American Contingent Claims with Physical Delivery under Small Proportional Transaction Costs

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Abstract

American options exercised by physical delivery of a portfolio of cash and underlying stock are considered in the binary tree model under small proportional transaction costs. Dynamic programming type recursive algorithms are developed for computing the ask and bid prices of such options, extending the Snell envelope construction. Representations of the ask and bid prices of American options with physical delivery in terms of maximax and, respectively, maximin martingale expectations of stopped option payoffs are also established in this setting.

Key words: American options, physical delivery, transaction costs, bid-ask spread, dynamic programming, Snell envelope.

1 Introduction

By an option with physical delivery we shall understand one that can be exercised by the delivery of a portfolio of cash and stock. In the presence of transaction costs there is a fundamental difference between options settled in cash and by physical delivery, even though they are equivalent in a market without friction.

For example, a call option enabling the holder to purchase a share of stock for the strike price K if the ask price S^a of stock is greater than or equal to K can be considered as a contingent claim exercised by the delivery of a portfolio $(-K1_{\{S^a \geq K\}}, 1_{\{S^a \geq K\}})$ of cash and stock. In the presence of transaction costs this is different than a call settled in cash, that is, by the delivery of a portfolio $((S^a - K)^+, 0)$. If the bid price S^b of stock is strictly less than S^a, then the latter option is more valuable than the former one.

American options with cash settlement under small proportional transaction costs in the binary tree model were studied by Tokarz and Zastawniak [TZ04]. In particular, algorithms resembling the standard construction of the Snell envelope were developed for computing the ask and bid prices of such options. In the present paper these results are extended to the case of American contingent claims with physical delivery.

Chalasani and Jha [CJ01] investigated American options with cash settlement under (not necessarily small) proportional transaction costs. They obtained general representations involving so-called randomised stopping times for the ask and bid prices of American contingent claims. However, no algorithmic procedure for computing the prices was proposed. Indeed, these authors commented that "The computation of the expressions for the upper and lower hedging prices appears non-trivial. It would be useful to design efficient algorithms for approximating the values of these expressions." (Chalasani and Jha [CJ01], p. 72.) Other papers devoted to American options under proportional transaction costs include Mercurio and Vorst [MV97], Kociński [Koc99], [Koc01], Perrakis and Lefoll [PL00], [PL04], and Jakubenas, Levental and Ryznar [JLR03].

Apparently, no algorithms or representations for the upper and lower hedging prices (i.e. the ask and bid prices) of American contingent claims with physical delivery under proportional transactions costs have been put forward to-date. Here we propose two algorithms extending the Snell envelope construction, one for computing the ask price and one for the bid price of an American contingent claim with physical delivery under small proportional transaction costs in the binary tree model. We also construct optimal hedging strategies for the option writer as well as for the seller, and establish representations of the ask and bid option prices in terms of maximax and, respectively, maximin martingale expectations of stopped payoffs over all stopping times and martingale probabilities (more precisely, stock price processes giving rise to martingale probabilities). In contrast to Chalasani and Jha [CJ01], our results suggest that ordinary rather than randomised stopping times may be sufficient, after all, to represent the ask and bid prices of American options under proportional transaction costs.

European contingent claims with physical delivery were studied, for example, by Jouini [Jou00] and Cvitanić and Karatzas [CK96]. Section 3.2 reproduces their general results in the binary tree setting under small transaction costs. However, the algorithmic procedure for computing the ask and bid prices for European options is new, as is the construction of a hedging strategy. Also, our proofs, including those of the known results, are entirely new. European contingent claims are treated separately form American options not just to serve as an introductory special case, but also for completeness, since some of the results obtained for European options are invoked later when dealing with American contingent claims. All constructions and results in this paper applying to American options are original.

As wider background to this topic we also refer to the extensive literature concerned with pricing and hedging European options under proportional transaction costs. This problem was studied by Merton [Mer90], Dermody and Rockafellar [DR91], Boyle and Vorst [BV92], Bensaid, Lesne, Pagès and Scheinkman [BLPS92], Edirsinghe, Naik and Uppal [ENU93], Jouini and Kallal [JK95], Kusuoka [Kus95], Naik [Nai95], Shirakawa and Konno [SK95], Soner, Shreve and Cvitanić [SSC95], Koehl, Pham and Touzi [KPT96], [KPT99], [KPT01], Cvitanić, Pham and Touzi [CPT99], Levental and Skorohod [LS97], Perrakis and Lefoll [PL97], Stettner [Ste97], [Ste00], Rutkowski [Rut98], Touzi [Tou99], Jouini [Jou00], Ortu [Ort01], Palmer [Pal01a], [Pal01b], Kociński [Koc04], and many others.

In the majority of these papers the authors assume a classical stock price process S_t under no-arbitrage conditions, and introduce transaction costs by multiplying S_t by constant factors $1+\lambda$ and $1-\mu$ for some positive λ, μ. Here we follow the more general approach of Jouini and Kallal [JK95] involving bid-ask spreads $S_t^b \leq S_t^a$ for the stock price. As pointed out by Jouini [Jou00], the spreads can be interpreted as proportional transaction costs, but can also be explained by the buying and selling of limit orders. Accordingly, S_t^a and S_t^b can be thought of as the prices ensuring liquidity in the stock market, that is, at which stock can be bought or, respectively, sold on demand. The spreads, therefore, include proportional transaction costs, but are not limited to them. The lack of arbitrage in a model with bid-ask spreads has been characterised by Jouini and Kallal [JK95] in terms of the existence of suitably defined martingale measures. We use their results here as our starting point. See also Ortu [Ort01].

The ask and bid prices of options studied in this paper have important implications. First of all, they provide arbitrage limits on the price at which American options are traded under transaction costs: A writer who could sell an option exercised by the delivery of a portfolio (ξ, ζ) of cash and stock for more than the ask price $\pi^a(\xi, \zeta)$ would be able to achieve arbitrage, as would a buyer who paid less than the bid price $\pi^b(\xi, \zeta)$ for the option. Indeed $\pi^a(\xi, \zeta)$ and $\pi^b(\xi, \zeta)$ are the lowest and, respectively, the highest prices with this property; see Definition 4.1. Moreover, $\pi^a(\xi, \zeta)$ and $\pi^b(\xi, \zeta)$ ensure liquidity in the options market. An option (ξ, ζ) can be purchased on demand for $\pi^a(\xi, \zeta)$ because any option writer who receives this amount will be able to hedge a short position in the option. Similarly, the option can be sold on demand for $\pi^b(\xi, \zeta)$ because any option buyer will be able hedge a shorted amount $\pi^b(\xi, \zeta)$ against a long position in the option. As a result, $\pi^a(\xi, \zeta)$ and $\pi^b(\xi, \zeta)$ play a similar role for options as the ask and bid prices S^a and S^b for stock.

The paper is organised as follows: In Section 2 we describe the model, introduce some notation, basic notions and facts, and specify the small proportional transaction costs assumption. In Section 3 we deal with European contingent claims with physical delivery. The main results concerning American options with physical delivery are contained in Section 4. Section 5 serves as an appendix providing a couple of technical propositions, and Section 6 concludes.

2 Model Specifications and Basic Properties

We adopt the binary tree model with trading times $t = 0, \ldots, T$ for some fixed positive integer T. The corresponding probability space Ω consists of sequences $\omega^1\omega^2\cdots\omega^T$ with $\omega^1, \ldots, \omega^T \in \{\mathrm{u}, \mathrm{d}\}$, where u and d stand for *up* and *down*. We take $\mathcal{F}$ to be the σ-field consisting of all subsets of Ω, and $\mathbb{Q}$ to be a probability measure on $\mathcal{F}$ such that $\mathbb{Q}\{\omega\} > 0$ for each $\omega \in \Omega$.

For each $t = 1, \ldots, T$ we define a random variable $\eta_t : \Omega \ni \omega^1\omega^2\cdots\omega^T \mapsto \omega^t \in \{\mathrm{u}, \mathrm{d}\}$. A *node* $\omega_t = \omega^1\omega^2\cdots\omega^t$ of the tree at time $t = 0, \ldots, T$, with $\omega^1, \ldots, \omega^t \in \{\mathrm{u}, \mathrm{d}\}$, will be identified with the event $\{\omega \in \Omega : \eta^1 = \omega^1, \ldots, \eta^t = \omega^t\}$. In particular, ω_0 will be identified with Ω. The family of all nodes ω_t at time t will be denoted by Ω_t. We take a filtration $\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$, where $\mathcal{F}_t$ is the σ-field generated by the family Ω_t for each $t = 0, \ldots, T$. We shall often identify $\mathcal{F}_t$-measurable random variables on Ω with random variables on Ω_t.

The market model will consist of a risky and a risk-free security, a *stock* and a *bond*. Trading in stock is subject to proportional transaction costs. At any time $t = 0, \ldots, T$ a share can be bought for the *ask* price S_t^a or sold for the *bid* price S_t^b, where $S_t^a \geq S_t^b > 0$. The price processes S^a and S^b are adapted to the filtration.

Without loss of generality we can assume the bond to be a risk-free security with zero interest rate, the bond price being 1 for all $t = 0, \ldots, T$. Equivalently, all prices can be regarded as discounted prices.

2.1 Self-Financing Strategies, Arbitrage, Martingale Measures

The time $t = 0, \ldots, T$ liquidation value of a portfolio (ξ, ζ) of cash and stock will be defined by

$$\vartheta_t(\xi, \zeta) = \xi + \zeta^+ S_t^b - \zeta^- S_t^a.$$

Definition 2.1 By a *self-financing strategy* we shall understand a pair (α, β) of predictable processes α_t, β_t representing positions in cash and stock for $t = 0, \ldots, T$ such that $\beta_0 = 0$ and

$$\vartheta_t(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) \geq 0 \tag{2.1}$$

for each $t = 0, \ldots, T-1$. The set of such strategies will be denoted by $\Phi(S^a, S^b)$.

