Maximal subsemigroups of the semigroup of all mappings on an infinite set

James Mitchell
joint work with J. East (Sydney)
and Y. Péresse (St Andrews)

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What's the problem?

Let S be a semigroup or group and let $T < S$. Then T is *maximal* if

$$T \leqslant U < S \Rightarrow T = U.$$

Equivalently, $\langle T, s \rangle = S$ for all $s \in S \setminus T$.

One way to understand the structure of S is to understand the subsemigroup or subgroup structure.

Starting point: understand the maximal subsemigroups!

We concentrate on:

- S_Ω - the symmetric group on a set Ω;
- Ω^Ω - the full transformation semigroup on Ω.

If $|\Omega| = n \in \mathbb{N}$, then we write S_n and n^n.

Finite permutation groups

Theorem (O'Nan–Scott '79)

A maximal subgroup of S_n or A_n is one of the following:

- $S_k \times S_{n-k}$ (*intransitive*)
- $S_k \wr S_m$ with $mk = n$, $m > 1$, $k > 1$ (*imprimitive*)
- $S_k \wr S_m$ in its product action where $m^k = n$, $m \geq 5$, $k > 1$ (*wreath*)
- $AGL(d, p)$ where p prime and $p^d = n$ (*affine*)
- $T^k \cdot (\mathrm{Out}(T) \times S_k)$ where T non-abelian simple and $|T|^{k-1} = n$ (*diagonal*)
- an almost simple group G in some primitive action – $T \leqslant G \leqslant \mathrm{Aut}(T)$ where T non-abelian simple (*almost simple*).

Finite transformation semigroups

If $k \leqslant n$, then write $I_k = \{ f \in n^n : |(n)f| \leqslant k \}$.

Theorem (Trivial)

A maximal subsemigroup of the full transformation semigroup n^n is one of the following:

- $S_n \cup I_{n-2}$;
- $G \cup I_{n-1}$ *where G is a maximal subgroup of S_n.*

Proof.

If $f \in n^n$ such that $|(n)f| = k \leqslant n-1$, then

$$\langle S_n, f \rangle = S_n \cup I_k.$$

This implies that the subsemigroups in the theorem are maximal.

If M is maximal, then $M \cap S_n = S_n$ or $=$ a maximal subgroup (since I_{n-1} is an ideal). □

Some infinite permutation groups

If Ω is an infinite set, then $\{\Sigma_1, \ldots, \Sigma_n\}$ is a *finite partition* of Ω if $\Sigma_1, \ldots, \Sigma_n$ partition Ω and $|\Sigma_i| = |\Omega \setminus \Sigma_i| = |\Omega|$.

If $\Sigma \subseteq \Omega$ is arbitrary, then define:

Pointwise stabilizer:
$S_{(\Sigma)} = S_{\Omega\setminus\Sigma} = \{\, f \in S_\Omega \ : \ (\sigma)f = \sigma \ (\forall \sigma \in \Sigma)\,\}$

Setwise stabilizer: $S_{\{\Sigma\}} = \{\, f \in S_\Omega \ : \ (\sigma)f \in \Sigma \ (\forall \sigma \in \Sigma)\,\}$

Stabilizer of finite partition:

$$\mathrm{Stab}(\Sigma_1, \ldots, \Sigma_n) = \{\, f \in S_\Omega \ : \ (\forall i)(\exists j)(\Sigma_i f = \Sigma_j)\,\} \cong S_\Omega \wr S_n$$

Lemma

If $\Gamma_1, \Gamma_2 \subseteq \Omega$ *and* $|\Gamma_1 \cap \Gamma_2| = \min\{|\Gamma_1|, |\Gamma_2|\}$, *then*

$$S_{\Gamma_1 \cup \Gamma_2} = \langle\, S_{\Gamma_1}, S_{\Gamma_2}\,\rangle.$$

Infinite symmetric groups - intransitive case

$G \leqslant S_\Omega$ intransitive $\Rightarrow \exists \Sigma \subseteq \Omega$ such that $\Sigma^G = \Sigma \Rightarrow G \leqslant S_{\{\Sigma\}}$

Proposition

$S_{\{\Sigma\}}$ is maximal if and only if $|\Sigma| < \infty$ or $|\Omega \setminus \Sigma| < \infty$.

