \hat{V}_t(\mathrm{u}) = V_t^v(\mathrm{u}) \\ \hat{Z}_t(\mathrm{u}) = Z_t^v(\mathrm{u})\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
\begin{array}{c}\hat{S}_{t-1} = S_{t-1}^u \\ \hat{V}_{t-1} = V_{t-1}^u \\ \hat{Z}_{t-1} = Z_{t-1}^u\end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & & & \begin{array}{c}\hat{S}_{t+1}(\mathrm{ud}) = S_{t+1}^h(\mathrm{ud}) \\ \hat{V}_{t+1}(\mathrm{ud}) = V_{t+1}^h(\mathrm{ud}) \\ \hat{Z}_{t+1}(\mathrm{ud}) = Z_{t+1}^h(\mathrm{ud})\end{array} \\
 & & \begin{array}{c}\hat{S}_t(\mathrm{d}) = S_t^w(\mathrm{d}) \\ \hat{V}_t(\mathrm{d}) = V_t^w(\mathrm{d}) \\ \hat{Z}_t(\mathrm{d}) = Z_t^w(\mathrm{d})\end{array} & &
\end{array}$$

and

$$V_{t-1}^u = \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t), \quad V_t^v(\mathrm{u}) = \mathbb{E}_t^{vgh}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u}).$$

Observe that

$$\hat{\beta}_t = \frac{\hat{Z}_t(\mathrm{u}) - \hat{Z}_t(\mathrm{d})}{\hat{S}_t(\mathrm{u}) - \hat{S}_t(\mathrm{d})} = \frac{Z_t^v(\mathrm{u}) - Z_t^w(\mathrm{d})}{S_t^v(\mathrm{u}) - S_t^w(\mathrm{d})},$$

$$\hat{\beta}_{t+1}(\mathrm{u}) = \frac{\hat{Z}_{t+1}(\mathrm{uu}) - \hat{Z}_{t+1}(\mathrm{ud})}{\hat{S}_{t+1}(\mathrm{uu}) - \hat{S}_{t+1}(\mathrm{ud})} = \frac{Z_{t+1}^g(\mathrm{uu}) - Z_{t+1}^h(\mathrm{ud})}{S_{t+1}^g(\mathrm{uu}) - S_{t+1}^h(\mathrm{ud})}.$$

By (4.19), for any $r \in \{a, b\}$

$$\mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}_{t-1}^{urw}(\mathbf{Z}_t; \mathbf{Z}_t),$$

which after a few transformations can be written as

$$\hat{\beta}_t \left(S_t^r(\mathrm{u}) - S_t^v(\mathrm{u})\right) \geq Z_t^r(\mathrm{u}) - Z_t^v(\mathrm{u}).$$

Next, again by (4.19), for any $r \in \{a, b\}$

$$V_t^r(\mathrm{u}) \geq \mathbb{E}_t^{rgh}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u}).$$

It follows that

$$\begin{aligned} V_t^r(\mathrm{u}) - V_t^v(\mathrm{u}) &\geq \mathbb{E}_t^{rgh}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u}) - \mathbb{E}_t^{vgh}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\mathrm{u}) \\ &= \hat{\beta}_{t+1}(\mathrm{u}) \left(S_t^r(\mathrm{u}) - S_t^v(\mathrm{u})\right). \end{aligned}$$

As a result, since $Z_t^r(\mathrm{u}) \geq V_t^r(\mathrm{u})$,

$$\begin{aligned} (\hat{\beta}_{t+1}(\mathrm{u}) - \hat{\beta}_t) \left(S_t^r(\mathrm{u}) - S_t^v(\mathrm{u})\right) &\leq \left(V_t^r(\mathrm{u}) - V_t^v(\mathrm{u})\right) - \left(Z_t^r(\mathrm{u}) - Z_t^v(\mathrm{u})\right) \\ &\leq Z_t^v(\mathrm{u}) - V_t^v(\mathrm{u}) \end{aligned}$$

for any $r \in \{a, b\}$, which implies (4.24) at node $\omega_{t-1}\mathrm{u}$. The argument to verify (4.24) at node $\omega_{t-1}\mathrm{d}$ is similar.

This, therefore, proves (4.24), and consequently (2.4), at all nodes for all times $t = 0, \ldots, T-1$.

(b) Using (4.22), we can write the inequality $\vartheta_t(\hat{\alpha}, \hat{\beta}) \geq Y_t$ as

$$\hat{\beta}_t^+(\hat{S}_t - S_t^b) + \hat{\beta}_t^-(S_t^a - \hat{S}_t) \leq \hat{Z}_t - Y_t,$$

or, equivalently, as

$$\hat{\beta}_t(S_t^r - \hat{S}_t) \geq Y_t - \hat{Z}_t \quad \forall r \in \{a, b\}. \tag{4.25}$$

We shall verify (4.25) for each $t = 0, \ldots, T$.

For $t = 0$ this is trivially satisfied because $\hat{\beta}_0 = 0$ and $\hat{Z}_0 \geq Y_0$. For any $t = 1, \ldots, T$ and any $\omega_{t-1} \in \Omega_{t-1}$ consider the tree fragment

$$\begin{array}{ccc} & & \omega_{t-1}\mathrm{u} \\ & \nearrow & \\ \omega_{t-1} & & \\ & \searrow & \\ & & \omega_{t-1}\mathrm{d} \end{array}$$