Observe that the self-financing condition (2.1) is equivalent to

$$\forall u \in \{a, b\} : \alpha_t + \beta_t S_t^u \geq \alpha_{t+1} + \beta_{t+1} S_t^u.$$

Definition 2.2 We say that a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ is a *martingale measure* if there is a martingale S under $\mathbb{P}$ such that $S_t^b \leq S_t \leq S_t^a$ for each $t = 0, \ldots, T$. By $\mathcal{S}$ we shall denote the set of such martingales S, and by $\mathcal{P}$ the set of the corresponding martingale measures $\mathbb{P}$.

Lemma 2.1 *If $(\alpha,\beta)\in\Phi(S^a,S^b)$ and $S\in\mathcal{S}$ is a martingale under $\mathbb{P}\in\mathcal{P}$, then $\alpha+\beta S$ is a supermartingale under $\mathbb{P}$.*

Proof Since $S_t^b\le S_t\le S_t^a$, the self-financing condition (2.1) implies that

$$\alpha_t-\alpha_{t+1}\ge(\beta_t-\beta_{t+1})^-S_t^a-(\beta_t-\beta_{t+1})^+S_t^b\ge-(\beta_t-\beta_{t+1})S_t.$$

As a result, for each $t=0,\dots,T-1$

$$\mathbb{E}(\alpha_{t+1}+\beta_{t+1}S_{t+1}|\mathcal{F}_t)=\alpha_{t+1}+\beta_{t+1}S_t\le\alpha_t+\beta_tS_t,$$

where $\mathbb{E}$ denotes the expectation under $\mathbb{P}$. ■

Definition 2.3 By an *arbitrage opportunity* we understand a strategy $(\alpha,\beta)\in\Phi(S^a,S^b)$ such that $\alpha_0\le0$, $\vartheta_T(\alpha_T,\beta_T)\ge0$ and $\mathbb{Q}\{\vartheta_T(\alpha_T,\beta_T)>0\}>0$.

The following result, obtained by Jouini and Kallal [JK95], who used a slightly different notion of arbitrage, referred to as 'free lunch' in their work, is also valid under the above definition of an arbitrage opportunity in the present setting, as shown in Tokarz [Tok04]. See also Ortu [Ort01].

Theorem 2.2 (Jouini and Kallal [JK95]) *There is no arbitrage opportunity if an only if $\mathcal{P}$ is non-empty or, equivalently, $\mathcal{S}$ is non-empty.*

2.2 Small Proportional Transaction Costs

For any $t=0,\dots,T-1$ and any node $\omega_t\in\Omega_t$ the corresponding single-step subtree of stock prices can be depicted as

$$\begin{array}{ccc} & & S_{t+1}^a(\omega_t\mathrm{u}) \\ & & S_{t+1}^b(\omega_t\mathrm{u}) \\ S_t^a(\omega_t) & \nearrow & \\ S_t^b(\omega_t) & \searrow & \\ & & S_{t+1}^a(\omega_t\mathrm{d}) \\ & & S_{t+1}^b(\omega_t\mathrm{d}) \end{array}$$

Throughout this paper we shall work under the following assumption, which means that the bid-ask spreads do not overlap in any single-step tree fragment as above.

Assumption (small transaction costs) *For each $t=0,\dots,T-1$ and each $\omega_t\in\Omega_t$*

$$S_{t+1}^b(\omega_t\mathrm{d})\le S_{t+1}^a(\omega_t\mathrm{d})<S_t^b(\omega_t)\le S_t^a(\omega_t)<S_{t+1}^b(\omega_t\mathrm{u})\le S_{t+1}^a(\omega_t\mathrm{u}).\quad(2.2)$$

It follows that $\mathcal{S}$ is non-empty. In particular, $S^a,S^b\in\mathcal{S}$. Consequently, the set of martingale measures $\mathcal{P}$ is also non-empty, and no arbitrage opportunity exists under the small transaction costs assumption (2.2).

2.3 Notation

Here we introduce some notation, which will be used throughout this paper. For any $u, v, w \in \{a, b\}$, any $t = 0, \ldots, T-1$ and $\omega_t \in \Omega_t$, and any $\mathcal{F}_{t+1}$-measurable $\mathbb{R}^2$-valued random variables $\mathbf{G} = (G^a, G^b)$ and $\mathbf{H} = (H^a, H^b)$ we put

$$\mathbb{E}_t^{uvw}(\mathbf{G}; \mathbf{H}|\omega_t) = p_t^{uvw}(\omega_t) G^v(\omega_t \mathrm{u}) + (1 - p_t^{uvw}(\omega_t)) H^w(\omega_t \mathrm{d}),$$

where

$$p_t^{uvw}(\omega_t) = \frac{S_t^u(\omega_t) - S_{t+1}^w(\omega_t \mathrm{d})}{S_{t+1}^v(\omega_t \mathrm{u}) - S_{t+1}^w(\omega_t \mathrm{d})}.$$

We shall write $\mathbb{E}_t^{uvw}(\mathbf{G}; \mathbf{H})$ to denote the $\mathcal{F}_t$-measurable random variable $\omega_t \mapsto \mathbb{E}_t^{uvw}(\mathbf{G}; \mathbf{H}|\omega_t)$.

This notation is slightly more complicated than necessary for European options or in the case of the writer of an American option, for which we shall always have $\mathbf{G} = \mathbf{H}$ in $\mathbb{E}_t^{uvw}(\mathbf{G}; \mathbf{H}|\omega_t)$. However it will become necessary to allow $\mathbf{G} \neq \mathbf{H}$ when discussing the buyer's case for an American option.

It will sometimes prove convenient to write $\mathbf{S}_t = (S_t^a, S_t^b)$ for any $t = 0, \ldots, T$. Observe that

$$S_t^u = \mathbb{E}_t^{uvw}(\mathbf{S}_{t+1}; \mathbf{S}_{t+1})$$

for each $u, v, w \in \{a, b\}$ and each $t = 0, \ldots, T-1$.

3 European Options with Physical Delivery

Let us consider a European option to be exercised at time T by the delivery of a portfolio (ξ, ζ) of cash and underlying stock. Here ξ and ζ are $\mathcal{F}_T$-measurable random variables. The portfolio (ξ, ζ) will be referred to as the option payoff.

Definition 3.1 The *upper hedging price* (*ask price*) and the *lower hedging price* (*bid price*) of a European option with payoff (ξ, ζ) are defined, respectively, by

$$\pi^a(\xi, \zeta) = \min\{\alpha_0 \mid (\alpha, \beta) \in \Phi(S^a, S^b), \vartheta_T(\alpha_T - \xi, \beta_T - \zeta) \geq 0\},$$
$$\pi^b(\xi, \zeta) = \max\{-\alpha_0 \mid (\alpha, \beta) \in \Phi(S^a, S^b), \vartheta_T(\alpha_T + \xi, \beta_T + \zeta) \geq 0\}.$$

The minimum and maximum are attained because the corresponding sets are closed and, respectively, bounded below and above in the discrete setting. Observe that

$$\pi^a(\xi, \zeta) = -\pi^b(-\xi, -\zeta). \tag{3.3}$$

3.1 Algorithm for the Ask and Bid Prices

Algorithm 3.1 For a European option with payoff (ξ, ζ) and exercise time T we construct an $\mathbb{R}^2$-valued process $\mathbf{Z} = (Z^a, Z^b)$ by backward induction:

- For each $u \in \{a, b\}$ we put

$$Z_T^u = \xi + \zeta S_T^u;$$

- For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ we put

$$Z^u_{t-1} = \max_{v,w\in\{a,b\}} \mathbb{E}^{uvw}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t).$$

The processes Z^a, Z^b constructed in this algorithm will be used in Theorem 3.2 to compute the ask and bid prices $\pi^a(\xi, \zeta)$ and $\pi^b(\xi, \zeta)$ of a European option (ξ, ζ) with physical delivery.

Lemma 3.1 *Let $t = 1, \ldots, T$ and let (γ, δ) be an $\mathbb{R}^2$-valued $\mathcal{F}_{t-1}$-measurable random variable. Then the following conditions are equivalent:*

(a) *For each $u \in \{a, b\}$*

$$\gamma + \delta S^u_{t-1} \geq Z^u_{t-1};$$

(b) *There is an $\mathbb{R}^2$-valued $\mathcal{F}_{t-1}$-measurable random variable (ρ, σ) such that for each $u \in \{a, b\}$*

$$\begin{aligned} \gamma + \delta S^u_{t-1} &\geq \rho + \sigma S^u_{t-1}, \\ \rho + \sigma S^u_t &\geq Z^u_t. \end{aligned}$$

Proof **(a)$\Longrightarrow$(b).** By Proposition 5.1 there are $g, h \in \{a, b\}$ such that $Z^u_{t-1} = \mathbb{E}^{ugh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t)$ for each $u \in \{a, b\}$. We put

$$\sigma = \frac{Z^g_t(\mathrm{u}) - Z^h_t(\mathrm{d})}{S^g_t(\mathrm{u}) - S^h_t(\mathrm{d})}, \quad \rho = Z^g_t(\mathrm{u}) - \sigma S^g_t(\mathrm{u}) = Z^h_t(\mathrm{d}) - \sigma S^h_t(\mathrm{d}).$$

