Semigroups from digraphs

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Outline

Arcs, digraphs, and semigroups

Length of words: results

Length of words: conjectures
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**Apcs**

We are in $\text{Sing}_n$, the semigroup of singular transformations of $[n] = \{1, \ldots, n\}$.

An **arc** is any transformation of the form $(a \rightarrow b)$ for distinct $a, b \in [n]$, such that for any $v \in [n]$:

$$v(a \rightarrow b) = \begin{cases} b & \text{if } v = a, \\ v & \text{otherwise}. \end{cases}$$

Let $D$ be a **digraph** on $[n]$. We then view $D \subseteq \text{Sing}_n$ and we are interested in $\langle D \rangle$. 


Example 1

Let $D$ be the transitive tournament on $n$ vertices.

Then $\langle D \rangle = OI_n = \{ \alpha : v \leq v\alpha \}$.

E.g. $\alpha = (5, 2, 4, 5, 5) = (1 \to 5)(4 \to 5)(3 \to 4)$. 
Example 2

Let $D$ be the undirected path on $n$ vertices.

Then $\langle D \rangle = O_n = \{ \alpha : u \leq v \Rightarrow u \alpha \leq v \alpha \}$.

Let $D$ be the directed path on $n$ vertices.

Then $\langle D \rangle = C_n = \{ \alpha : v \leq v \alpha , u \leq v \Rightarrow u \alpha \leq v \alpha \}$.
Example 3

Let $D = K_n$ be the clique on $n$ vertices.

Theorem (Howie 66)

$\langle K_n \rangle = \text{Sing}_n$. 
Previous results

(You, Yang 02; Yang, Yang 06; Yang, Yang 09)

Different properties of $\langle D \rangle$, such as:

- $\text{Arcs}(\langle D \rangle) = D \cup \{(a \rightarrow b) : (b \rightarrow a) \text{ lies in a cycle of } D\}$.
- When $\langle D_1 \rangle = \langle D_2 \rangle$.
- When $\langle D_1 \rangle \cong \langle D_2 \rangle$ as semigroups.
- Classification of when $\langle D \rangle$ is regular.
More classifications

(East, G, Mitchell -in preparation)

Classification of when $\langle D \rangle$

- is inverse, or completely regular, or commutative, or simple, or 0-simple;
- is a semilattice, or a band, or a rectangular band;
- is $\mathcal{H}$-trivial, or $\mathcal{L}$-trivial, or $\mathcal{R}$-trivial, or $\mathcal{I}$-trivial;
- has a unique $\mathcal{L}$-class, or has a unique $\mathcal{R}$-class;
- has a left zero, a right zero, a zero.
Example of classification

Theorem (East, G, Mitchell)

Let $D$ be a connected digraph. Then the following are equivalent:

(i) $\langle D \rangle \cong (2^{[n-1]} \setminus \emptyset, \cup)$;
(ii) $\langle D \rangle$ is a semilattice;
(iii) $\langle D \rangle$ is inverse;
(iv) $\langle D \rangle$ is commutative;
(v) $D$ is a fan.
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Length of words: conjectures
We study the length of words $w \in D^*$ that express $\alpha \in \langle D \rangle$.

For any $D$ and $\alpha \in \langle D \rangle$, let

$$\ell(D, \alpha) := \min \{\text{length}(w) : w \in D^*, w = \alpha\}.$$  

We are interested in the longest elements:

$$\ell(D) := \max \{\ell(D, \alpha) : \alpha \in \langle D \rangle\}.$$
Acyclic digraphs

Let $Q_n$ be the following acyclic digraph.

![Diagram of an acyclic digraph with nodes 1 to 4 and edges 1->2, 2->3, 3->4, and 4->5, 4->6.]

Theorem (Cameron, Castillo-Ramirez, G, Mitchell 16)

For any acyclic digraph $A$ on $[n]$, 

$$\ell(A) \leq \ell(Q_n) = \frac{1}{2}(n^2 - 3n + 4).$$
Results for the clique \( K_n \)

**Theorem (Iwahori 77; Howie 80)**

\[
l(K_n, \alpha) = n - \text{fix}(\alpha) + \text{cycl}(\alpha),
\]

where \( \text{fix}(\alpha) = \{ v : v\alpha = v \} \) and \( \text{cycl}(\alpha) \) is the number of cyclic components of \( \alpha \).

Easy to maximise:

\[
l(K_n) = \left\lfloor \frac{3n - 3}{2} \right\rfloor.
\]

In (Cameron, Castillo-Ramirez, G, Mitchell 16), we characterise the digraphs \( D \) such that \( l(D, \alpha) = l(K_n, \alpha) \) for all \( \alpha \in \langle D \rangle \).
Strong tournaments

Theorem (Howie 78)
For any $n \geq 3$, $\langle D \rangle = \text{Sing}_n$ iff $D$ contains a strong tournament.

Hence, strong tournaments are the “almighty” ones: the minimal arc generating sets of $\text{Sing}_n$. Let

$$
\ell_{\text{min}}(n) := \min\{\ell(D) : D \text{ is a strong tournament on } [n]\},
$$

$$
\ell_{\text{max}}(n) := \max\{\ell(D) : D \text{ is a strong tournament on } [n]\}.
$$

Theorem (Cameron, Castillo-Ramirez, G, Mitchell 16)
For any $n \geq 3$,

$$
2n - 3 \leq \ell_{\text{min}}(n) \leq 10n - 8,
$$

$$
\frac{1}{4}n^2 - 2 \leq \ell_{\text{max}}(n) \leq 6n^2 - 15n + 1.
$$
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The “bad” tournament

Let $\pi_n$ be the tournament below.

Conjecture (Cameron, Castillo-Ramirez, G, Mitchell 16)

*For any $n \geq 3$,*

$$
\ell_{\text{max}}(n) = \ell(\pi_n, \alpha) = \frac{n^2 + 3n - 6}{2},
$$

*where $\alpha = (n, n-1, \ldots, 2, n)$.*
The “good” tournament

Let \( n = 2m + 1 \) and \( \kappa_n \) be the circulant tournament \( \{ (v \rightarrow v + [m]) \} \).

Conjecture (Cameron, Castillo-Ramirez, G, Mitchell 16)

For any \( n \geq 3 \), \( \ell(\kappa_n) = \ell_{\text{min}}(n) \).
More general conjectures and open problems

**Conjecture (Cameron, Castillo-Ramirez, G, Mitchell 16)**
*The diameter of any semigroup generated by a digraph is polynomial.*
*More precisely, $\exists c$ such that for any $D$ on $n$ large enough, $\ell(D) \leq n^c$.***

**Question (G)**
*The diameter of any transformation semigroup generated by idempotents is polynomial.*
*For any set $T \subseteq E(\text{Sing}_n)$ ($n$ large enough), $\ell(T) \leq n^c$.***

**Question (Kornhauser, Miller, Spirakis 84)**
*The diameter of any permutation group is polynomial.*