Basic Stochastic Processes
by Zdzisław Brzeźniak and Tomasz Zastawniak
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Corrections in the 2nd printing

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Page and line numbers refer to the 2nd printing of the book.
A list of corrections in the 1st printing is also available. Please see
`http://www-users.york.ac.uk/~tz506/bsp`
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`bz506@york.ac.uk` or `tz506@york.ac.uk`
Your feedback will be greatly appreciated!

viii `insert at the end of Preface:`

As this book is going into its 3rd printing, we would like to thank our students and readers for their support and feedback. In particular, we wish to express our gratitude to Iaonnis Emmanouil of the University of Athens and to Brett T. Reynolds and Chris N. Reynolds of the University of Wales in Swansea for their extensive and meticulous lists of remarks and valuable suggestions, which helped us to improve the current version of *Basic Stochastic Processes*.

We would greatly appreciate further feedback from our readers, who are invited to visit the Web Page `http://www-users.york.ac.uk/~tz506/bsp` for more information and to check the latest corrections in the book.

Zdzisław Brzeźniak and Tomasz Zastawniak
Kingston upon Hull, June 2000

5_{4-3} `Replace the text of Exercise 1.6 by:`

Show that if ξ has discrete distribution with values $x_1, x_2, \ldots$, then F_ξ is constant on each interval $(s,t]$ not containing any of the x_i's and has jumps of size $P\{\xi = x_i\}$ at each x_i.

13^{1-7} `Replace Solution 1.6 by:`

If $s < t$ are real numbers such that $x_i \notin (s,t]$ for any i, then

$$F_\xi(t) - F_\xi(s) = P\{\xi \le t\} - P\{\xi \le s\} = P\{\xi \in (s,t]\} = 0,$$

i.e. $F_\xi(s) = F_\xi(t)$. Because F_ξ is non-decreasing, this means that F_ξ is constant on $(s,t]$. To show that F_ξ has a jump of size $P\{\xi = x_i\}$ at each x_i, we compute

$$\begin{aligned}\lim_{t \searrow x_i} F_\xi(t) - \lim_{s \nearrow x_i} F_\xi(s) &= \lim_{t \searrow x_i} P\{\xi \le t\} - \lim_{s \nearrow x_i} P\{\xi \le s\} \\ &= P\{\xi \le x_i\} - P\{\xi < x_i\} = P\{\xi = x_i\}.\end{aligned}$$

21_{11} $\int_B E(\xi)\, dP$ `should be` $\int_B \xi\, dP$

25_{10} $\eta = 1 - |2x - 1|$ `should be` $\eta(x) = 1 - |2x - 1|$

29_6 $B = \Omega$ `should be` $A = \Omega$

35^{11-12} `the sentence`
Indeed, if A is such a set, then $A = \{\eta \in 2A - 1\} \in \sigma(A)$.
`should be replaced by`
Indeed, if A is such a set, then $A = \{\eta \in B\}$, where $B = \{2x : x \in A \cap [0, \frac{1}{2}]\}$ is a Borel set, so $A \in \sigma(\eta)$.

37^4 $F(y)$ `should be` $F : \mathbb{R} \to \mathbb{R}$

37_2 $F(y)$ `should be` $F : \mathbb{R} \to \mathbb{R}$

59_5 $\bigcap_{n=1}^{\infty} P\{\tau > 2Kn\}$ `should be` $P\left(\bigcap_{n=1}^{\infty} \{\tau > 2Kn\}\right)$

60^8 `replace` 2) `by` 1)

72_{10-1} `replace the last 10 lines on page 72 by:`
for any $a < b$. Since the set of all pairs of rational numbers $a < b$ is countable, the event

$$A = \bigcap_{a<b \text{ rational}} \left\{ \lim_{n\to\infty} U_n[a, b] < \infty \right\} \tag{4.3}$$

has probability 1. (The intersection of countably many events has probability 1 if each of these events has probability 1.)