Here and in the expressions to follow we omit the argument $\omega_{t-1} \in \Omega_{t-1}$ for brevity. Then, for each $u \in \{a, b\}$

$$\rho + \sigma S^u_{t-1} = \rho + \sigma \mathbb{E}^{ugh}_{t-1}(\mathbf{S}_t; \mathbf{S}_t) = \mathbb{E}^{ugh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t) = Z^u_{t-1} \leq \gamma + \delta S^u_{t-1},$$

where we have used (a) in the last inequality. Next, observe that for each $r \in \{a, b\}$

$$\mathbb{E}^{ugh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}^{urh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t) \quad \text{and} \quad \mathbb{E}^{ugh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}^{ugr}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t),$$

which can be transformed, respectively, into

$$\rho + \sigma S^r_t(\mathrm{u}) \geq Z^r_t(\mathrm{u}) \quad \text{and} \quad \rho + \sigma S^r_t(\mathrm{d}) \geq Z^r_t(\mathrm{d}),$$

which gives $\rho + \sigma S^r_t \geq Z^r_t$ for each $r \in \{a, b\}$, as required.
(b)$\Longrightarrow$(a). Fix any $u \in \{a, b\}$ and take $g, h \in \{a, b\}$ such that $Z^u_{t-1} = \mathbb{E}^{ugh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t)$. Then, using (b), we obtain

$$Z^u_{t-1} = \mathbb{E}^{ugh}_{t-1}(\mathbf{Z}_t; \mathbf{Z}_t) \leq \rho + \sigma \mathbb{E}^{ugh}_{t-1}(\mathbf{S}_t; \mathbf{S}_t) = \rho + \sigma S^u_{t-1} \leq \gamma + \delta S^u_{t-1},$$

completing the proof. ■

Theorem 3.2 *Under the small transaction costs assumption* (2.2) *the ask and bid prices of a European option* (ξ,ζ) *with physical delivery can be computed as*

$$\pi^a(\xi,\zeta) = \max\{Z_0^a, Z_0^b\},$$
$$\pi^b(\xi,\zeta) = -\pi^a(-\xi,-\zeta),$$

where Z^a, Z^b *are constructed in Algorithm* 3.1.

Proof The second equality has already been obtained, see (3.3).

To prove the first equality, take a strategy $(\alpha,\beta) \in \Phi(S^a,S^b)$ such that $\vartheta_T(\alpha_T - \xi, \beta_T - \zeta) \geq 0$ and $\alpha_0 = \pi^a(\xi,\zeta)$. We claim that for each $t = 0, ..., T$ and for each $u \in \{a,b\}$

$$\alpha_t + \beta_t S_t^u \geq Z_t^u.$$

We shall prove this claim by backward induction on t. Since $\vartheta_T(\alpha_T - \xi, \beta_T - \zeta) \geq 0$, for each $u \in \{a,b\}$

$$\alpha_T + \beta_T S_T^u \geq \xi + \zeta S_T^u = Z_T^u,$$

so the claim is satisfied for $t = T$. Now suppose that the claim is valid for some $t = 1, \ldots, T$. Because $(\alpha,\beta) \in \Phi(S^a,S^b)$ we therefore know that for each $u \in \{a,b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq \alpha_t + \beta_t S_{t-1}^u,$$
$$\alpha_t + \beta_t S_t^u \geq Z_t^u.$$

It follows by Lemma 3.1 that for each $u \in \{a,b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq Z_{t-1}^u.$$

The claim has been verified. It implies that $\alpha_0 = \alpha_0 + \beta_0 S_0^u \geq Z_0^u$ for each $u \in \{a,b\}$. Since $\alpha_0 = \pi^a(\xi,\zeta)$, it follows that

$$\pi^a(\xi,\zeta) \geq \max\{Z_0^a, Z_0^b\}.$$

To prove the reverse inequality put $\alpha_0 = \max\{Z_0^a, Z_0^b\}$ and $\beta_0 = 0$. It follows that for each $u \in \{a,b\}$

$$\alpha_0 + \beta_0 S_0^u \geq Z_0^u.$$

Then, according to Lemma 3.1, there exists an $\mathcal{F}_0$-measurable portfolio (α_1,β_1) such that for each $u \in \{a,b\}$

$$\alpha_0 + \beta_0 S_0^u \geq \alpha_1 + \beta_1 S_0^u,$$
$$\alpha_1 + \beta_1 S_1^u \geq Z_1^u.$$

Next, by Lemma 3.1 there is an $\mathcal{F}_1$-measurable portfolio (α_2,β_2) such that for each $u \in \{a,b\}$

$$\alpha_1 + \beta_1 S_1^u \geq \alpha_2 + \beta_2 S_1^u,$$
$$\alpha_2 + \beta_2 S_2^u \geq Z_2^u.$$

Proceeding in this way by induction, we can construct a strategy (α, β) such that for each $u \in \{a, b\}$

$$\alpha_{t-1} + \beta_{t-1} S^u_{t-1} \geq \alpha_t + \beta_t S^u_{t-1}$$

for each $t = 1, \ldots, T$ and

$$\alpha_t + \beta_t S^u_t \geq Z^u_t$$

for each $t = 0, \ldots, T$. This means that $(\alpha, \beta) \in \Phi(S^a, S^b)$ and $\alpha_T + \beta_T S^u_T \geq Z^u_T = \xi + \zeta S^u_T$ for each $u \in \{a, b\}$, so that $\vartheta_T(\alpha_T - \xi, \beta_T - \zeta) \geq 0$. It follows that

$$\pi^a(\xi, \zeta) \leq \alpha_0 = \max\{Z^a_0, Z^b_0\},$$

completing the proof. ■

3.2 Martingale Representations for the Ask and Bid Prices

Theorem 3.3 *Under assumption* (2.2)*, the ask and bid prices of a European option to be exercised at time T by physical delivery of a portfolio (ξ, ζ) can be represented as*

$$\pi^a(\xi, \zeta) = \max_{S \in \mathcal{S}} \mathbb{E}(\xi + \zeta S_T), \tag{3.4}$$

$$\pi^b(\xi, \zeta) = \min_{S \in \mathcal{S}} \mathbb{E}(\xi + \zeta S_T), \tag{3.5}$$

where $\mathbb{E}$ is the expectation under the probability measure $\mathbb{P} \in \mathcal{P}$ that turns $S \in \mathcal{S}$ into a martingale.

Proof To verify (3.4) we construct processes $\hat{S}, \hat{Z}$ such that:

- For some $u \in \{a, b\}$

  $$\hat{S}_0 = S^u_0, \quad \hat{Z}_0 = Z^u_0,$$

  and

  $$Z^u_0 = \max\{Z^a_0, Z^b_0\}. \tag{3.6}$$

- For each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$ there are $v, w \in \{a, b\}$ such that

  $$\hat{S}_{t+1}(\omega_t \mathrm{u}) = S^v_{t+1}(\omega_t \mathrm{u}), \quad \hat{S}_{t+1}(\omega_t \mathrm{d}) = S^w_{t+1}(\omega_t \mathrm{d}),$$
  $$\hat{Z}_{t+1}(\omega_t \mathrm{u}) = Z^v_{t+1}(\omega_t \mathrm{u}), \quad \hat{Z}_{t+1}(\omega_t \mathrm{d}) = Z^w_{t+1}(\omega_t \mathrm{d}),$$

  and

  $$Z^u_t(\omega_t) = \mathbb{E}^{uvw}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1} | \omega_t) \tag{3.7}$$

  for each $u \in \{a, b\}$. Such v, w exist by Proposition 5.1.

The processes $\hat{S}, \hat{Z}$ may not be unique. The lack of uniqueness may arise whenever there is more than one pair $v, w \in \{a, b\}$ satisfying (3.7), or there is more than one $u \in \{a, b\}$ such that (3.6) holds. In such cases we can choose any $\hat{S}, \hat{Z}$ satisfying the conditions above.

Because of the small transaction costs assumption (2.2), $\hat{S} \in \mathcal{S}$. Let $\hat{\mathbb{P}} \in \mathcal{P}$ be the probability measure turning $\hat{S}$ into a martingale, and let $\hat{\mathbb{E}}$ denote the expectation under $\hat{\mathbb{P}}$. We claim that $\hat{Z}$ is a martingale under $\hat{\mathbb{P}}$. Indeed, by the construction of $\hat{S}, \hat{Z}$, for any $t = 0, \ldots, T-1$ and any $\omega_t \in \Omega_t$ there are $u, v, w \in \{a, b\}$ such that

$$\begin{array}{lll} \hat{S}_t(\omega_t) = S^u_t(\omega_t), & \hat{S}_{t+1}(\omega_t \mathrm{u}) = S^v_{t+1}(\omega_t \mathrm{u}), & \hat{S}_{t+1}(\omega_t \mathrm{d}) = S^w_{t+1}(\omega_t \mathrm{d}), \\ \hat{Z}_t(\omega_t) = Z^u_t(\omega_t), & \hat{Z}_{t+1}(\omega_t \mathrm{u}) = Z^v_{t+1}(\omega_t \mathrm{u}), & \hat{Z}_{t+1}(\omega_t \mathrm{d}) = Z^w_{t+1}(\omega_t \mathrm{d}), \end{array}$$

and (3.7) holds. Thus

$$\hat{Z}_t(\omega_t) = Z^u_t(\omega_t) = \mathbb{E}^{uvw}_t(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1}|\omega_t) = \hat{\mathbb{E}}(\hat{Z}_{t+1}|\omega_t),$$

which verifies the claim. Since $Z^u_T = \xi + \zeta S^u_T$ for each $u \in \{a, b\}$, it follows that $\hat{Z}_T = \xi + \zeta \hat{S}_T$. Because $\hat{Z}$ is a martingale under $\hat{\mathbb{P}}$ we obtain

$$\hat{\mathbb{E}}(\xi + \zeta \hat{S}_T) = \hat{\mathbb{E}}(\hat{Z}_T) = \hat{Z}_0.$$