We shall verify (4.25) at node $\omega_{t-1}\mathrm{u}$. For brevity, ω_{t-1} will be omitted in the expressions below. By the construction of $\hat{S}, \hat{V}, \hat{Z}$ there are $u, v, w \in \{a, b\}$ such that at the respective nodes

$$\begin{array}{ccc} & & \begin{array}{l} \hat{S}_t(\mathrm{u}) = S_t^v(\mathrm{u}) \\ \hat{V}_t(\mathrm{u}) = V_t^v(\mathrm{u}) \\ \hat{Z}_t(\mathrm{u}) = Z_t^v(\mathrm{u}) \end{array} \\ \begin{array}{l} \hat{S}_{t-1} = S_{t-1}^u \\ \hat{V}_{t-1} = V_{t-1}^u \\ \hat{Z}_{t-1} = Z_{t-1}^u \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & \begin{array}{l} \hat{S}_t(\mathrm{d}) = S_t^w(\mathrm{d}) \\ \hat{V}_t(\mathrm{d}) = V_t^w(\mathrm{d}) \\ \hat{Z}_t(\mathrm{d}) = Z_t^w(\mathrm{d}) \end{array} \end{array}$$

and

$$V_{t-1}^u = \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t).$$

Observe that

$$\hat{\beta}_t = \frac{\hat{Z}_t(\mathrm{u}) - \hat{Z}_t(\mathrm{d})}{\hat{S}_t(\mathrm{u}) - \hat{S}_t(\mathrm{d})} = \frac{Z_t^v(\mathrm{u}) - Z_t^w(\mathrm{d})}{S_t^v(\mathrm{u}) - S_t^w(\mathrm{d})}.$$

By (4.19), for any $r \in \{a, b\}$

$$\mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}_{t-1}^{urw}(\mathbf{Z}_t; \mathbf{Z}_t),$$

which can be transformed into

$$\hat{\beta}_t \left(S_t^r(\mathrm{u}) - S_t^v(\mathrm{u})\right) \geq Z_t^r(\mathrm{u}) - Z_t^v(\mathrm{u}) \geq Y_t(\mathrm{u}) - Z_t^v(\mathrm{u}) \geq 0.$$

This implies (4.25) at node $\omega_{t-1}\mathrm{u}$. The argument at node $\omega_{t-1}\mathrm{d}$ is similar.

We have therefore verified (4.25), and consequently the hedging condition $\vartheta_t(\hat{\alpha}, \hat{\beta}) \geq Y_t$ at each node for every $t = 0, \ldots, T$. ■

We are now in a position to summarise our results obtained so far in the following theorem, providing various representations of the ask price $\pi^a(Y)$ of an American option Y.

Theorem 4.4 *In a model with proportional transaction costs subject to the small transaction costs assumption* (2.3) *the ask price* $\pi^a(Y)$ *of an American option* Y *can be represented as*

$$\pi^a(Y) = \hat{\alpha}_0 = \hat{Z}_0 = \max\{Z_0^a, Z_0^b\} = \hat{\mathbb{E}}(Y_{\hat{\tau}}) = \max_{\tau \in \mathcal{T}} \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}(Y_\tau).$$

In particular, this implies the correctness of Algorithm 4.1 for computing $\pi^a(Y)$ *asserted in Claim 4.1.*

Proof By the definition (4.17) of $\pi^a(Y)$ and Lemma 4.3 we know that $\pi^a(Y) \leq \hat{\alpha}_0$. Next, $\hat{\alpha}_0 = \hat{Z}_0$ by the construction of $\hat{\alpha}_0$, $\hat{Z}_0 = \hat{\mathbb{E}}(Y_{\hat{\tau}})$ according to

Lemma 4.2, and $\hat{\mathbb{E}}(Y_{\hat{\tau}}) \leq \max_{\tau \in \mathcal{T}} \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}(Y_\tau)$ since $\hat{\tau} \in \mathcal{T}$ and $\hat{\mathbb{P}} \in \mathcal{P}$. Moreover, $\max_{\tau \in \mathcal{T}} \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}(Y_\tau) \leq \pi^a(Y)$ by Lemma 4.1. Finally, $\hat{Z}_0 = \max\{Z_0^a, Z_0^b\}$ by the construction of $\hat{Z}$. This completes the proof. ■

Algorithm 4.1 for the ask price of an American option is similar to the standard iterative construction of the Snell envelope. However, in this form it does not readily extend to the bid price case. We shall now present an alternative and somewhat more complicated version of the algorithm, which will be extended to the bid price case in the next section.

Algorithm 4.1a (ask price of American option, reformulated) Denote by $\mathbf{Y} = (Y^a, Y^b)$ the $\mathbb{R}^2$-valued process with $Y^a = Y^b = Y$, where Y is the payoff process of an American option with expiry time T, and construct an $\mathbb{R}^2$-valued process $\mathbf{V} = (V^a, V^b)$ by backward induction as follows:

1. Put
$$V_T^a = V_T^b = Y_T;$$
2. For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ take
$$V_{t-1}^u = \max_{\mathbf{M},\mathbf{N} \in \{\mathbf{Y}_t, \mathbf{V}_t\}} \max_{v,w \in \{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{M}; \mathbf{N}).$$

Claim 4.1a The ask price of the American option is given by

$$\pi^a(Y) = \max\{Y_0, \max\{V_0^a, V_0^b\}\}.$$

Proof Observe that V_t^u is equal to that given by (4.19) and that $Z_t^u = \max\{Y_t, V_t^u\}$ for each $t = 0, \ldots, T$ and each $u \in \{a, b\}$. As a result, this version of the algorithm can also be used to compute the ask price as $\pi^a(Y) = \max\{Z_0^a, Z_0^b\} = \max\{Y_0, \max\{V_0^a, V_0^b\}\}$. ■

4.2 Bid Price Algorithm

Algorithm 4.1a can be modified to apply to the bid price case as follows.

Algorithm 4.2 (bid price of American option) Denote by $\mathbf{Y} = (Y^a, Y^b)$ the $\mathbb{R}^2$-valued process with $Y^a = Y^b = Y$, where Y is the payoff process of an American option with expiry time T, and construct an $\mathbb{R}^2$-valued process $\mathbf{U} = (U^a, U^b)$ by backward induction:

1. We put
$$U_T^a = U_T^b = Y_T;$$
2. For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ we take
$$U_{t-1}^u = \max_{\mathbf{M},\mathbf{N} \in \{\mathbf{Y}_t, \mathbf{U}_t\}} \min_{v,w \in \{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{M}; \mathbf{N}). \tag{4.26}$$

Claim 4.2 The bid price of the American option is given by

$$\pi^b(Y) = \max\{Y_0, \min\{U_0^a, U_0^b\}\}.$$

The claim will be proved in Theorem 4.8.