Proof.

($\Rightarrow$) $|\Sigma| = |\Omega| = |\Omega \setminus \Sigma| \Rightarrow S_{\{\Sigma\}} < \mathrm{Stab}(\Sigma, \Omega \setminus \Sigma) < S_\Omega$.

($\Leftarrow$) S_Ω is transitive and primitive $\Rightarrow S_{\{\alpha\}}$ is maximal for all $\alpha \in \Omega$

Proceed by induction:

- $\Gamma_1 := \Omega \setminus \Sigma$ and $f \in S_\Omega \setminus S_{\{\Sigma\}}$
- $\exists \alpha \in \Sigma$ such that $(\alpha)f \notin \Sigma$
- $\langle S_{\{\Sigma\}}, f \rangle \geq \langle S_{\Gamma_1}, f^{-1} S_{\Gamma_1} f \rangle = \langle S_{\Gamma_1}, S_{\Gamma_1 f^{-1}} \rangle = S_{\Gamma_1 \cup \Gamma_1 f^{-1}}$
- $S_{\{\Sigma \setminus \{\alpha\}\}} = S_{\{\Sigma\}} S_{\Gamma_1 \cup \Gamma_1 f^{-1}} \leqslant \langle S_{\{\Sigma\}}, S_{\Gamma_1 \cup \Gamma_1 f^{-1}} \rangle \leqslant \langle S_{\{\Sigma\}}, f \rangle$.
- $S_{\{\Sigma \setminus \{\alpha\}\}}$ maximal and $S_{\{\Sigma\}} \setminus S_{\{\Sigma \setminus \{\alpha\}\}} \neq \emptyset$
- $S_\Omega = \langle S_{\{\Sigma \setminus \{\alpha\}\}}, S_{\{\Sigma\}} \rangle \leqslant \langle S_{\{\Sigma\}}, f \rangle$ and so $S_{\{\Sigma\}}$ maximal. $\square$

Infinite symmetric groups - imprimitive case I

$\mathrm{Stab}(\Sigma_1, \ldots, \Sigma_n)$ is imprimitive, is it maximal?

Let $\alpha \in \Sigma_1$ and $\beta \in \Sigma_2$. Then $\langle\, \mathrm{Stab}(\Sigma_1, \ldots, \Sigma_n), (\alpha\, \beta)\,\rangle \neq S_\Omega$.

If $\Sigma, \Gamma \subseteq \Omega$, then Σ is *almost equal* Γ if

$|\Sigma \setminus \Gamma| + |\Gamma \setminus \Sigma| < |\Omega|$ and we write $\Sigma \approx \Gamma$.

If $BS_\Omega = \{\, f \in S_\Omega \;:\; |\,\mathrm{supp}(f)| < |\Omega|\,\}$, then

$$\langle\, \mathrm{Stab}(\Sigma_1, \ldots, \Sigma_n), BS_\Omega\,\rangle = \mathrm{Stab}(\Sigma_1, \ldots, \Sigma_n) \cdot BS_\Omega$$
$$= \mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n) = \{f \in S_\Omega : (\forall i)(\exists j)((\Sigma_i)f \approx \Sigma_j)\} \neq S_{\mathbb{N}}.$$

Infinite symmetric groups - imprimitive case II

Theorem (Ball '66)

$\mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n)$ *is maximal for all* $n \geq 2$.

Proof.