As a result,

$$\hat{\mathbb{E}}(\xi + \zeta \hat{S}_T) = \hat{Z}_0 = \max\{Z^a_0, Z^b_0\} = \pi^a(\xi, \zeta),$$

where the last equality holds by Theorem 3.2. This proves that

$$\pi^a(\xi, \zeta) \leq \max_{S \in \mathcal{S}} \mathbb{E}(\xi + \zeta S_T).$$

To prove the reverse inequality take any $S \in \mathcal{S}$ and a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\alpha_0 = \pi^a(\xi, \zeta)$ and $\vartheta_T(\alpha_T - \xi, \beta_T - \zeta) \geq 0$. The last inequality implies that $\xi + \zeta S^u_T \leq \alpha_T + \beta_T S^u_T$ for each $u \in \{a, b\}$. Because $S^b_T \leq S_T \leq S^a_T$ it follows that $\xi + \zeta S_T \leq \alpha_T + \beta_T S_T$. By Lemma 2.1 the process $\alpha + \beta S$ is a supermartingale under the probability measure $\mathbb{P} \in \mathcal{P}$ that turns S into a martingale. As a result,

$$\mathbb{E}(\xi + \zeta S_T) \leq \mathbb{E}(\alpha_T + \beta_T S_T) \leq \alpha_0 + \beta_0 S_0 = \alpha_0 = \pi^a(\xi, \zeta),$$

where $\mathbb{E}$ is the expectation under $\mathbb{P}$. Since $S \in \mathcal{S}$ is arbitrary, it follows that

$$\max_{S \in \mathcal{S}} \mathbb{E}(\xi + \zeta S_T) \leq \pi^a(\xi, \zeta).$$

This proves (3.4). Finally, by (3.3),

$$\pi^b(\xi, \zeta) = -\pi^a(-\xi, -\zeta) = -\max_{S \in \mathcal{S}} \mathbb{E}(-\xi - \zeta S_T) = \min_{S \in \mathcal{S}} \mathbb{E}(\xi + \zeta S_T),$$

verifying (3.5). ■

4 American Options with Physical Delivery

We shall consider an American option to be exercised by the delivery of a portfolio (ξ_τ, ζ_τ) of cash and stock at any stopping time τ chosen by the option holder such that $0 \leq \tau \leq T$, where (ξ, ζ) is an $\mathbb{R}^2$-valued adapted process. By $\mathcal{T}$ we denote the family of such stopping times τ. We shall refer to (ξ, ζ) as the payoff process and to T as the expiry time.

Definition 4.1 The *upper hedging price* (*ask price*) and the *lower hedging price* (*bid price*) of an American option with physical delivery, with payoff process (ξ, ζ) and expiry time T, are defined, respectively, by

$$\pi^a(\xi,\zeta) = \min\{\alpha_0 \mid (\alpha,\beta) \in \Phi(S^a,S^b), \forall\tau \in \mathcal{T} : \vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \geq 0\},$$
$$\pi^b(\xi,\zeta) = \max\{-\alpha_0 \mid (\alpha,\beta) \in \Phi(S^a,S^b), \exists\tau \in \mathcal{T} : \vartheta_\tau(\alpha_\tau + \xi_\tau, \beta_\tau + \zeta_\tau) \geq 0\}.$$

The minimum and maximum are attained because the corresponding sets are closed and, respectively, bounded below and above in the discrete setting.

4.1 Algorithm for the Ask Price

In what follows it will prove convenient to use processes X^a, X^b such that for each $u \in \{a, b\}$ and each $t = 0, \ldots, T$

$$X_t^u = \xi_t + \zeta_t S_t^u. \tag{4.8}$$

Algorithm 4.1 Given an American option with payoff process (ξ, ζ) and expiry time T we construct an $\mathbb{R}^2$-valued process $\mathbf{Z} = (Z^a, Z^b)$ by backward induction:

- For each $u \in \{a, b\}$ we put
$$Z_T^u = V_T^u = X_T^u;$$
- For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ we put
$$Z_{t-1}^u = \max\{X_{t-1}^u, V_{t-1}^u\},$$
where
$$V_{t-1}^u = \max_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t).$$

The processes V^a, V^b correspond to the value of continuation in the standard Snell envelope construction, whereas Z^a, Z^b correspond to the Snell envelope itself. They will be used in Theorem 4.2 to compute the ask price $\pi^a(\xi, \zeta)$ of an American option (ξ, ζ) with physical delivery.

Lemma 4.1 *Let $t = 1, \ldots, T$ and let (γ, δ) be an $\mathbb{R}^2$-valued $\mathcal{F}_{t-1}$-measurable random variable. Then the following conditions are equivalent:*

(a) *For each* $u \in \{a, b\}$

$$\gamma + \delta S_{t-1}^u \geq V_{t-1}^u;$$

(b) *There is an* $\mathbb{R}^2$*-valued* $\mathcal{F}_{t-1}$*-measurable random variable* (ρ, σ) *such that for each* $u \in \{a, b\}$

$$\begin{aligned} \gamma + \delta S_{t-1}^u &\geq \rho + \sigma S_{t-1}^u, \\ \rho + \sigma S_t^u &\geq Z_t^u. \end{aligned}$$

Proof (a)$\Longrightarrow$(b). By Proposition 5.1 there are $g, h \in \{a, b\}$ such that $V_{t-1}^u = \mathbb{E}_{t-1}^{ugh}(\mathbf{Z}_t; \mathbf{Z}_t)$ for each $u \in \{a, b\}$. We put

$$\sigma = \frac{Z_t^g(\mathrm{u}) - Z_t^h(\mathrm{d})}{S_t^g(\mathrm{u}) - S_t^h(\mathrm{d})}, \quad \rho = Z_t^g(\mathrm{u}) - \sigma S_t^g(\mathrm{u}) = Z_t^h(\mathrm{d}) - \sigma S_t^h(\mathrm{d}).$$

Here and in the expressions to follow we omit the argument $\omega_{t-1} \in \Omega_{t-1}$ for brevity. Then, for each $u \in \{a, b\}$

$$\rho + \sigma S_{t-1}^u = \rho + \sigma \mathbb{E}_{t-1}^{ugh}(\mathbf{S}_t; \mathbf{S}_t) = \mathbb{E}_{t-1}^{ugh}(\mathbf{Z}_t; \mathbf{Z}_t) = V_{t-1}^u \leq \gamma + \delta S_{t-1}^u,$$

where we have used (a) in the last inequality. Next, observe that for each $r \in \{a, b\}$

$$\mathbb{E}_{t-1}^{ugh}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}_{t-1}^{urh}(\mathbf{Z}_t; \mathbf{Z}_t) \quad \text{and} \quad \mathbb{E}_{t-1}^{ugh}(\mathbf{Z}_t; \mathbf{Z}_t) \geq \mathbb{E}_{t-1}^{ugr}(\mathbf{Z}_t; \mathbf{Z}_t),$$

which can be transformed, respectively, into

$$\rho + \sigma S_t^r(\mathrm{u}) \geq Z_t^r(\mathrm{u}) \quad \text{and} \quad \rho + \sigma S_t^r(\mathrm{d}) \geq Z_t^r(\mathrm{d}),$$

which gives $\rho + \sigma S_t^r \geq Z_t^r$ for each $r \in \{a, b\}$, as required.
(b)$\Longrightarrow$(a). Fix any $u \in \{a, b\}$ and take $g, h \in \{a, b\}$ such that $V_{t-1}^u = \mathbb{E}_{t-1}^{ugh}(\mathbf{Z}_t; \mathbf{Z}_t)$. Then, using (b), we obtain

$$V_{t-1}^u = \mathbb{E}_{t-1}^{ugh}(\mathbf{Z}_t; \mathbf{Z}_t) \leq \rho + \sigma \mathbb{E}_{t-1}^{ugh}(\mathbf{S}_t; \mathbf{S}_t) = \rho + \sigma S_{t-1}^u \leq \gamma + \delta S_{t-1}^u,$$

completing the proof. ■

Theorem 4.2 *Under the small transaction costs assumption* (2.2) *the ask price of an American option* (ξ, ζ) *with physical delivery can be computed as*

$$\begin{aligned} \pi^a(\xi, \zeta) &= \max\{Z_0^a, Z_0^b\} \\ &= \max\big\{ \max\{X_0^a, X_0^b\}, \max\{V_0^a, V_0^b\} \big\}, \end{aligned}$$

where Z^a, Z^b *and* V^a, V^b *are constructed in Algorithm* 4.1 *and* X^a, X^b *are given by* (4.8).

Remark 4.1 The expression $\max\big\{\max\{X_0^a, X_0^b\}, \max\{V_0^a, V_0^b\}\big\}$ for the ask price $\pi^a(\xi,\zeta)$ is unnecessarily complicated and could be simplified, but is presented in this form to emphasise an analogy with that for the bid price $\pi^b(\xi,\zeta)$ in Theorem 4.4.

Proof The second equality follows immediately since $Z_0^u = \max\{X_0^u, V_0^u\}$ for each $u \in \{a,b\}$.