Let us construct $\mathbb{R}$-valued processes $\check{S}, \check{U}$ and an $\mathbb{R}^2$-valued process $\check{\mathbf{U}}$ such that:

1. For some $\mathbf{M} \in \{\mathbf{Y}_0, \mathbf{U}_0\}$ and $u \in \{a, b\}$

$$\begin{aligned} \check{S}_0 &= S_0^u, \\ \check{U}_0 &= M^u, \\ \check{\mathbf{U}}_0 &= \mathbf{M}, \end{aligned}$$

   and

$$M^u = \min\{M^a, M^b\} = \max\{Y_0, \min\{U_0^a, U_0^b\}\}. \tag{4.27}$$

2. For each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$ there are $\mathbf{K}, \mathbf{L} \in \{\mathbf{Y}_{t+1}, \mathbf{U}_{t+1}\}$ and $v, w \in \{a, b\}$ such that

$$\begin{aligned} \check{S}_{t+1}(\omega_t\mathrm{u}) &= S_{t+1}^v(\omega_t\mathrm{u}), & \check{S}_{t+1}(\omega_t\mathrm{d}) &= S_{t+1}^w(\omega_t\mathrm{d}), \\ \check{U}_{t+1}(\omega_t\mathrm{u}) &= K^v(\omega_t\mathrm{u}), & \check{U}_{t+1}(\omega_t\mathrm{d}) &= L^w(\omega_t\mathrm{d}), \\ \check{\mathbf{U}}_{t+1}(\omega_t\mathrm{u}) &= \mathbf{K}(\omega_t\mathrm{u}), & \check{\mathbf{U}}_{t+1}(\omega_t\mathrm{u}) &= \mathbf{L}(\omega_t\mathrm{u}), \end{aligned}$$

   and

$$U_t^u(\omega_t) = \min_{c,d \in \{a,b\}} \mathbb{E}_t^{ucd}(\mathbf{K}; \mathbf{L}|\omega_t) = \mathbb{E}_t^{uvw}(\mathbf{K}; \mathbf{L}|\omega_t) \tag{4.28}$$

   for each $u \in \{a, b\}$. The existence of such $\mathbf{K}, \mathbf{L}$ and v, w follows, respectively, from Propositions 6.2 and 6.1.

Remark 4.2 The processes $\check{S}, \check{U}, \check{\mathbf{U}}$ may not be unique. The lack of uniqueness may arise whenever there is more than one pair $\mathbf{K}, \mathbf{L} \in \{\mathbf{Y}_{t+1}, \mathbf{U}_{t+1}\}$ or more than one pair $v, w \in \{a, b\}$ such that (4.28) holds for each $u \in \{a, b\}$, or there is more then one $\mathbf{M} \in \{\mathbf{Y}_0, \mathbf{U}_0\}$ or more than one $u \in \{a, b\}$ such that (4.27) holds. In such cases we can choose any $\check{S}, \check{U}, \check{\mathbf{U}}$ satisfying the conditions above.

Let $\check{\mathbb{P}}$ be the probability measure turning $\check{S}$ into a martingale. It is well defined and equivalent to $\mathbb{Q}$ because, by the small transaction costs assumption (2.3), $\check{S}_{t+1}(\omega_t\mathrm{u}) > \check{S}_t(\omega_t) > \check{S}_{t+1}(\omega_t\mathrm{d})$ for each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$. Since $S_t^b \leq \check{S}_t \leq S_t^b$ for each $t = 0, \ldots, T$, we have $\check{\mathbb{P}} \in \mathcal{P}$.

We also define a stopping time $\check{\tau} \in \mathcal{T}$ by

$$\check{\tau} = \min\{t \,|\, \check{\mathbf{U}}_t = \mathbf{Y}_t\}.$$

Observe that the definition of $\check{\tau}$ involves the $\mathbb{R}^2$-valued processes $\check{\mathbf{U}}$ and $\mathbf{Y}$. This is in contrast to the stopping time $\hat{\tau}$, which was introduced in Section 4.1 in connection with the ask price of an American option and defined in terms of $\mathbb{R}$-valued processes, namely $\hat{Z}$ and Y.

Lemma 4.5 **(a)** *The stopped process $\check{U}_{\check{\tau}\wedge t}$ is a martingale under $\check{\mathbb{P}}$;*

(b) $\check{\mathbb{E}}(Y_{\check{\tau}}) = \check{U}_0$.

Proof **(a)** By the construction of $\check{S}, \check{U}, \check{\mathbf{U}}$, for any $t = 0, \ldots, T-1$ and any $\omega_t \in \Omega_t$ there are $u, v, w \in \{a, b\}$, $\mathbf{K}, \mathbf{L} \in \{\mathbf{Y}_{t+1}, \mathbf{U}_{t+1}\}$ and $\mathbf{M} \in \{\mathbf{Y}_t, \mathbf{U}_t\}$ such that

$$\begin{array}{lll}
\check{S}_t(\omega_t) = S_t^u(\omega_t), & \check{S}_{t+1}(\omega_t \mathrm{u}) = S_{t+1}^v(\omega_t \mathrm{u}), & \check{S}_{t+1}(\omega_t \mathrm{d}) = S_{t+1}^w(\omega_t \mathrm{d}), \\
\check{U}_t(\omega_t) = M^u(\omega_t), & \check{U}_{t+1}(\omega_t \mathrm{u}) = K^v(\omega_t \mathrm{u}), & \check{U}_{t+1}(\omega_t \mathrm{d}) = L^w(\omega_t \mathrm{d}), \\
\check{\mathbf{U}}_t(\omega_t) = \mathbf{M}(\omega_t), & \check{\mathbf{U}}_{t+1}(\omega_t \mathrm{u}) = \mathbf{K}(\omega_t \mathrm{u}), & \check{\mathbf{U}}_{t+1}(\omega_t \mathrm{d}) = \mathbf{L}(\omega_t \mathrm{d}),
\end{array}$$

and (4.28) holds. If $t < \check{\tau}$ on ω_t, then $\mathbf{M}(\omega_t) = \mathbf{U}_t(\omega_t)$, so that $M^u(\omega_t) = U_t^u(\omega_t)$, and we obtain

$$\check{U}_t(\omega_t) = U_t^u(\omega_t) = \mathbb{E}_t^{uvw}(\mathbf{K}; \mathbf{L}|\omega_t) = \check{\mathbb{E}}(\check{U}_{t+1}|\omega_t).$$