We claim that the sequence ξ_n converges a.s. to a limit ξ. Consider the set

$$B = \{\liminf_n \xi_n < \limsup_n \xi_n\} \subset \Omega$$

on which the sequence ξ_n fails to converge. Then for any $\omega \in B$ there are rational numbers a, b such that

$$\liminf_n \xi_n(\omega) < a < b < \limsup_n \xi_n(\omega),$$

implying that $\lim_{n\to\infty} U_n[a, b](\omega) = \infty$. This means that B and the event A in (4.3) are disjoint, so $P(B) = 0$, since $P(A) = 1$, which proves the claim.

79^{1-2} `the sentence`
Because this σ-field is generated by the family of sets $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots$ it follows that $\xi = 1_A$ a.s.
`should be replaced by`
The family $\mathcal{G}$ consisting of all sets $B \in \mathcal{F}$ such that $\int_B \xi\, dP = \int_B 1_A\, dP$ is a σ-field containing $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots$. As a result, $\mathcal{G}$ contains the σ-field $\mathcal{F}_\infty$ generated by the family $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots$. By Lemma 2.1 it follows that $\xi = 1_A$ a.s.

81_{10-1} `Solution 4.5 should be replaced by:`
In Example 3.4 it was verified that $\xi_n = E(\xi|\mathcal{F}_n)$ is a martingale. Let $\varepsilon > 0$. By Lemma 4.2 there is a $\delta > 0$ such that

$$P(A) < \delta \Longrightarrow \int_A |\xi|\, dP < \varepsilon.$$

By Jensen's inequality $|\xi_n| \leq E(|\xi|\,|\mathcal{F}_n)$ a.s., so

$$E(|\xi|) \geq E(|\xi_n|) \geq \int_{\{|\xi_n| \geq M\}} |\xi_n|\, dP \geq M P\{|\xi_n| > M\}.$$

If we take $M > E(|\xi|)/\delta$, then

$$P\{|\xi_n| > M\} < \delta.$$

Since $\{|\xi_n| > M\} \in \mathcal{F}_n$, it follows that

$$\int_{\{|\xi_n|>M\}} |\xi_n|\, dP \leq \int_{\{|\xi_n|>M\}} E(|\xi|\,|\mathcal{F}_n)\, dP = \int_{\{|\xi_n|>M\}} |\xi|\, dP < \varepsilon,$$

proving that $\xi_n = E(\xi|\mathcal{F}_n)$ is a uniformly integrable sequence.

89^{15} $P(\xi_{n+1} = 0|\xi_n = 1) = \dfrac{P(\xi_{n+1} = 0, \xi_n = 1)}{P(\xi_n = 1)}$
`should be`
$P(\xi_{n+1} = 1|\xi_n = 0) = \dfrac{P(\xi_{n+1} = 1, \xi_n = 0)}{P(\xi_n = 0)}$

93^5 $P(\eta_1 = 0) = q = 1 - p$ `should be` $P(\eta_1 = -1) = q = 1 - p$

93^9 $\eta_0 = 0$ `should be` $\xi_0 = 0$

93^9 $\eta_0 = i$ `should be` $\xi_0 = i$

94_4 $p_n = P(\xi_n = 0)$ `should be` $p_0(n) = P(\xi_n = 0)$

96_4 $j \geq i \geq 0$ `should be` $j, i \geq 0$

97^{5-7} `replace Propostion 5.6 by:`
The probability of survival in Exercise 5.12, part 2) equals 0 if $\lambda \leq 1$, and $1 - \hat{r}^k$ if $\lambda > 1$, where k is the initial Vugiel population and $\hat{r} \in (0, 1)$ is a solution to

$$r = e^{(r-1)\lambda}. \tag{5.28}$$

98^{15} probability of extinction is certain
`should be`
probability of extinction is 1

100^5 $\lambda > 0$ `should be` $\lambda > 1$

100_{12} $(n+1)$th `should be` $(n+1)$st

100_5 ± 1 `should be` -1

103^7 $F_{jj}(x) \to \sum_n f_n(j|j) = 1$
`should be`
$F_{jj}(x) \to \sum_n f_n(j|j) = 1$ as $x \nearrow 1$