To prove the first equality take a strategy $(\alpha,\beta) \in \Phi(S^a,S^b)$ such that $\vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \geq 0$ for each $\tau \in \mathcal{T}$ and $\alpha_0 = \pi^a(\xi,\zeta)$. We claim that for each $t = 0, ..., T$ and for each $u \in \{a,b\}$

$$\alpha_t + \beta_t S_t^u \geq Z_t^u.$$

We shall prove this claim by backward induction on t. Since $\vartheta_T(\alpha_T - \xi_T, \beta_T - \zeta_T) \geq 0$, for each $u \in \{a,b\}$

$$\alpha_T + \beta_T S_T^u \geq \xi_T + \zeta_T S_T^u = X_T^u = Z_T^u,$$

so the claim is satisfied for $t = T$. Now suppose that the claim is valid for some $t = 1, \ldots, T$. Because $(\alpha,\beta) \in \Phi(S^a,S^b)$ we therefore know that for each $u \in \{a,b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq \alpha_t + \beta_t S_{t-1}^u,$$
$$\alpha_t + \beta_t S_t^u \geq Z_t^u.$$

It follows by Lemma 4.1 that for each $u \in \{a,b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq V_{t-1}^u.$$

Since $\vartheta_T(\alpha_{t-1} - \xi_{t-1}, \beta_{t-1} - \zeta_{t-1}) \geq 0$, for each $u \in \{a,b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq \xi_{t-1} + \zeta_{t-1} S_{t-1}^u = X_{t-1}^u.$$

As a result, for each $u \in \{a,b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq \max\{X_{t-1}^u, V_{t-1}^u\} = Z_{t-1}^u.$$

The claim has been verified. It implies that $\alpha_0 = \alpha_0 + \beta_0 S_0^u \geq Z_0^u$ for each $u \in \{a,b\}$. Since $\alpha_0 = \pi^a(\xi,\zeta)$, it follows that

$$\pi^a(\xi,\zeta) \geq \max\{Z_0^a, Z_0^b\}.$$

To prove the reverse inequality put $\alpha_0 = \max\{Z_0^a, Z_0^b\}$ and $\beta_0 = 0$. Then for each $u \in \{a,b\}$

$$\alpha_0 + \beta_0 S_0^u \geq Z_0^u.$$

According to Lemma 4.1, there is an $\mathcal{F}_0$-measurable portfolio (α_1,β_1) such that for each $u \in \{a,b\}$

$$\alpha_0 + \beta_0 S_0^u \geq \alpha_1 + \beta_1 S_0^u,$$
$$\alpha_1 + \beta_1 S_1^u \geq Z_1^u \geq V_1^u.$$

Next, again by Lemma 4.1, there is an $\mathcal{F}_1$-measurable portfolio (α_2, β_2) such that for each $u \in \{a, b\}$

$$\alpha_1 + \beta_1 S_1^u \geq \alpha_2 + \beta_2 S_1^u,$$
$$\alpha_2 + \beta_2 S_2^u \geq Z_2^u \geq V_2^u.$$

Proceeding in this manner by induction, we can construct a strategy (α, β) such that for each $u \in \{a, b\}$

$$\alpha_{t-1} + \beta_{t-1} S_{t-1}^u \geq \alpha_t + \beta_t S_{t-1}^u$$

for each $t = 1, \ldots, T$, and

$$\alpha_t + \beta_t S_t^u \geq Z_t^u$$

for each $t = 0, \ldots, T$. In particular, this means that $(\alpha, \beta) \in \Phi(S^a, S^b)$ and $\vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \geq 0$ for each $\tau \in \mathcal{T}$. It follows that

$$\pi^a(\xi, \zeta) \leq \max\{Z_0^a, Z_0^b\},$$

completing the proof. ■

4.2 Algorithm for the Bid Price

Algorithm 4.2 Given an American option with physical delivery, with payoff process (ξ, ζ) and expiry time T, we consider the $\mathbb{R}^2$-valued process $\mathbf{X} = (X^a, X^b)$ with X^a, X^b given by (4.8), and construct an $\mathbb{R}^2$-valued process $\mathbf{U} = (U^a, U^b)$ by backward induction as follows:

- For each $u \in \{a, b\}$ we put
$$U_T^u = X_T^u;$$
- For each $t = 1, \ldots, T$ and each $u \in \{a, b\}$ we put
$$U_{t-1}^u = \max_{\mathbf{V}, \mathbf{W} \in \{\mathbf{X}_t, \mathbf{U}_t\}} \min_{v, w \in \{a, b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{V}; \mathbf{W}).$$

The processes U^a, U^b will be used in Theorem 4.4 to compute the bid price $\pi^b(\xi, \zeta)$ of an American option (ξ, ζ) with physical delivery. They correspond to the value of continuation in the standard Snell envelope construction.

We define a stopping time $\check{\tau} \in \mathcal{T}$ by putting

$$\check{\tau} = t \text{ on } A_t \setminus (A_0 \cup \cdots \cup A_{t-1}) \tag{4.9}$$

for $t = 0, \ldots, T$, where

$$A_0 = \{\min\{X_0^a, X_0^b\} \geq \min\{U_0^a, U_0^b\}\}$$

and

$$A_t = \{\eta^t = \mathrm{u}, \exists \mathbf{W} \in \{\mathbf{X}_t, \mathbf{U}_t\} : U^u_{t-1} = \min_{v,w\in\{a,b\}} \mathbb{E}^{uvw}_{t-1}(\mathbf{X}_t; \mathbf{W})\}$$
$$\cup \{\eta^t = \mathrm{d}, \exists \mathbf{W} \in \{\mathbf{X}_t, \mathbf{U}_t\} : U^u_{t-1} = \min_{v,w\in\{a,b\}} \mathbb{E}^{uvw}_{t-1}(\mathbf{W}; \mathbf{X}_t)\}$$

for $t = 1, \ldots, T$. These sets satisfy $\Omega = A_0 \cup \cdots \cup A_T$ and, by Proposition 5.1, are in fact independent of $u \in \{a, b\}$.

Lemma 4.3 *Let $t = 1, \ldots, T$ and let (γ, δ) be an $\mathbb{R}^2$-valued $\mathcal{F}_{t-1}$-measurable random variable. Then the following conditions are equivalent:*

(a) *For each $u \in \{a, b\}$*

$$\gamma + \delta S^u_{t-1} \geq -U^u_{t-1} \ \text{on}\ \{\check{\tau} > t - 1\};$$

(b) *There is an $\mathbb{R}^2$-valued $\mathcal{F}_{t-1}$-measurable random variable (ρ, σ) such that for each $u \in \{a, b\}$*

$$\gamma + \delta S^u_{t-1} \geq \rho + \sigma S^u_{t-1},$$
$$\rho + \sigma S^u_t \geq -X^u_t \ \text{on}\ \{\check{\tau} = t\},$$
$$\rho + \sigma S^u_t \geq -U^u_t \ \text{on}\ \{\check{\tau} > t\}.$$

Proof (a)$\Longrightarrow$(b). Take $\mathbf{G}, \mathbf{H} \in \{\mathbf{X}_t, \mathbf{U}_t\}$ and $g, h \in \{a, b\}$ such that for each $u \in \{a, b\}$

$$U^u_{t-1} = \min_{v,w\in\{a,b\}} \mathbb{E}^{uvw}_{t-1}(\mathbf{G}; \mathbf{H}) = \mathbb{E}^{ugh}_{t-1}(\mathbf{G}; \mathbf{H}).$$

Such $\mathbf{G}, \mathbf{H}$ and g, h exist by Propositions 5.1 and 5.2. We put

$$\sigma = 0, \quad \rho = \gamma + \delta^+ S^b_{t-1} - \delta^- S^a_{t-1}$$

on $\{\check{\tau} < t\}$, and

$$\sigma = -\frac{G^g(\mathrm{u}) - H^h(\mathrm{d})}{S^g_t(\mathrm{u}) - S^h_t(\mathrm{d})}, \quad \rho = -G^g(\mathrm{u}) - \sigma S^g_t(\mathrm{u}) = -H^h(\mathrm{d}) - \sigma S^h_t(\mathrm{d})$$

on $\{\check{\tau} \geq t\}$. Here and in the expressions to follow we omit the argument $\omega_{t-1} \in \Omega_{t-1}$ for brevity. Then, for each $u \in \{a, b\}$

$$\rho + \sigma S^u_{t-1} = \gamma + \delta^+ S^b_{t-1} - \delta^- S^a_{t-1} \leq \gamma + \delta S^u_{t-1}$$

on $\{\check{\tau} < t\}$, and

$$\rho + \sigma S^u_{t-1} = \rho + \sigma \mathbb{E}^{ugh}_{t-1}(\mathbf{S}_t; \mathbf{S}_t) = -\mathbb{E}^{ugh}_{t-1}(\mathbf{G}; \mathbf{H}) = -U^u_{t-1} \leq \gamma + \delta S^u_{t-1}$$

on $\{\check{\tau} \geq t\}$. We have used (a) in the last inequality. Next, observe that for each $r \in \{a, b\}$

$$\mathbb{E}^{ugh}_{t-1}(\mathbf{G}; \mathbf{H}) \leq \mathbb{E}^{urh}_{t-1}(\mathbf{G}; \mathbf{H}) \quad \text{and} \quad \mathbb{E}^{ugh}_{t-1}(\mathbf{G}; \mathbf{H}) \leq \mathbb{E}^{ugr}_{t-1}(\mathbf{G}; \mathbf{H}),$$

which can be transformed, respectively, into

$$\rho + \sigma S_t^r(\mathrm{u}) \geq -G^r(\mathrm{u}) \quad \text{and} \quad \rho + \sigma S_t^r(\mathrm{d}) \geq -H^r(\mathrm{d})$$

on $\{\check{\tau} \geq t\}$. By the construction of $\check{\tau}$ we know that $\mathbf{G} = \mathbf{H} = \mathbf{U}_t$ on $\{\check{\tau} > t\}$, and we can select $\mathbf{G} = \mathbf{X}_t$ on $\{\check{\tau} = t, \eta^t = \mathrm{u}\}$ and $\mathbf{H} = \mathbf{X}_t$ on $\{\check{\tau} = t, \eta^t = \mathrm{d}\}$. It follows that $\rho + \sigma S_t^r \geq -X_t^r$ on $\{\check{\tau} = t\}$ and $\rho + \sigma S_t^r \geq -U_t^r$ on $\{\check{\tau} > t\}$ for each $r \in \{a, b\}$, as required.