This means that on $\{t < \check{\tau}\}$

$$\check{U}_{\check{\tau}\wedge t} = \check{\mathbb{E}}(\check{U}_{\check{\tau}\wedge(t+1)}|\mathcal{F}_t).$$

On the other hand, on $\{t \geq \check{\tau}\}$ we have $\check{U}_{\check{\tau}\wedge t} = \check{U}_{\check{\tau}\wedge(t+1)}$, and since $\check{U}_{\check{\tau}\wedge t}$ is $\mathcal{F}_t$-measurable,

$$\check{U}_{\check{\tau}\wedge t} = \check{\mathbb{E}}(\check{U}_{\check{\tau}\wedge t}|\mathcal{F}_t) = \check{\mathbb{E}}(\check{U}_{\check{\tau}\wedge(t+1)}|\mathcal{F}_t),$$

completing the proof that $\check{U}_{\check{\tau}\wedge t}$ is a martingale under $\check{\mathbb{P}}$.

(b) By the definition of $\check{\tau}$ we have $\check{\mathbf{U}}_{\check{\tau}} = \mathbf{Y}_{\check{\tau}}$, which implies $\check{U}_{\check{\tau}\wedge T} = \check{U}_{\check{\tau}} = Y_{\check{\tau}}$, so that $\check{\mathbb{E}}(Y_{\check{\tau}}) = \check{\mathbb{E}}(\check{U}_{\check{\tau}\wedge T}) = \check{U}_0$ by (a). ■

Next, we define predictable processes $\check{\alpha}, \check{\beta}$ such that $\check{\beta}_0 = 0$ and for each $t = 0, , \ldots, T$

$$\check{\alpha}_t + \check{\beta}_t \check{S}_t = -\check{U}_{\check{\tau}\wedge t}. \tag{4.29}$$

Since $\check{S}$ and the stopped process $\check{U}_{\check{\tau}\wedge t}$ are martingales under $\check{\mathbb{P}}$, it follows that for each $t = 0, \ldots, T-1$

$$\begin{aligned}
\check{\alpha}_{t+1} + \check{\beta}_{t+1}\check{S}_t &= \check{\alpha}_{t+1} + \check{\beta}_{t+1}\check{\mathbb{E}}(\check{S}_{t+1}|\mathcal{F}_t) \\
&= \check{\mathbb{E}}(\check{\alpha}_{t+1} + \check{\beta}_{t+1}\check{S}_{t+1}|\mathcal{F}_t) = -\check{\mathbb{E}}(\check{U}_{\check{\tau}\wedge(t+1)}|\mathcal{F}_t) = -\check{U}_{\check{\tau}\wedge t}.
\end{aligned} \tag{4.30}$$

Also observe that $\check{\beta}_t = 0$ and $\check{\alpha}_t = -\check{U}_{\check{\tau}}$ on $\{\check{\tau} < t\}$.

Lemma 4.6 **(a)** $(\check{\alpha}, \check{\beta}) \in \Phi(S^a, S^b)$,

(b) $\vartheta_{\check{\tau}}(\check{\alpha}, \check{\beta}) = -Y_{\check{\tau}}$.

Proof **(a)** By (4.29) and (4.30) the self-financing condition (2.4) can be written as

$$(\check{\beta}_{t+1} - \check{\beta}_t)^+(S_t^a - \check{S}_t) + (\check{\beta}_{t+1} - \check{\beta}_t)^-(\check{S}_t - S_t^b) \leq 0.$$

This, in turn, is equivalent to

$$(\check{\beta}_{t+1}-\check{\beta}_t)(S^r_t-\check{S}_t)\le 0 \quad \forall r\in\{a,b\}. \tag{4.31}$$

We shall demonstrate that $(\check{\alpha},\check{\beta})$ satisfies (4.31) for each $t=0,\dots,T-1$.

First, suppose that $t=0$. If $\check{\tau}=0$, then $\check{\beta}_0=\check{\beta}_1=0$ and (4.31) at $t=0$ follows trivially.

If $t=0$ and $\check{\tau}>0$, then we consider the tree fragment

$$\omega_0 \begin{array}{l} \nearrow \quad \omega_0\mathrm{u} \\ \searrow \quad \omega_0\mathrm{d} \end{array}$$

In the expressions below ω_0 will be omitted for brevity. By the construction of $\check{S},\check{U},\check{\mathbf{U}}$ there are $\mathbf{K},\mathbf{L}\in\{\mathbf{Y}_1,\mathbf{U}_1\}$, $\mathbf{M}\in\{\mathbf{Y}_0,\mathbf{U}_0\}$ and $u,v,w\in\{a,b\}$ such that at the respective nodes

$$\begin{array}{lcl} & & \check{S}_1(\mathrm{u})=S^v_1(\mathrm{u}) \\ & & \check{U}_1(\mathrm{u})=K^v(\mathrm{u}) \\ & & \check{\mathbf{U}}_1(\mathrm{u})=\mathbf{K}(\mathrm{u}) \\ \check{S}_0=S^u_0 & \nearrow & \\ \check{U}_0=M^u & & \\ \check{\mathbf{U}}_0=\mathbf{M} & \searrow & \\ & & \check{S}_1(\mathrm{d})=S^w_1(\mathrm{d}) \\ & & \check{U}_1(\mathrm{d})=L^w(\mathrm{d}) \\ & & \check{\mathbf{U}}_1(\mathrm{d})=\mathbf{L}(\mathrm{d}) \end{array}$$

and

$$U^u_0=\min_{c,d\in\{a,b\}}\mathbb{E}^{ucd}_0(\mathbf{K};\mathbf{L})=\mathbb{E}^{uvw}_0(\mathbf{K};\mathbf{L}). \tag{4.32}$$

Observe that

$$\check{\beta}_1=-\frac{\check{U}_1(\mathrm{u})-\check{U}_1(\mathrm{d})}{\check{S}_1(\mathrm{u})-\check{S}_1(\mathrm{d})}=-\frac{K^v(\mathrm{u})-L^w(\mathrm{d})}{S^v_1(\mathrm{u})-S^w_1(\mathrm{d})}.$$