- let $f \in S_\Omega \setminus \mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n)$
- $\exists i, j, k$ such that $|\Sigma_i f \cap \Sigma_j| = \infty$ and $|\Sigma_i f \cap \Sigma_k| = \infty$
- $S_{\Sigma_i f} \leqslant f^{-1}\,\mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n) f$
- $S_{\Sigma_j}, S_{\Sigma_k} \leqslant \mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n)$
- $S_{\Sigma_j \cup \Sigma_k} = \langle\, S_{\Sigma_j}, S_{\Sigma_i f}, S_{\Sigma_k} \,\rangle \leqslant \langle\, \mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n), f \,\rangle$
- $\mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n)$ is 2-transitive on $\Sigma_1, \ldots, \Sigma_n$

$$\begin{aligned} S_\Omega &\leqslant \langle\, S_{\Sigma_1 \cup \Sigma_2}, S_{\Sigma_2 \cup \Sigma_3}, \ldots, S_{\Sigma_{n-1} \cup \Sigma_n} \,\rangle \\ &\leqslant \langle\, \mathrm{AStab}(\Sigma_1, \ldots, \Sigma_n), f \,\rangle. \quad \square \end{aligned}$$

Filters and ideals - I

A *filter* $\mathcal{F}$ is a subset of the power set $\mathcal{P}(\Omega)$ such that

- $\emptyset \notin \mathcal{F}$
- if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An *ideal* $\mathcal{I}$ is a subset of $\mathcal{P}(\Omega)$ such that

- $\emptyset \in \mathcal{I}$ and $\Omega \notin \mathcal{I}$
- if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$
- if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

For example, if $\alpha \in \Omega$, then $\mathcal{F} = \{ A \subseteq \Omega : \alpha \in A \}$ is an filter. Such a filter $\mathcal{F}$ is called *principal*.

Filters and ideals - II

An *ultrafilter* is a filter not contained in any other filter.

If $\mathcal{F}$ is a filter on Ω, then the *stabilizer* of $\mathcal{F}$ in S_Ω is

$$S_{\{\mathcal{F}\}} = \{\, f \in S_\Omega \,:\, (\forall A \subseteq \Omega)(A \in \mathcal{F} \leftrightarrow (A)f \in \mathcal{F})\,\}.$$

Theorem (Richman '67)

If $\mathcal{F}$ is an ultrafilter, then

(a) $S_{\{\mathcal{F}\}}$ *has two orbits on infinite coinfinite subsets of* Ω

(b) $S_{\{\mathcal{F}\}}$ *is a maximal subgroup of* S_Ω

(c) $S_{\{\mathcal{F}\}} = \bigcup_{A \in \mathcal{F}} S_{(A)}$.

Corollary

There are $2^{2^{|\Omega|}}$ *non-conjugate maximal subgroups* S_Ω.

Some non-maximal ideals

Theorem

Let $\mathcal{I}$ be an ideal of Ω such that $S_{\{\mathcal{I}\}}$ has 3 orbits on infinite coinfinite subsets. Then $S_{\{\mathcal{I}\}}$ is maximal.

There exist ideals $\mathcal{I}$ on Ω such that $S_{\{\mathcal{I}\}}$ have 4 orbits on infinite coinfinite subsets and $S_{\{\mathcal{I}\}}$ is not maximal.

There are $2^{2^{|\Omega|}}$ maximal subgroups that are stabilizers of non-maximal ideals.

Example. Define

$$\mathcal{I} = \{ A \subseteq \mathbb{Q} \ : \ \mathbb{Q} \not\hookrightarrow A \} \text{ or}$$
$$\mathcal{I} = \{ A \subseteq \mathbb{Q} \ : \ A \text{ is nowhere dense} \}.$$

Then $\mathcal{I}$ is an ideal and $S_{\{\mathcal{I}\}}$ has 3 orbits on moieties.

Theorem (Macpherson & Neumann '90)

There exists a maximal subgroup of S_Ω that does not contain any $S_{(\Sigma)}$ for any $\Sigma \subseteq \mathbb{N}$.