(b)$\Longrightarrow$(a). Fix any $u \in \{a, b\}$ and take $\mathbf{G}, \mathbf{H} \in \{\mathbf{X}_t, \mathbf{U}_t\}$ and $g, h \in \{a, b\}$ such that

$$U_{t-1}^u = \min_{v,w \in \{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{G}; \mathbf{H}) = \mathbb{E}_{t-1}^{ugh}(\mathbf{G}; \mathbf{H}).$$

By the construction of $\check{\tau}$ we know that $\mathbf{G} = \mathbf{H} = \mathbf{U}_t$ on $\{\check{\tau} > t\}$, and we can select $\mathbf{G} = \mathbf{X}_t$ on $\{\check{\tau} = t, \eta^t = \mathrm{u}\}$ and $\mathbf{H} = \mathbf{X}_t$ on $\{\check{\tau} = t, \eta^t = \mathrm{d}\}$. By (b) it follows that

$$\rho + \sigma S_t^u(\mathrm{u}) \geq -G^u(\mathrm{u}) \quad \text{and} \quad \rho + \sigma S_t^u(\mathrm{d}) \geq -H^u(\mathrm{d})$$

on $\{\check{\tau} \geq t\}$. As a result,

$$-U_{t-1}^u = -\mathbb{E}_{t-1}^{ugh}(\mathbf{G}; \mathbf{H}) \leq \rho + \sigma \mathbb{E}_{t-1}^{ugh}(\mathbf{S}_t; \mathbf{S}_t) = \rho + \sigma S_{t-1}^u \leq \gamma + \delta S_{t-1}^u$$

on $\{\check{\tau} \geq t\} = \{\check{\tau} > t - 1\}$. ■

Theorem 4.4 *Under the small transaction costs assumption* (2.2) *the bid price of an American option* (ξ, ζ) *with physical delivery can be computed as*

$$\pi^b(\xi, \zeta) = \max\left\{ \min\{X_0^a, X_0^b\}, \min\{U_0^a, U_0^b\}\right\},$$

where U^a, U^b *are constructed in Algorithm* 4.2 *and* X^a, X^b *are given by* (4.8).

Proof Take a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\alpha_0 = -\pi^b(\xi, \zeta)$ and there is a stopping time $\tau \in \mathcal{T}$ such that $\vartheta_\tau(\alpha_\tau + \xi_\tau, \beta_\tau + \zeta_\tau) \geq 0$, that is,

$$\forall u \in \{a, b\} : \alpha_\tau + \beta_\tau S_\tau^u \geq -X_\tau^u. \tag{4.10}$$

Consider processes Z^a, Z^b constructed as in Algorithm 3.1 for the European option with payoff $(\kappa, \lambda) = (-\xi_\tau, -\zeta_\tau)$ and exercise time T. Observe that for each $u \in \{a, b\}$

$$Z_\tau^u = -X_\tau^u.$$

We claim that for each $u \in \{a, b\}$ and each $t = 0, \ldots, T$

$$Z_t^u \geq -U_t^u \text{ on } \{\tau > 0\}.$$

This can be proved by backward induction on t. Since $\{\tau > T\}$ is empty, the claim is trivially satisfied for $t = T$. Now suppose that the claim is valid for some $t = 1, \ldots, T$. For each node $\omega_{t-1} \in \Omega_{t-1}$ such that $\tau > t - 1$ at ω_{t-1} there are up to four possibilities:

1. If $\tau > t$ at ω_{t-1}u and at ω_{t-1}d, then $Z_t^v \geq -U_t^v$ at ω_{t-1}u and $Z_t^w \geq -U_t^w$ at ω_{t-1}d for each $v, w \in \{a,b\}$, so for each $u \in \{a,b\}$

$$Z_{t-1}^u = \max_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t) \geq - \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{U}_t; \mathbf{U}_t) \geq -U_{t-1}^u;$$

2. If $\tau = t$ at ω_{t-1}u and $\tau > t$ at ω_{t-1}d, then $Z_t^v = -X_t^v$ at ω_{t-1}u and $Z_t^w \geq -U_t^w$ at ω_{t-1}d for each $v, w \in \{a,b\}$, so for each $u \in \{a,b\}$

$$Z_{t-1}^u = \max_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t) \geq - \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{X}_t; \mathbf{U}_t) \geq -U_{t-1}^u;$$

3. If $\tau > t$ at ω_{t-1}u and $\tau = t$ at ω_{t-1}d, then $Z_t^v \geq -U_t^v$ at ω_{t-1}u and $Z_t^w = -X_t^w$ at ω_{t-1}d for each $v, w \in \{a,b\}$, so for each $u \in \{a,b\}$

$$Z_{t-1}^u = \max_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t) \geq - \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{U}_t; \mathbf{X}_t) \geq -U_{t-1}^u;$$

4. Finally, if $\tau = t$ at ω_{t-1}u and at ω_{t-1}d, then $Z_t^v = -X_t^v$ at ω_{t-1}u and $Z_t^w = -X_t^w$ at ω_{t-1}d for each $v, w \in \{a,b\}$, so for each $u \in \{a,b\}$

$$Z_{t-1}^u = \max_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{Z}_t; \mathbf{Z}_t) \geq - \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{X}_t; \mathbf{X}_t) \geq -U_{t-1}^u.$$

This verifies the claim. It follows that $Z_0^u = -X_0^u$ on $\{\tau = 0\}$ and $Z_0^u \geq -U_0^u$ on $\{\tau > 0\}$ for each $u \in \{a,b\}$. As a result,

$$-\max\{Z_0^a, Z_0^b\} \leq \max\left\{\min\{X_0^a, X_0^b\}, \min\{U_0^a, U_0^b\}\right\}.$$

Now put for each $t = 0, \ldots, T$

$$(\alpha_t', \beta_t') = \begin{cases} (\alpha_t, \beta_t) & \text{on } \{\tau \geq t\}, \\ (-\xi_\tau, -\zeta_\tau) & \text{on } \{\tau < t\}. \end{cases}$$

Since $(\alpha, \beta) \in \Phi(S^a, S^b)$ and (4.10) holds, it follows that $(\alpha', \beta') \in \Phi(S^a, S^b)$ and

$$\forall u \in \{a,b\} : \alpha_T' + \beta_T' S_T^u \geq -\xi_\tau - \zeta_\tau S_T^u = \kappa + \lambda S_T^u,$$

that is, $\vartheta_T(\alpha_T' - \kappa, \beta_T' - \lambda) \geq 0$. By the definition of the ask price $\pi^a(\kappa, \lambda)$ of the European option with payoff (κ, λ) and exercise time T it follows that

$$\pi^a(\kappa, \lambda) \leq \alpha_0' = \alpha_0 = -\pi^b(\xi, \zeta).$$

By Theorem 3.2, $\pi^a(\kappa, \lambda) = \max\{Z_0^a, Z_0^b\}$ and we obtain

$$\pi^b(\xi, \zeta) \leq \max\left\{\min\{X_0^a, X_0^b\}, \min\{U_0^a, U_0^b\}\right\}.$$

Next we shall prove the reverse inequality. If $\min\{X_0^a, X_0^b\} \geq \min\{U_0^a, U_0^b\}$, then we take a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\alpha_t = -\min\{X_0^a, X_0^b\}$ and $\beta_t = 0$ for each $t = 0, \ldots, T$. Then, for each $u \in \{a,b\}$

$$\alpha_0 + \beta_0 S_0^u \geq -X_0^u \text{ on } \{\check{\tau} = 0\}.$$

On the other hand, if $\min\{X_0^a, X_0^b\} < \min\{U_0^a, U_0^b\}$, we put $\alpha_0 = -\min\{U_0^a, U_0^b\}$ and $\beta_0 = 0$. Then, for each $u \in \{a, b\}$

$$\alpha_0 + \beta_0 S_0^u \geq -U_0^u \text{ on } \{\check{\tau} > 0\}$$

and, according to Lemma 4.3, there is an $\mathcal{F}_0$-measurable portfolio (α_1, β_1) such that for each $u \in \{a, b\}$

$$\begin{aligned} \alpha_0 + \beta_0 S_0^u &\geq \alpha_1 + \beta_1 S_0^u, \\ \alpha_1 + \beta_1 S_1^u &\geq -X_1^u \text{ on } \{\check{\tau} = 1\}, \\ \alpha_1 + \beta_1 S_1^u &\geq -U_1^u \text{ on } \{\check{\tau} > 1\}. \end{aligned}$$

Next, again by Lemma 4.3, there is an $\mathcal{F}_1$-measurable portfolio (α_2, β_2) such that for each $u \in \{a, b\}$

$$\begin{aligned} \alpha_1 + \beta_1 S_1^u &\geq \alpha_2 + \beta_2 S_1^u, \\ \alpha_2 + \beta_2 S_2^u &\geq -X_2^u \text{ on } \{\check{\tau} = 2\}, \\ \alpha_2 + \beta_2 S_2^u &\geq -U_2^u \text{ on } \{\check{\tau} > 2\}. \end{aligned}$$