Since $\check{\tau}>0$, we know that $\mathbf{M}=\mathbf{U}_0$, so $U^u_0=M^u\le M^r=U^r_0$ for each $r\in\{a,b\}$. By (4.32) and Proposition 6.2

$$U^r_0=\min_{c,d\in\{a,b\}}\mathbb{E}^{rcd}_0(\mathbf{K};\mathbf{L})\le\mathbb{E}^{rvw}_0(\mathbf{K};\mathbf{L}).$$

As a result, for each $r\in\{a,b\}$

$$\mathbb{E}^{uvw}_0(\mathbf{K};\mathbf{L})\le\mathbb{E}^{rvw}_0(\mathbf{K};\mathbf{L}),$$

which can be transformed into

$$\check{\beta}_1(S^r_0-S^u_0)\le 0.$$

Since $\check{\beta}_0=0$, this gives (4.31) for $t=0$ and $\check{\tau}>0$.

Next suppose that $t = 1, \dots, T-1$ and $\check{\tau} < t$. Then $\check{\beta}_t = \check{\beta}_{t+1} = 0$ and (4.31) is self evident.

If $t = 1, \dots, T-1$ and $\check{\tau} \geq t$, then consider the tree fragment

$$
\begin{array}{ccccc}
 & & & & \omega_{t-1}\mathrm{uu} \\
 & & \omega_{t-1}\mathrm{u} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
 & & & & \omega_{t-1}\mathrm{ud} \\
\omega_{t-1} & \begin{array}{c}\nearrow \\ \searrow\end{array} & & & \\
 & & \omega_{t-1}\mathrm{d} & &
\end{array}
$$

We shall verify (4.31) at ω_{t-1}u. For brevity ω_{t-1} will be omitted in the expressions below. By the construction of $\check{S}, \check{U}, \check{\mathbf{U}}$ there are $\mathbf{I}, \mathbf{J} \in \{\mathbf{Y}_{t+1}, \mathbf{U}_{t+1}\}$, $\mathbf{K}, \mathbf{L} \in \{\mathbf{Y}_t, \mathbf{U}_t\}$, $\mathbf{M} \in \{\mathbf{Y}_{t-1}, \mathbf{U}_{t-1}\}$ and $u, v, w, g, h \in \{a, b\}$ such that at the respective nodes

$$
\begin{array}{ccccc}
 & & & & \begin{array}{l} \check{S}_{t+1}(\mathrm{uu}) = S^g_{t+1}(\mathrm{uu}) \\ \check{U}_{t+1}(\mathrm{uu}) = I^g(\mathrm{uu}) \\ \check{\mathbf{U}}_{t+1}(\mathrm{uu}) = \mathbf{I}(\mathrm{uu}) \end{array} \\
 & & \begin{array}{l} \check{S}_t(\mathrm{u}) = S^v_t(\mathrm{u}) \\ \check{U}_t(\mathrm{u}) = K^v(\mathrm{u}) \\ \check{\mathbf{U}}_t(\mathrm{u}) = \mathbf{K}(\mathrm{u}) \end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & \\
\begin{array}{l} \check{S}_{t-1} = S^u_{t-1} \\ \check{U}_{t-1} = M^u \\ \check{\mathbf{U}}_{t-1} = \mathbf{M} \end{array} & \begin{array}{c}\nearrow \\ \searrow\end{array} & & & \begin{array}{l} \check{S}_{t+1}(\mathrm{ud}) = S^h_{t+1}(\mathrm{ud}) \\ \check{U}_{t+1}(\mathrm{ud}) = J^h(\mathrm{ud}) \\ \check{\mathbf{U}}_{t+1}(\mathrm{ud}) = \mathbf{J}(\mathrm{ud}) \end{array} \\
 & & \begin{array}{l} \check{S}_t(\mathrm{d}) = S^w_t(\mathrm{d}) \\ \check{U}_t(\mathrm{d}) = L^w(\mathrm{d}) \\ \check{\mathbf{U}}_t(\mathrm{d}) = \mathbf{L}(\mathrm{d}) \end{array} & &
\end{array}
$$

and

$$U^u_{t-1} = \min_{c,d} \mathbb{E}^{ucd}_{t-1}(\mathbf{K}; \mathbf{L}) = \mathbb{E}^{uvw}_{t-1}(\mathbf{K}; \mathbf{L}), \tag{4.33}$$

$$U^v_t(\mathrm{u}) = \min_{c,d} \mathbb{E}^{vcd}_t(\mathbf{I}; \mathbf{J}|\mathrm{u}) = \mathbb{E}^{vgh}_t(\mathbf{I}; \mathbf{J}|\mathrm{u}). \tag{4.34}$$

Observe that

$$\check{\beta}_t = -\frac{\check{U}_t(\mathrm{u}) - \check{U}_t(\mathrm{d})}{\check{S}_t(\mathrm{u}) - \check{S}_t(\mathrm{d})} = -\frac{K^v(\mathrm{u}) - L^w(\mathrm{d})}{S^v_t(\mathrm{u}) - S^w_t(\mathrm{d})}.$$

From (4.33) we deduce that for each $r \in \{a, b\}$

$$\mathbb{E}^{uvw}_{t-1}(\mathbf{K}; \mathbf{L}) \leq \mathbb{E}^{urw}_{t-1}(\mathbf{K}; \mathbf{L}),$$

which can be transformed into

$$-\check{\beta}_t \left(S^r_t(\mathrm{u}) - S^v_t(\mathrm{u})\right) \leq K^r(\mathrm{u}) - K^v(\mathrm{u}). \tag{4.35}$$