Theorem (Brazil, Covington, Penttila, Praeger, Woods '94)

Let G be a maximal subgroup of S_Ω such that $S_{(\Sigma)} \leqslant S_\Omega$ for some Σ such that $|\Omega \setminus \Sigma| = |\Omega|$. Then

(i) $G = \mathrm{AStab}(\mathcal{P})$ *for some finite partition $\mathcal{P}$ of Ω*

(ii) $G = S_{\{\mathcal{F}\}}$ *for some specific type of filter $\mathcal{F}$.*

Infinite symmetric groups - wreath case

There is an analogue of this case but I'm not going to talk about it...

> *'It seems hopeless to try to prove an analogue of the O'Nan–Scott Theorem in the infinite case.'*

Containment in maximal subgroups

Theorem (Zorn's Lemma)

Let G be any (semi)group and let $H \leqslant G$ such that $\exists K \subseteq G$ with $|K| < \infty$ and $\langle H, K \rangle = G$. Then H is contained in a maximal (semi)subgroup of G.

Theorem (Macpherson & Praeger '90)

Let G be a subgroup of $S_{\mathbb{N}}$ that is not highly transitive. Then G is contained in a maximal subgroup.

Theorem (Baumgartner, Shelah, Thomas '93)

It is consistent and independent of ZFC that $\exists G \leqslant S_{\mathbb{N}}$ not contained in any maximal subgroup.

Infinite transformation semigroups - preliminaries

The functions with finite image:

$$\mathfrak{F} = \{ f \in \Omega^\Omega : |\Omega f| < \infty \}.$$

A subsemigroup S of Ω^Ω is *dense* if for all finite $\Sigma \subseteq \Omega$ and for all $f \in \Sigma^\Sigma$ there exists $g \in S$ such that $g|_\Sigma = f$.

Lemma

If M is a maximal subsemigroup of Ω^Ω, then M is dense.

Proof.

$M \neq \Omega^\Omega \Rightarrow M \leqslant M \cup \mathfrak{F} \neq \Omega^\Omega$. □

Proposition (Macpherson & Praeger '90)

Let S be a countable subsemigroup of Ω^Ω. Then S is contained in a maximal subsemigroup of Ω^Ω.

Infinite transformation semigroups - parameters

If $f \in \Omega^\Omega$ and $\Sigma \subseteq \Omega$ such that $f|_\Sigma$ is injective and $\Sigma f = \Omega f$, then Σ is a *transversal* of f.

$$\begin{aligned} d(f) &= |\Omega \setminus \Omega f| \\ c(f) &= |\Omega \setminus \Sigma| \text{ where } \Sigma \text{ is any transversal of } f \\ k(f,\mu) &= |\{\, \alpha \in \Omega \,:\, |\alpha f^{-1}| \geq \mu \,\}| \end{aligned}$$

where $\mu \leqslant |\Omega|$.

Theorem (Howie, Higgins, Ruškuc '98)
Let Ω be an infinite set and let $f, g \in \Omega^\Omega$ such that $c(f) = 0$, $d(f) = |\Omega|$, $d(g) = 0$, and $k(g, |\Omega|) = |\Omega|$. Then $\langle\, S_\Omega, f, g \,\rangle = \Omega^\Omega$.

Maximal subsemigroups containing the symmetric group

Theorem (East, M., Péresse '11)

Let Ω be any infinite set and let $M \lneq \Omega^\Omega$ such that $S_\Omega \leqslant M$. If $|\Omega|$ is regular, then M is maximal if and only if M is one of:

$$\{\, f \in \Omega^\Omega \,:\, c(f) < \mu \text{ or } d(f) \geq \mu \,\} \text{ for some } \aleph_0 \leqslant \mu \leqslant |\Omega|;$$

$$\{\, f \in \Omega^\Omega \,:\, c(f) = 0 \text{ or } d(f) > 0 \,\};$$

$$\{\, f \in \Omega^\Omega \,:\, c(f) \geq \mu \text{ or } d(f) < \mu \,\} \text{ for some } \aleph_0 \leqslant \mu \leqslant |\Omega|;$$

$$\{\, f \in \Omega^\Omega \,:\, c(f) > 0 \text{ or } d(f) = 0 \,\}$$

$$\{\, f \in \Omega^\Omega \,:\, k(f, |\Omega|) < |\Omega| \,\}.$$

If $|\Omega|$ is a singular cardinal, then M is maximal if and only if M is one of the first four subsemigroups above or

$$\{\, f \in \Omega^\Omega \,:\, (\exists \nu < |\Omega|)\,(k(f, \nu) < |\Omega|) \,\}.$$