Proceeding in this way by induction, we can construct a strategy (α, β) such that for each $t = 1, \ldots, T$ and for each $u \in \{a, b\}$

$$\begin{aligned} \alpha_{t-1} + \beta_{t-1} S_{t-1}^u &\geq \alpha_t + \beta_t S_{t-1}^u \\ \alpha_t + \beta_t S_t^u &\geq -X_t^u \text{ on } \{\check{\tau} = t\}, \\ \alpha_t + \beta_t S_t^u &\geq -U_t^u \text{ on } \{\check{\tau} > t\}. \end{aligned}$$

We have proved that there is a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\alpha_{\check{\tau}} + \beta_{\check{\tau}} S_{\check{\tau}}^u \geq -X_{\check{\tau}}^u$ for each $u \in \{a, b\}$, that is, such that $\vartheta_{\check{\tau}}(\alpha_{\check{\tau}} + \xi_{\check{\tau}}, \beta_{\check{\tau}} + \zeta_{\check{\tau}}) \geq 0$. It follows that

$$\pi^b(\xi, \zeta) \geq -\alpha_0 = \max\left\{ \min\{X_0^a, X_0^b\}, \min\{U_0^a, U_0^b\} \right\},$$

completing the proof. ■

4.3 Martingale Representations for the Ask and Bid Prices

Theorem 4.5 *Under assumption* (2.2)*, the ask price of an American option* (ξ, ζ) *with physical delivery can be represented as*

$$\pi^a(\xi, \zeta) = \max_{\tau \in \mathcal{T}} \max_{S \in \mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau),$$

where $\mathbb{E}$ *is the expectation under the probability measure* $\mathbb{P} \in \mathcal{P}$ *that turns* $S \in \mathcal{S}$ *into a martingale.*

Proof We begin by constructing processes $\hat{S}, \hat{V}, \hat{Z}$ such that:

- For some $u \in \{a, b\}$
$$\hat{S}_0 = S_0^u,$$
$$\hat{V}_0 = V_0^u,$$
$$\hat{Z}_0 = Z_0^u,$$
and
$$Z_0^u = \max\{Z_0^a, Z_0^b\}. \tag{4.11}$$
- For each $t = 0, \ldots, T-1$ and each $\omega_t \in \Omega_t$ there are $v, w \in \{a, b\}$ such that
$$\begin{aligned} \hat{S}_{t+1}(\omega_t \mathrm{u}) &= S_{t+1}^v(\omega_t \mathrm{u}), \quad & \hat{S}_{t+1}(\omega_t \mathrm{d}) &= S_{t+1}^w(\omega_t \mathrm{d}), \\ \hat{V}_{t+1}(\omega_t \mathrm{u}) &= V_{t+1}^v(\omega_t \mathrm{u}), \quad & \hat{V}_{t+1}(\omega_t \mathrm{d}) &= V_{t+1}^w(\omega_t \mathrm{d}), \\ \hat{Z}_{t+1}(\omega_t \mathrm{u}) &= Z_{t+1}^v(\omega_t \mathrm{u}), \quad & \hat{Z}_{t+1}(\omega_t \mathrm{d}) &= Z_{t+1}^w(\omega_t \mathrm{d}), \end{aligned}$$
and
$$V_t^u(\omega_t) = \mathbb{E}_t^{uvw}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1} | \omega_t) \tag{4.12}$$
for each $u \in \{a, b\}$. Such v, w exist by Proposition 5.1.

The processes $\hat{S}, \hat{V}, \hat{Z}$ may not be unique. The lack of uniqueness may arise whenever there is more than one pair $v, w \in \{a, b\}$ satisfying (4.12), or there is more than one $u \in \{a, b\}$ such that (4.11) holds. In such cases we can choose any $\hat{S}, \hat{V}, \hat{Z}$ satisfying the conditions above.

Because of the small transaction costs assumption (2.2), $\hat{S} \in \mathcal{S}$. Let $\hat{\mathbb{P}} \in \mathcal{P}$ be the probability measure turning $\hat{S}$ into a martingale, and let $\hat{\mathbb{E}}$ denote the expectation under $\hat{\mathbb{P}}$. We claim that $\hat{Z}$ is the Snell envelope of the process $\xi + \zeta \hat{S}$ under $\hat{\mathbb{P}}$. Indeed, by the construction of $\hat{S}, \hat{V}, \hat{Z}$, for any $t = 0, \ldots, T-1$ and any $\omega_t \in \Omega_t$ there are $u, v, w \in \{a, b\}$ such that

$$\begin{aligned} \hat{S}_t(\omega_t) &= S_t^u(\omega_t), \quad & \hat{S}_{t+1}(\omega_t \mathrm{u}) &= S_{t+1}^v(\omega_t \mathrm{u}), \quad & \hat{S}_{t+1}(\omega_t \mathrm{d}) &= S_{t+1}^w(\omega_t \mathrm{d}), \\ \hat{V}_t(\omega_t) &= V_t^u(\omega_t), \quad & \hat{V}_{t+1}(\omega_t \mathrm{u}) &= V_{t+1}^v(\omega_t \mathrm{u}), \quad & \hat{V}_{t+1}(\omega_t \mathrm{d}) &= V_{t+1}^w(\omega_t \mathrm{d}), \\ \hat{Z}_t(\omega_t) &= Z_t^u(\omega_t), \quad & \hat{Z}_{t+1}(\omega_t \mathrm{u}) &= Z_{t+1}^v(\omega_t \mathrm{u}), \quad & \hat{Z}_{t+1}(\omega_t \mathrm{d}) &= Z_{t+1}^w(\omega_t \mathrm{d}), \end{aligned}$$

and (4.12) holds. Thus

$$\hat{V}_t(\omega_t) = V_t^u(\omega_t) = \mathbb{E}_t^{uvw}(\mathbf{Z}_{t+1}; \mathbf{Z}_{t+1} | \omega_t) = \hat{\mathbb{E}}(\hat{Z}_{t+1} | \omega_t).$$

Because $Z_T^u = \xi_T + \zeta_T S_T^u$ for each $u \in \{a, b\}$, we have $\hat{Z}_T = \xi_T + \zeta_T \hat{S}_T$. Since $Z_t^u = \max\{\xi_t + \zeta_t S_t^u, V_t^u\}$ for each $u \in \{a, b\}$, it follows that

$$\hat{Z}_t = \max\{\xi_t + \zeta_t \hat{S}_t, \hat{V}_t\} = \max\{\xi_t + \zeta_t \hat{S}_t, \hat{\mathbb{E}}(\hat{Z}_{t+1} | \mathcal{F}_t)\}$$

for each $t = 0, \ldots, T-1$, proving the claim. Next, we define a stopping time $\hat{\tau} \in \mathcal{T}$ by
$$\hat{\tau} = \min\{\, t \,|\, \hat{Z}_t = \xi_t + \zeta_t \hat{S}_t \}.$$
Because $\hat{Z}$ is the Snell envelope of $\xi + \zeta \hat{S}$ under $\hat{\mathbb{P}}$ it follows that

$$\hat{\mathbb{E}}(\xi_{\hat{\tau}} + \zeta_{\hat{\tau}} \hat{S}_{\hat{\tau}}) = \hat{Z}_0 = \max\{Z_0^a, Z_0^b\} = \pi^a(\xi, \zeta),$$

where the last equality holds because of Theorem 4.2. This proves that

$$\pi^a(\xi,\zeta) \le \max_{\tau\in\mathcal{T}} \max_{S\in\mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau).$$

To prove the reverse inequality take any $S \in \mathcal{S}$ and a strategy $(\alpha,\beta) \in \Phi(S^a, S^b)$ such that $\alpha_0 = \pi^a(\xi,\zeta)$ and $\vartheta_\tau(\alpha_\tau - \xi_\tau, \beta_\tau - \zeta_\tau) \ge 0$ for each $\tau \in \mathcal{T}$. Then $\xi_\tau + \zeta_\tau S^u_\tau \le \alpha_\tau + \beta_\tau S^u_\tau$ for each $u \in \{a,b\}$ and each $\tau \in \mathcal{T}$. Because $S^b_\tau \le S_\tau \le S^a_\tau$ it follows that $\xi_\tau + \zeta_\tau S_\tau \le \alpha_\tau + \beta_\tau S_\tau$. By Lemma 2.1 the process $\alpha + \beta S$ is a supermartingale under the probability measure $\mathbb{P} \in \mathcal{P}$ that turns S into a martingale. As a result,

$$\mathbb{E}(\xi_\tau + \zeta_\tau S_\tau) \le \mathbb{E}(\alpha_\tau + \beta_\tau S_\tau) \le \alpha_0 + \beta_0 S_0 = \alpha_0 = \pi^a(\xi,\zeta),$$

where $\mathbb{E}$ is the expectation under $\mathbb{P}$. Since $S \in \mathcal{S}$ and $\tau \in \mathcal{T}$ are arbitrary,

$$\max_{\tau\in\mathcal{T}} \max_{S\in\mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau) \le \pi^a(\xi,\zeta).$$

This completes the proof. ■

Theorem 4.6 *Under assumption* (2.2)*, the bid price of an American option* (ξ,ζ) *with physical delivery can be represented as*

$$\pi^b(\xi,\zeta) = \max_{\tau\in\mathcal{T}} \min_{S\in\mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau),$$

where $\mathbb{E}$ *is the expectation under the probability measure* $\mathbb{P} \in \mathcal{P}$ *turning* $S \in \mathcal{S}$ *into a martingale.*