103_3 (5.73) `should be` (5.43)

106^{11} $\varepsilon := p(j|i)p(i|j)$ `should be` $\varepsilon := p_m(j|i)p_n(i|j)$

106^{15} $\sum_{r,s\in S} p_m(i|s)p_k(s|r)p_n(r|i)$ `should be` $\sum_{r,s\in S} p_n(i|s)p_k(s|r)p_m(r|i)$

106_{11-1} `These lines should be replaced by the following:`
To prove 5) it is enough to show that $d(i) \le d(j)$. Using the first inequality derived above, we have

$$p_{n+k+m}(i|i) \ge \varepsilon p_k(j|j)$$

for all $k \in \mathbb{N}$. From this inequality we can draw two conclusions:
(a) $d(i)|n+m$, since by taking $k=0$ we get $p_{n+m}(i|i) > 0$;
(b) if $p_k(j|j) > 0$, then $p_{n+k+m}(i|i) > 0$.
From (a) and (b) we can see that $d(i)|k$ provided that $p_k(j|j) > 0$. This proves what is required.

107_{11} $p(j|i) > 0$ for all $i, j \in C$
`should be`
for all $i, j \in C$ there exists an $n \in \mathbb{N}$ such that $p_n(j|i) > 0$

108^2 $R = S \setminus C$ `should be` $R = S \setminus T$

108^2 If $i \leftrightarrow j$, the both `should be` If $i \leftrightarrow j$, then both

113^5 `the line`
where $F_{ji}(x)$ is defined in (5.39);
`should be replaced by`
where $F_{ji}(1)$ is the probability that the chain will ever visit state j if it starts at i, see (5.39), and where m_j is the mean recurrence time of state j, see (5.44);

113_{11-10} if each $i \in S$ is ergodic.
`should be`
if each $i \in S$ is ergodic, i.e. each state $i \in S$ is positive, recurrent and aperiodic.

113_8 ergodic Markov chain
`should be`
ergodic irreducible Markov chain

113_5 `replace the hint by:`
Use Theorem 5.5. You may assume as a known fact that if j is recurrent and $i \leftrightarrow j$, then $F_{ji}(1) = 1$.

114^{12} $\sum\sum_i$ `should be` $\sum_i$

114_{10-7} `replace Exercise 5.36 by:`
Prove that if there exists an invariant measure for the Markov chain in Exercise 5.14, then $\lambda' := \sum_{j=0}^{\infty} jq'_j < 1$. Assuming that the converse is also true, conclude that the chain is ergodic if and only if $\lambda' < 1$. Show that if such an invariant measure exists, then it is unique.

114_6 `replace the hint by:`
Suppose that $\pi = \sum_{j=0}^{\infty} \pi_j \delta_j$ is an invariant measure. Write down an infinite system of linear equations for π_j. If you don't know how to follow, look at the solution.

115^{14} $p_{kn_0}(j|i) = q_n(j|i)$ `should be` $p_{kn_0}(j|i) = q_k(j|i)$

116^6 $\max_k p_n(j|k)$ `should be` $\min_k p_n(j|k)$

116^8 $\min_i p_n(j|i)$ `should be` $\min_i p_{n+1}(j|i)$

116^{12} $\min_k p_n(j|k)$ `should be` $\max_k p_n(j|k)$

116^{14} $\max_i p_n(j|i)$ `should be` $\max_i p_{n+1}(j|i)$

116^{15} $m_j(n)$ `should be` $m_n(j)$

116_4 $\varepsilon \sum_{s\in S} p_n(j|s)\varepsilon p_n(s|j)$ `should be` $\varepsilon \sum_{s\in S} p_n(j|s) p_n(s|j)$

116_2 Chapman-Kolmogorov equations
`should be`
by the Chapman-Kolmogorov equations