The countable case

Theorem (East, M., Péresse '11)

Let $M \lneq \mathbb{N}^{\mathbb{N}}$ such that $S_{\mathbb{N}} \leqslant M$. Then M is maximal if and only if M is one of:

$$\{ f \in \Omega^{\Omega} \ : \ c(f) < \infty \text{ or } d(f) = \infty \}$$
$$\{ f \in \Omega^{\Omega} \ : \ c(f) = 0 \text{ or } d(f) > 0 \}$$
$$\{ f \in \Omega^{\Omega} \ : \ c(f) = \infty \text{ or } d(f) < \infty \}$$
$$\{ f \in \Omega^{\Omega} \ : \ c(f) > 0 \text{ or } d(f) = 0 \}$$
$$\{ f \in \Omega^{\Omega} \ : \ k(f, \aleph_0) < \infty \}.$$

Koppitz independently proved that the semigroups in the above theorem are maximal.

Stabilizers of finite sets

Theorem (East, M., Péresse '11)

Let $S := S_\Omega$, let $\Sigma \subseteq \Omega$ be finite, and let $M \leqslant \Omega^\Omega$ such that $M \cap S_\Omega = S_{\{\Sigma\}}$. Then M is maximal if and only if M is one of:

$$\{f \in \Omega^\Omega : \quad d(f) \geq \mu \text{ or } \Sigma \not\subseteq \Omega f \text{ or } ((\Omega \setminus \Sigma)f \subseteq \Omega \setminus \Sigma \text{ and } c(f) < \mu)\}$$

$$\{f \in \Omega^\Omega : \quad (\Omega \setminus \Sigma)f \subseteq \Omega \setminus \Sigma \text{ or } \Sigma \not\subseteq \Omega f\} \cup \mathfrak{F}$$

$$\{f \in \Omega^\Omega : \quad \Sigma f \subseteq \Sigma \text{ or } |\Sigma f| < |\Sigma|\} \cup \mathfrak{F}$$

$$\{f \in \Omega^\Omega : \quad c(f) \geq \mu \text{ or } |\Sigma f| < |\Sigma| \text{ or } (\Sigma f = \Sigma \text{ and } d(f) < \mu)\}.$$

Almost stabilizers of finite partitions

Let $\mathcal{P} = \{A_1, A_2, \ldots, A_n\}$ where $n \geq 2$ be a finite partition of $\mathbb{N}$ and let $f \in \mathbb{N}^{\mathbb{N}}$. Then define $\rho_f \subseteq \{1, 2, \ldots, n\}^2$ by

$$\rho_f = \{\, (i, j) \;:\; |A_i f \cap A_j| = \infty \,\}$$

$$\rho_f^{-1} = \{\, (i, j) \;:\; (j, i) \in \rho_f \,\}$$

A binary relation σ is *total* if for all $\alpha \in \Omega$ there exists $\beta \in \Omega$ such that $(\alpha, \beta) \in \sigma$.