Now if $\check{\tau} = t$ at ω_{t-1}u, then $\check{\beta}_{t+1}(\mathrm{u}) = 0$ and $\mathbf{K}(\mathrm{u}) = \mathbf{Y}_t(\mathrm{u})$, that is, $K^r(\mathrm{u}) = K^v(\mathrm{u}) = Y_t(\mathrm{u})$, so that (4.35) implies (4.31) at node ω_{t-1}u. If, on the other hand, $\check{\tau} > t$ at ω_{t-1}u, then

$$\check{\beta}_{t+1}(\mathrm{u}) = -\frac{\check{U}_{t+1}(\mathrm{uu}) - \check{U}_{t+1}(\mathrm{ud})}{\check{S}_{t+1}(\mathrm{uu}) - \check{S}_{t+1}(\mathrm{ud})} = -\frac{I^g(\mathrm{uu}) - J^h(\mathrm{ud})}{S^g_{t+1}(\mathrm{uu}) - S^h_{t+1}(\mathrm{ud})}.$$

From (4.34), by Proposition 6.2, we know that for each $r \in \{a, b\}$

$$U^r_t(\mathrm{u}) = \min_{c,d} \mathbb{E}^{rcd}_t(\mathbf{I}; \mathbf{J}|\mathrm{u}) \leq \mathbb{E}^{rgh}_t(\mathbf{I}; \mathbf{J}|\mathrm{u}).$$

As a result, by (4.33)

$$U^r_t(\mathrm{u}) - U^v_t(\mathrm{u}) \leq \mathbb{E}^{rgh}_t(\mathbf{I}; \mathbf{J}|\mathrm{u}) - \mathbb{E}^{vgh}_t(\mathbf{I}; \mathbf{J}|\mathrm{u}),$$

which can be transformed into

$$U^r_t(\mathrm{u}) - U^v_t(\mathrm{u}) \leq -\check{\beta}_{t+1}(\mathrm{u}) \left(S^r_t(\mathrm{u}) - S^v_t(\mathrm{u})\right). \tag{4.36}$$

But $\check{\tau} > t$ at ω_{t-1}u also means that $\mathbf{K}(\mathrm{u}) = \mathbf{U}_t(\mathrm{u})$, that is, $K^r(\mathrm{u}) = U^r_t(\mathrm{u})$ and $K^v(\mathrm{u}) = U^v_t(\mathrm{u})$, so (4.35) and (4.36) imply (4.31) at node ω_{t-1}u. In a similar manner, one can verify (4.31) at each node ω_{t-1}d for $t = 1, \ldots, T-1$ in the case when $\check{\tau} \geq t$, completing the proof of (a).

(b) We want to show that $\check{\alpha}_{\check{\tau}} + \check{\beta}^+_{\check{\tau}} S^b_{\check{\tau}} - \check{\beta}^-_{\check{\tau}} S^a_{\check{\tau}} = -Y_{\check{\tau}}$. Because $\check{\alpha}_{\check{\tau}} + \check{\beta}_{\check{\tau}} \check{S}_{\check{\tau}} = -\check{U}_{\check{\tau}}$ and $\check{U}_{\check{\tau}} = Y_{\check{\tau}}$ this equality can be written as

$$\check{\beta}^+_{\check{\tau}} \left(\check{S}_{\check{\tau}} - S^b_{\check{\tau}}\right) + \check{\beta}^-_{\check{\tau}} \left(S^a_{\check{\tau}} - \check{S}_{\check{\tau}}\right) = 0,$$

which is equivalent to

$$\check{\beta}_{\check{\tau}}(S^r_{\check{\tau}} - \hat{S}_{\check{\tau}}) \geq 0 \quad \forall r \in \{a, b\}. \tag{4.37}$$

We shall show that $(\check{\alpha}, \check{\beta})$ satisfies (4.37).

If $\check{\tau} = 0$, then (4.37) follows immediately since $\check{\beta}_0 = 0$.

Suppose that $\check{\tau} > 0$. If $\check{\tau} = t$ at a node ω_{t-1}u, then we shall consider the tree fragment

$$\begin{array}{ccc} & & \omega_{t-1}\mathrm{u} \\ & \nearrow & \\ \omega_{t-1} & & \\ & \searrow & \\ & & \omega_{t-1}\mathrm{d} \end{array}$$

and we shall verify (4.37) at ω_{t-1}u. For brevity, ω_{t-1} will be omitted in the expressions below. By the construction of $\check{S}, \check{U}, \check{\mathbf{U}}$ there are $\mathbf{K}, \mathbf{L} \in \{\mathbf{Y}_t, \mathbf{U}_t\}$,

$\mathbf{M} \in \{\mathbf{Y}_{t-1}, \mathbf{U}_{t-1}\}$ and $u, v, w \in \{a, b\}$ such that at the respective nodes

$$\begin{array}{ccc} & & \begin{array}{l} \check{S}_t(\mathrm{u}) = S_t^v(\mathrm{u}) \\ \check{U}_t(\mathrm{u}) = K^v(\mathrm{u}) \\ \check{\mathbf{U}}_t(\mathrm{u}) = \mathbf{K}(\mathrm{u}) \end{array} \\ \begin{array}{l} \check{S}_{t-1} = S_{t-1}^u \\ \check{U}_{t-1} = M^u \\ \check{\mathbf{U}}_{t-1} = \mathbf{M} \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & \begin{array}{l} \check{S}_t(\mathrm{d}) = S_t^w(\mathrm{d}) \\ \check{U}_t(\mathrm{d}) = L^w(\mathrm{d}) \\ \check{\mathbf{U}}_t(\mathrm{d}) = \mathbf{L}(\mathrm{d}) \end{array} \end{array}$$

and

$$U_{t-1}^u = \min_{c,d} E_{t-1}^{ucd}(\mathbf{K}; \mathbf{L}) = E_{t-1}^{uvw}(\mathbf{K}; \mathbf{L}).$$

Observe that

$$\check{\beta}_t = -\frac{\check{U}_t(\mathrm{u}) - \check{U}_t(\mathrm{d})}{\check{S}_t(\mathrm{u}) - \check{S}_t(\mathrm{d})} = -\frac{K_t^v(\mathrm{u}) - L_t^w(\mathrm{d})}{S_t^v(\mathrm{u}) - S_t^w(\mathrm{d})}.$$

It follows that for all $r \in \{a, b\}$

$$E_{t-1}^{uvw}(\mathbf{K}; \mathbf{L}) \leq E_{t-1}^{urw}(\mathbf{K}; \mathbf{L}),$$

which can be transformed into

$$-\check{\beta}_t \left(S_t^r(\mathrm{u}) - S_t^v(\mathrm{u})\right) \leq K^r(\mathrm{u}) - K^v(\mathrm{u}). \tag{4.38}$$