116_1 $p_k(s|i) \ge \varepsilon$ `should be` $p(s|i) \ge \varepsilon$

117^2 $m_n(j) \sum_{s\in S}(1-\varepsilon)$ `should be` $(1-\varepsilon)m_n(j)$

117^9 m_{n+1} `should be` $m_{n+1}(j)$

117^{11} m_n `should be` $m_n(j)$

124^9 equals ± 1 or 0 `should be` equals -1 or 0

124_{10} $j \ge i \ge 1$ `should be` $i \ge 1$, $j \ge i - 1$

124_{10-7} `the sentences`
On the other hand, $P(\xi_{n+1} = j|\xi_n = 0) = \frac{\lambda^j}{j!}e^{-\lambda}$ if $j \ge i = 0$. Finally, if $j = i-1$, $i \ge 1$, then $P(\xi_{n+1} = i-1|\xi_n = i) = pe^{-\lambda}$.
`should be replaced by`
On the other hand, if $j \ge 0$, then $P(\xi_{n+1} = j|\xi_n = 0) = \frac{\lambda^j}{j!}e^{-\lambda}$.

124_5 `formula (5.71) should be replaced by`

$$p(j|i) = \begin{cases} q_j, & \text{if } i = 0,\ j \in \mathbb{N}, \\ q'_{j-i+1}, & \text{if } i \geq 1,\ j \geq i-1, \\ 0, & \text{otherwise,} \end{cases} \tag{5.71}$$

124_3 `formula (5.72) should be replaced by`

$$q'_k = \begin{cases} pq_k, & \text{if } k = 0, \\ (1-p)q_{k-1} + pq_k, & \text{if } k \geq 1. \end{cases} \tag{5.72}$$

127^4 $P_{00}(x) = \sum_{n=0}^{\infty} f_n(0|0)x^n$ `should be` $P_{00}(x) = \sum_{n=0}^{\infty} p_n(0|0)x^n$

130_6 $\sum_{i \in S} p_n(j|i)\mu_i \leq \mu_i$ `should be` $p_n(j|i)\mu_i \leq \mu_i$

132^2 $\Pi(x) = \sum_{j=0}^{\infty} \pi_j x^i$ `should be` $\Pi(x) = \sum_{j=0}^{\infty} \pi_j x^j$

132^3 $Q(x) = \sum_{j=0}^{\infty} q_j x^i$ `should be` $Q(x) = \sum_{j=0}^{\infty} q_j x^j$

132^4 $G(x) = \sum_{j=0}^{\infty} q'_j x^i$ `should be` $G(x) = \sum_{j=0}^{\infty} q'_j x^j$

132_{13-7} `these lines should be replaced by:`
1) if $\lambda' \neq 1$, then $\Pi(x) \to \pi_0 \frac{1+\lambda-\lambda'}{1-\lambda'}$;
2) if $\lambda' = 1$, then $\Pi(x) \to \infty$.
In case 1) $\frac{1+\lambda-\lambda'}{1-\lambda'} \geq 1$, since $\pi_0 \leq 1$. It follows that $\frac{\lambda}{1-\lambda'} \geq 0$, and so $\lambda' < 1$. Moreover, in this case

$$\pi_0 = \frac{1-\lambda'}{1-\lambda'+\lambda}.$$

The above argument shows that if there exists an invariant measure, then it is unique. Now suppose that an invariant measure exists (and then it is unique).

137^2 $\left(\frac{q}{p}\right) ix$ `should be` $\left(\frac{q}{p}\right)^i x$

137^3 $x = \phi(i) - \phi(0)$ `should be` $x = \phi(1) - \phi(0)$

137_1 `insert` $=$ `before` $\dfrac{(\frac{q}{p})^i - 1}{(\frac{q}{p})^a - 1}$

154^5 $\{(x,y) : x - y \in A\}$ `should be` $\{(x,y) : y - x \in A\}$

154^6 $\{y : x - y \in A\}$ `should be` $\{y : y - x \in A\}$

154^7 $p(t-s, x, x-u)$ `should be` $p(t-s, x, x+u)$

165_{15} $P\{\eta_1 < s\}$ `should be` $P\{\eta_1 < s_1\}$

173^5 $t \mapsto V(t) = W(t+T)$ `should be` $t \mapsto V(t) = W(t+T) - W(T)$

175^9 $2t$ `in the denominator should be` $2t^2$

175^{11} t `in the denominator should be` t^2

175_1 $p(t-s, W(t), y)$ `should be` $p(t-s, W(s), y)$

176^2 $p(t-s, W(t), y)$ `should be` $p(t-s, W(s), y)$

181_{8-7} the approximations should consist only of random variables adapted to
`should be`
the approximations of the integrand should consist only of processes adapted to

181_4 depend only by what has happened up to time t, not on any future events.
`should be`
depend only on what has happened up to time t, but not on any future events.