Theorem (East, M., Péresse '11)

Let M be a subsemigroup of $\mathbb{N}^{\mathbb{N}}$ such that $M \cap S_{\mathbb{N}} = \mathrm{AStab}(\mathcal{P})$. Then M is maximal if and only if M is one of:

$$\mathrm{AStab}(\mathcal{P}) \cup \{\, f \in \mathbb{N}^{\mathbb{N}} \;:\; \rho_f \textit{ is not total} \,\}$$
$$\mathrm{AStab}(\mathcal{P}) \cup \{\, f \in \mathbb{N}^{\mathbb{N}} \;:\; \rho_f^{-1} \textit{ is not total} \,\}.$$

Ultrafilters

Theorem (East, M., Péresse '11)

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $M \leqslant \mathbb{N}^{\mathbb{N}}$ such that $M \cap S_{\mathbb{N}} = S_{\{\mathcal{F}\}}$. Then M is maximal if and only if M is one of:

$$\{\, f \in \mathbb{N}^{\mathbb{N}} \,:\, (\forall A \subseteq \mathbb{N})(A \in \mathcal{F} \to Af \in \mathcal{F} \text{ or } c(f|_A) > 0) \,\}$$

$$\{\, f \in \mathbb{N}^{\mathbb{N}} \,:\, (\forall A \subseteq \mathbb{N})(A \notin \mathcal{F} \to Af \notin \mathcal{F} \text{ or } c(f|_A) > 0)(A \notin \mathcal{F} \to Af \notin$$

Corollary

There are $2 \times 2^{2^{\aleph_0}}$ non-conjugate maximal subsemigroups of $\mathbb{N}^{\mathbb{N}}$.

A non-ultrafilter

Theorem (East, M., Péresse '11)
Let $A \subseteq \mathbb{N}$ be infinite coinfinite $\mathbb{N}$ and let

$$M = \{ f \in \mathbb{N}^{\mathbb{N}} : |Af \cap (\mathbb{N} \setminus A)| < \infty \}.$$

Then M is a maximal subsemigroup of $\mathbb{N}^{\mathbb{N}}$.

Note that $M \cap S_{\mathbb{N}}$ is not a subgroup of $S_{\mathbb{N}}$.

In fact, $M \cap S_{\mathbb{N}}$ is a generating set for $S_{\mathbb{N}}$.

3 orbits on infinite coinfinite sets

Theorem (East, M., Péresse '11)

Let $\mathcal{F}$ be a filter such that $S_{\{\mathcal{F}\}}$ has 3 orbits on infinite coinfinite sets, let $\mathcal{I}$ be the ideal corresponding to $\mathcal{F}$, and let $M \leqslant \Omega^\Omega$ such that $S_{\{\mathcal{F}\}} \leqslant M \neq S_\Omega$. Then M is maximal if and only if M is one of:

$$\{ f \in \mathbb{N}^{\mathbb{N}} : (\forall A \subseteq \mathbb{N})(A \in \mathcal{F} \to Af \in \mathcal{F} \text{ or } c(f|_A) > 0) \}$$
$$\{ f \in \mathbb{N}^{\mathbb{N}} : (\forall A \subseteq \mathbb{N})(A \in \mathcal{I} \to Af \in \mathcal{I} \text{ or } c(f|_A) > 0) \}$$
$$\{ f \in \mathbb{N}^{\mathbb{N}} : (\forall A \subseteq \mathbb{N})(A \in \mathcal{F} \to Af^{-1} \in \mathcal{F} \text{ or } c(f|_A) > 0) \}$$
$$\{ f \in \mathbb{N}^{\mathbb{N}} : (\forall A \subseteq \mathbb{N})(A \in \mathcal{I} \to Af^{-1} \in \mathcal{I} \text{ or } c(f|_A) > 0) \}.$$

For some examples some of these semigroups are equal, and for other examples they are distinct.

Open problems

Open Problem

Does there exist a maximal subsemigroup M of $\mathbb{N}^{\mathbb{N}}$ such that $M \cap S_{\mathbb{N}}$ is not a maximal subsemigroup of $S_{\mathbb{N}}$?

Open Problem

Can we prove that there does not exist a maximal subsemigroup M of $\mathbb{N}^{\mathbb{N}}$ such that $M \cap S_{\mathbb{N}}$ is trivial or $\{\, f \in S_{\mathbb{N}} : |\operatorname{supp}(f)| < \infty \,\}$?