Because $\check{\tau} = t$ at $\omega_{t-1}\mathrm{u}$, we know that $\mathbf{K}(\mathrm{u}) = \mathbf{Y}_t(\mathrm{u})$, so that $K^r(\mathrm{u}) = K^v(\mathrm{u}) = Y_t(\mathrm{u})$, and (4.38) implies (4.37) at $\omega_{t-1}\mathrm{u}$, as required. The argument to verify (4.37) if $\check{\tau} = t$ at a node $\omega_{t-1}\mathrm{d}$ is similar, completing the proof. ∎

Lemma 4.7

$$\max_{\tau \in \mathcal{T}} \min_{\mathbb{P} \in \mathcal{P}} \mathbb{E}(Y_\tau) \leq \check{U}_0.$$

Proof We take any $\tau \in \mathcal{T}$, consider a European option with payoff $X = -Y_\tau$ expiring at time T, and construct processes Z^a, Z^b as in Algorithm 3.1. Observe that for each $u \in \{a, b\}$

$$-Z_\tau^u = Y_\tau.$$

We claim that for each $u \in \{a, b\}$ and for each $t = 0, \ldots, T$

$$-Z_t^u \leq U_t^u$$

on $\{\tau > t\}$. This claim can be proved by backward induction on t. Since $\{\tau > T\}$ is empty, the claim is trivially satisfied for $t = T$. Now suppose that the claim is valid for some $t = 1, \ldots, T$. At each node $\omega_{t-1} \in \Omega_{t-1}$ such that $\tau > t - 1$ at ω_{t-1} (that is, $\omega_{t-1} \subset \{\tau > t - 1\}$) there are up to four possibilities:

1. If $\tau > t$ at ω_{t-1}u and at ω_{t-1}d, then $-Z_t^v \leq U_t^v$ at ω_{t-1}u and $-Z_t^w \leq U_t^w$ at ω_{t-1}d for each v, w, so by (3.10) and (4.26) for each u

$$-Z_{t-1}^u \leq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{U}_t; \mathbf{U}_t) \leq U_{t-1}^u;$$

2. If $\tau = t$ at ω_{t-1}u and $\tau > t$ at ω_{t-1}d, then $-Z_t^v = Y_t$ at ω_{t-1}u and $-Z_t^w \leq U_t^w$ at ω_{t-1}d for each v, w, so by (3.10) and (4.26) for each u

$$-Z_{t-1}^u \leq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Y}_t; \mathbf{U}_t) \leq U_{t-1}^u;$$

3. If $\tau > t$ at ω_{t-1}u and $\tau = t$ at ω_{t-1}d, then $-Z_t^v \leq U_t^v$ at ω_{t-1}u and $-Z_t^w = Y_t$ at ω_{t-1}d for each v, w, so by (3.10) and (4.26) for each u

$$-Z_{t-1}^u \leq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{U}_t; \mathbf{Y}_t) \leq U_{t-1}^u;$$

4. Finally, if $\tau = t$ at ω_{t-1}u and at ω_{t-1}d, then $-Z_t^v = Y_t$ at ω_{t-1}u and $-Z_t^w = Y_t$ at ω_{t-1}d for each v, w, so by (3.10) and (4.26) for each u

$$-Z_{t-1}^u \leq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Y}_t; \mathbf{Y}_t) \leq U_{t-1}^u.$$

This verifies the claim. It follows that $-Z_0^u \leq U_0^u$ on $\{\tau > 0\}$ for each u. Moreover, $-Z_0^u = Y_0$ on $\{\tau = 0\}$ for each u. As a result,

$$-\max\{Z_0^a, Z_0^b\} = \min\{-Z_0^a, -Z_0^b\} \leq \max\{Y_0, \min\{U_0^a, U_0^b\}\} = \check{U}_0.$$

By Theorem 3.4 we therefore obtain

$$\min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(Y_\tau) = -\max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(-Y_\tau) = -\max\{Z_0^a, Z_0^b\} \leq \check{U}_0,$$

completing the proof because of the arbitrariness of τ. ∎

Theorem 4.8 *In a model with proportional transaction costs subject to the small transaction costs assumption* (2.3) *the bid price* $\pi^b(Y)$ *of an American option* Y *can be represented as*

$$\pi^b(Y) = -\check{\alpha}_0 = \check{U}_0 = \max\{Y_0, \min\{U_0^a, U_0^b\}\} = \check{\mathbb{E}}(Y_{\check{\tau}}) = \max_{\tau\in\mathcal{T}} \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(Y_\tau).$$

In particular, this implies the correctness of Algorithm 4.2 for computing $\pi^b(Y)$ *asserted in Claim 4.2.*

Proof By the definition (4.18) of $\pi^b(Y)$ and Lemma 4.6 we know that $\pi^b(Y) \geq -\check{\alpha}_0$. Moreover, $-\check{\alpha}_0 = \check{U}_0$ by the construction of $\check{\alpha}_0$, $\check{U}_0 \geq \max_{\tau\in\mathcal{T}} \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(Y_\tau)$ by Lemma 4.7, and $\max_{\tau\in\mathcal{T}} \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}(Y_\tau) \geq \pi^b(Y)$ by Lemma 4.1. Finally, $\check{U}_0 = \check{\mathbb{E}}(Y_{\check{\tau}})$ by Lemma 4.5, and $\check{U}_0 = \max\{Y_0, \min\{U_0^a, U_0^b\}\}$ by the construction of $\check{U}_0$, completing the proof. ∎

5 Concluding Remarks

We have extended the construction of the Snell envelope to compute the bid and ask prices of American contingent claims under small proportional transaction costs. In addition, we have provided iterative constructions of optimal hedging strategies for the seller as well as for the buyer of an American option. As a special case, we have also considered European options in the same setting. The pricing algorithms are based on backward induction, involving the solution of an optimisation problem at each tree node, and can be viewed as dynamic programming procedures. An interesting new feature of these algorithms is that it is necessary to keep track of two quantities at each node, rather than a single one as in the well known case with no friction. Otherwise, the algorithms have many features in common with the recursive construction of the Snell envelope, to which they in fact reduce in the absence of transaction costs.