189^{10} form `should be` from

189_5 Exercise 7.4 `should be` Exercise 7.6

190_6 `insert the following sentence at the end of the hint:`
You may also need the following identity:

$$\left(a^2-b^2\right)^2 = (a-b)^4 + 4\,(a-b)^3\,b + 4\,(a-b)^2\,b^2.$$

198_{10} $|f_n(t) - f(t)|^2 \le 2C^2$ `should be` $|f_n(t) - f(t)|^2 \le 4C^2$

198_7 $2TC^2$ `should be` $4TC^2$

199_{7-6} $\sum_{i=0}^{n-1}(\Delta_i^n W)^2 \to t$ `should be` $\sum_{i=0}^{n-1}(\Delta_i^n W)^2 \to T$

199_5 $\sum_{i=0}^{n_k-1}(\Delta_i^{n_k} W)^2 \to t$ `should be` $\sum_{i=0}^{n_k-1}(\Delta_i^{n_k} W)^2 \to T$

200^7 $x \in [-n-1, n+1]$ `should be` $x \notin [-n-1, n+1]$

201^{12} will replaced `should be` will be replaced

201^{14} where a belongs to M_t^2 and b to $\mathcal{L}_t^1$
`should be`
where a belongs to $\mathcal{L}_t^1$ and b to M_t^2

204_{12} form `should be` from

205^4 To obtain a solution
`should be replaced by`
Once we have shown that such a $\xi \in M_T^2$ exists, to obtain a solution

205_7 are strict contractions.
`should be replaced by`
are strict contractions with contracting constants α_1 and α_2 such that $\alpha_1 + \alpha_2 < 1$.

206^2 $\frac{C^2}{\lambda}||\xi-\zeta||_\lambda$ `should be` $\frac{C^2}{\lambda}||\xi-\zeta||^2_\lambda$

206^4 $\lambda > C^2$ `should be` $\lambda > C^2/\varepsilon$

206^4 Φ_2 is a strict contraction.
`should be replaced by`
Φ_2 is a strict contraction with contracting constant $\leq \varepsilon$.

210^8 $E(\eta_j\zeta_j\Delta_j^2)$ `should be` $E(\eta_j\zeta_j|\Delta_j W|^2)$

210^8 $E(\Delta_j^2)$ `should be` $E(|\Delta_j W|^2)$

214_2 `the line`
in L^2 as $n \to \infty$, since by the Cauchy-Schwartz inequality
`should be replaced by`
in L^2 as $n \to \infty$. Indeed, by the classical inequality $\left|\sum_{i=0}^{n-1} a_i\right|^2 \leq n\sum_{i=0}^{n-1}|a_i|^2$ and by the Cauchy-Schwartz inequality

215_2 $4\left(E(W_t-W_s)^2W_s^2\right)$ `should be` $4E\left((W_t-W_s)^2W_s^2\right)$

219_{13} $F(t,W(t)) = nW(t)^{n-1}$ `should be` $F'_x(t,W(t)) = nW(t)^{n-1}$

219_9 $F(t,W(t)) = nW(t)^{n-1}$ `should be` $F'_x(t,W(t)) = nW(t)^{n-1}$

220^1 $\sigma\varepsilon^{\alpha t}F'_x(t,x)$ `should be` $\sigma e^{\alpha t}F'_x(t,x)$

220^2 $\sigma\varepsilon^{\alpha t}F'_x(t,x)$ `should be` $\sigma e^{\alpha t}F'_x(t,x)$