Proof Take any $\tau \in \mathcal{T}$ and consider a European option with payoff $(\kappa,\lambda) = (\xi_\tau,\zeta_\tau)$ and exercise time T. We claim that

$$\pi^b(\kappa,\lambda) \le \pi^b(\xi,\zeta),$$

where $\pi^b(\kappa,\lambda)$ denotes the bid price of the European option (κ,λ) and where $\pi^b(\xi,\zeta)$ is the bid price of the American option (ξ,ζ). Take a strategy $(\alpha,\beta) \in \Phi(S^a,S^b)$ such that $\pi^b(\kappa,\lambda) = -\alpha_0$ and $\vartheta_T(\alpha_T + \kappa, \beta_T + \lambda) \ge 0$. If follows that for each $u \in \{a,b\}$

$$\alpha_T + \beta_T S^u_T \ge -\kappa - \lambda S^u_T = -\xi_\tau - \xi_\tau S^u_T.$$

For each fixed $u \in \{a,b\}$ there is a measure $\mathbb{P}^u \in \mathcal{P}$ turning the process $S^u \in \mathcal{S}$ into a martingale. By Lemma 2.1 we know that $\alpha + \beta S^u$ is a supermartingale under $\mathbb{P}^u$. It follows that for each $u \in \{a,b\}$

$$\alpha_\tau + \beta_\tau S^u_\tau \ge \mathbb{E}^u(\alpha_T + \beta_T S^u_T|\mathcal{F}_\tau) \ge \mathbb{E}^u(-\xi_\tau - \xi_\tau S^u_T|\mathcal{F}_\tau) = -\xi_\tau - \xi_\tau S^u_\tau,$$

where $\mathbb{E}^u$ is the expectation under $\mathbb{P}^u$. It means that $\vartheta_\tau(\alpha_\tau+\xi_\tau, \beta_\tau+\zeta_\tau) \ge 0$, and so $\pi^b(\kappa,\lambda) = -\alpha_0 \le \pi^b(\xi,\zeta)$. The claim has been verified. Now by Theorem 3.3

$$\begin{aligned}\pi^b(\xi,\zeta) \ge \pi^b(\kappa,\lambda) &= \min_{S\in\mathcal{S}} \mathbb{E}(\kappa + \lambda S_T) = \min_{S\in\mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_T) \\ &= \min_{S\in\mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau \mathbb{E}(S_T|\mathcal{F}_\tau)) = \min_{S\in\mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau).\end{aligned}$$

Since $\tau \in \mathcal{T}$ is arbitrary, it follows that

$$\pi^b(\xi, \zeta) \geq \max_{\tau \in \mathcal{T}} \min_{S \in \mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau).$$

To prove the reverse inequality take any $S \in \mathcal{S}$ and a strategy $(\alpha, \beta) \in \Phi(S^a, S^b)$ such that $\alpha_0 = -\pi^b(\xi, \zeta)$ and there is a $\tau \in \mathcal{T}$ such that $\vartheta_\tau(\alpha_\tau + \xi_\tau, \beta_\tau + \zeta_\tau) \geq 0$, and therefore $-\xi_\tau - \zeta_\tau S^u_\tau \leq \alpha_\tau + \beta_\tau S^u_\tau$ for each $u \in \{a, b\}$. Because $S^b_\tau \leq S_\tau \leq S^a_\tau$ it follows that $-\xi_\tau - \zeta_\tau S_\tau \leq \alpha_\tau + \beta_\tau S_\tau$. By Lemma 2.1 the process $\alpha + \beta S$ is a supermartingale under the probability measure $\mathbb{P} \in \mathcal{P}$ that turns S into a martingale. As a result,

$$\mathbb{E}(-\xi_\tau - \zeta_\tau S_\tau) \leq \mathbb{E}(\alpha_\tau + \beta_\tau S_\tau) \leq \alpha_0 + \beta_0 S_0 = \alpha_0 = -\pi^b(\xi, \zeta),$$

where $\mathbb{E}$ is the expectation under $\mathbb{P}$. Since $S \in \mathcal{S}$ is arbitrary,

$$\pi^b(\xi, \zeta) \leq \min_{S \in \mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau),$$

which implies that

$$\pi^b(\xi, \zeta) \leq \max_{\tau \in \mathcal{T}} \min_{S \in \mathcal{S}} \mathbb{E}(\xi_\tau + \zeta_\tau S_\tau),$$

completing the proof. ∎

5 Appendix: Two Technical Propositions

Here we shall state and outline the proofs of two technical propositions, which have already appeared in Tokarz and Zastawniak [TZ04], and which are given here for completeness. Full proofs, which are elementary but tedious, can be found in [Tok04].

Proposition 5.1 *Under assumption* (2.2)*, for any $t = 1, \ldots, T$, any $\mathbb{R}^2$-valued $\mathcal{F}_t$- measurable random variables $\mathbf{A} = (A^a, A^b)$ and $\mathbf{B} = (B^a, B^b)$ and any $c, d \in \{a, b\}$ the following conditions are equivalent:*

(a) $\min_{v,w \in \{a,b\}} \mathbb{E}^{avw}_{t-1}(\mathbf{A}; \mathbf{B}) = \mathbb{E}^{acd}_{t-1}(\mathbf{A}; \mathbf{B})$;

(b) $\min_{v,w \in \{a,b\}} \mathbb{E}^{bvw}_{t-1}(\mathbf{A}; \mathbf{B}) = \mathbb{E}^{bcd}_{t-1}(\mathbf{A}; \mathbf{B})$.

The conditions remain equivalent to one another if the two minima are replaced by maxima.

Proof Outline The main steps of the proof are:

- For any $c, d, e, f, u \in \{a, b\}$ such that $c \neq e$ and $d \neq f$ show that the inequalities

$$\mathbb{E}_{t-1}^{ucd}(\mathbf{A};\mathbf{B}) \leq \mathbb{E}_{t-1}^{ued}(\mathbf{A};\mathbf{B}), \tag{5.13}$$
$$\mathbb{E}_{t-1}^{ucd}(\mathbf{A};\mathbf{B}) \leq \mathbb{E}_{t-1}^{ucf}(\mathbf{A};\mathbf{B}) \tag{5.14}$$

  imply

$$\mathbb{E}_{t-1}^{ucd}(\mathbf{A};\mathbf{B}) \leq \mathbb{E}_{t-1}^{uef}(\mathbf{A};\mathbf{B}),$$

  and deduce that (5.13), (5.14) are equivalent to

$$\min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{A};\mathbf{B}) = \mathbb{E}_{t-1}^{ucd}(\mathbf{A};\mathbf{B}).$$

- Verify that for any $c, d, e, f \in \{a, b\}$ the inequalities (5.13), (5.14) with $u = a$ are equivalent to (5.13), (5.14) with $u = b$.

The equivalence of (a) and (b) follows directly from these two steps. ■

Proposition 5.2 *Under assumption* (2.2)*, for any* $t = 1, \dots, T$*, any* $\mathbb{R}^2$*-valued* $\mathcal{F}_t$*- measurable random variables* $\mathbf{A} = (A^a, A^b)$ *and* $\mathbf{B} = (B^a, B^b)$ *and any* $\mathbf{C}, \mathbf{D} \in \{\mathbf{A}, \mathbf{B}\}$ *the following conditions are equivalent:*

(a) $\max_{\mathbf{V},\mathbf{W}\in\{\mathbf{A},\mathbf{B}\}} \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{avw}(\mathbf{V};\mathbf{W}) = \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{avw}(\mathbf{C};\mathbf{D})$*;*

(b) $\max_{\mathbf{V},\mathbf{W}\in\{\mathbf{A},\mathbf{B}\}} \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{bvw}(\mathbf{V};\mathbf{W}) = \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{bvw}(\mathbf{C};\mathbf{D})$*.*

Proof Outline The main steps of the proof are:

- For any $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in \{\mathbf{A}, \mathbf{B}\}$ such that $\mathbf{C} \neq \mathbf{E}$ and $\mathbf{D} \neq \mathbf{F}$ and for any $u \in \{a, b\}$ show that the inequalities

$$\min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{C};\mathbf{D}) \geq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{E};\mathbf{D}), \tag{5.15}$$
$$\min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{C};\mathbf{D}) \geq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{C};\mathbf{F}) \tag{5.16}$$

  imply

$$\min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{C};\mathbf{D}) \geq \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{E};\mathbf{F}),$$

  and deduce that (5.15), (5.16) are equivalent to

$$\max_{\mathbf{V},\mathbf{W}\in\{\mathbf{A},\mathbf{B}\}} \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{V};\mathbf{W}) = \min_{v,w\in\{a,b\}} \mathbb{E}_{t-1}^{uvw}(\mathbf{C};\mathbf{D}).$$

- Verify that for any $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F} \in \{\mathbf{A}, \mathbf{B}\}$ the inequalities (5.15), (5.16) with $u = a$ are equivalent to (5.15), (5.16) with $u = b$.

The equivalence of (a) and (b) follows directly from these two steps. ■

6 Conclusions and Outlook

It has been demonstrated that the ask price $\pi^a(\xi, \zeta)$ of an American contingent claim (ξ, ζ) with physical delivery under small proportional transaction costs can be computed by a procedure resembling the standard construction of the Snell envelope. A suitable algorithm has also been proposed for the bid price $\pi^b(\xi, \zeta)$. A distinctive feature of the algorithms developed under small transaction costs is that two quantities need to be tracked at each tree node, rather than a single one as in the standard iterative construction of the Snell envelope in a friction free case. Moreover, representations of the ask and bid process in terms maximax and maximin martingale expectations of stopped payoffs have been established.

A natural question arises as to what will happen if the small costs assumption (2.2) is relaxed, so that only the no-arbitrage condition prevails. Results in Tokarz [Tok04], which apply to European options only, suggest that the algorithms for American options will need to be modified by keeping track of more than two quantities at each tree node whenever the bid-ask spreads for the stock price at adjacent nodes overlap. We conjecture that the maximax and maximin martingale expectation representations for the ask and bid option prices in Theorems 4.5 and 4.6 will remain valid in the general case.

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