We have only considered contingent claims with cash delivery and assumed implicitly that the option buyer or seller have no position in the underlying asset initially. This would, of course, be no consequence in a friction free market. However, in the presence of transaction costs the algorithms would need to be adapted for an option buyer or seller with a long or short initial position in the underlying or to allow for options with physical delivery or, generally, mixed delivery with the payoff being a portfolio of cash and underlying stock.

A natural next step is to try to construct similar algorithms under relaxed assumptions of small proportional transaction costs. In the case of European options this is addressed in Tokarz [Tok04] for arbitrary proportional transaction costs, as long as the model remains free of arbitrage. The procedure is similar to the algorithms in the present paper, except that one has to keep track of more than two quantities at any node for which the underlying bid-ask spread overlaps with that at an adjacent node.

Other directions in which the algorithms could be extended in future work involve more general models, including incomplete ones, and other kinds of friction, such as, for example, different lending and borrowing rates, differently priced short and long positions, short selling restrictions, or fixed transaction costs.

6 Appendix

Here we shall state and outline the proofs of two technical propositions concerning a single-step tree

$$\begin{array}{ccc} & & \begin{array}{c} S_1^a(\mathrm{u}) \\ S_1^b(\mathrm{u}) \end{array} \\ \begin{array}{c} S_0^a \\ S_0^b \end{array} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & \begin{array}{c} S_1^a(\mathrm{d}) \\ S_1^b(\mathrm{d}) \end{array} \end{array}$$

with small proportional transaction costs, i.e. subject to assumption (2.3). The results also apply to any single-step fragment of a larger binary tree model, which is how they are used in the preceding sections. The full details of the proofs, which are elementary but somewhat tedious, can be found in [Tok04].

Consider any two $\mathbb{R}^2$-valued random variables $\mathbf{A} = (A^a, A^b)$ and $\mathbf{B} = (B^a, B^b)$ in the single-step model. Then the following propositions hold.

Proposition 6.1 *For any $c, d \in \{a, b\}$ the following conditions are equivalent:*

(a) $\min_{v,w\in\{a,b\}} \mathbb{E}_0^{avw}(\mathbf{A};\mathbf{B}) = \mathbb{E}_0^{acd}(\mathbf{A};\mathbf{B})$;

(b) $\min_{v,w\in\{a,b\}} \mathbb{E}_0^{bvw}(\mathbf{A};\mathbf{B}) = \mathbb{E}_0^{bcd}(\mathbf{A};\mathbf{B})$.

The conditions remain equivalent if the minima are replaced by maxima.

Proof Outline The main steps of the proof are:

1. For any $c, d, e, f, u \in \{a, b\}$ such that $c \neq e$ and $d \neq f$ show that the inequalities

$$\mathbb{E}_0^{ucd}(\mathbf{A};\mathbf{B}) \leq \mathbb{E}_0^{ued}(\mathbf{A};\mathbf{B}), \tag{6.39}$$

$$\mathbb{E}_0^{ucd}(\mathbf{A};\mathbf{B}) \leq \mathbb{E}_0^{ucf}(\mathbf{A};\mathbf{B}) \tag{6.40}$$

   imply

$$\mathbb{E}_0^{ucd}(\mathbf{A};\mathbf{B}) \leq \mathbb{E}_0^{uef}(\mathbf{A};\mathbf{B}),$$

   and deduce that (6.39), (6.40) are equivalent to

$$\min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{A};\mathbf{B}) = \mathbb{E}_0^{ucd}(\mathbf{A};\mathbf{B}).$$

2. Verify that for any $c, d, e, f \in \{a, b\}$ the inequalities (6.39), (6.40) with $u = a$ are equivalent to (6.39), (6.40) with $u = b$.

The equivalence of (a) and (b) follows directly from steps 1 and 2 above. ■

Proposition 6.2 *For any $\mathbf{C}, \mathbf{D} \in \{\mathbf{A}, \mathbf{B}\}$ the following conditions are equivalent:*

(a) $\max_{\mathbf{V},\mathbf{W}\in\{\mathbf{A},\mathbf{B}\}} \min_{v,w\in\{a,b\}} \mathbb{E}_0^{avw}(\mathbf{V};\mathbf{W}) = \min_{v,w\in\{a,b\}} \mathbb{E}_0^{avw}(\mathbf{C};\mathbf{D})$;

(b) $\max_{\mathbf{V},\mathbf{W}\in\{\mathbf{A},\mathbf{B}\}} \min_{v,w\in\{a,b\}} \mathbb{E}_0^{bvw}(\mathbf{V};\mathbf{W}) = \min_{v,w\in\{a,b\}} \mathbb{E}_0^{bvw}(\mathbf{C};\mathbf{D})$.

Proof Outline The main steps of the proof are:

1. For any $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in \{\mathbf{A}, \mathbf{B}\}$ such that $\mathbf{E} \neq \mathbf{C}$ and $\mathbf{F} \neq \mathbf{D}$ and for any $u \in \{a, b\}$ show that the inequalities

$$\min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{C};\mathbf{D}) \geq \min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{E};\mathbf{D}), \tag{6.41}$$

$$\min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{C};\mathbf{D}) \geq \min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{C};\mathbf{F}) \tag{6.42}$$

imply

$$\min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{C};\mathbf{D}) \geq \min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{E};\mathbf{F}),$$

and deduce that (6.41), (6.42) are equivalent to

$$\max_{\mathbf{V},\mathbf{W}\in\{\mathbf{A},\mathbf{B}\}} \min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{V};\mathbf{W}) = \min_{v,w\in\{a,b\}} \mathbb{E}_0^{uvw}(\mathbf{C};\mathbf{D}).$$

2. Verify that for any $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in \{\mathbf{A}, \mathbf{B}\}$ the inequalities (6.41), (6.42) with $u = a$ are equivalent to (6.41), (6.42) with $u = b$.

The equivalence of (a) and (b) follows directly from steps 1 and 2 above. ■

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