Semigroups from digraphs

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Outline

Arcs, digraphs, and semigroups

Length of words: results

Length of words: conjectures

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Arcs

We are in Sing_n, the semigroup of singular transformations of $[n] = \{1, \ldots, n\}$.

An **arc** is any transformation of the form $(a \to b)$ for distinct $a, b \in [n]$, such that for any $v \in [n]$:

$$v(a \to b) = \begin{cases} b & \text{if } v = a, \\ v & \text{otherwise.} \end{cases}$$

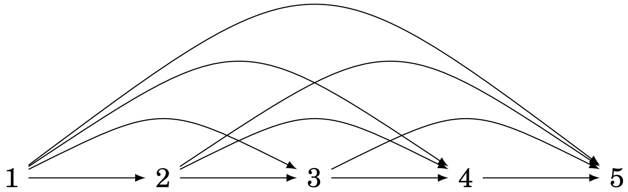
Let D be a **digraph** on $[n]$. We then view $D \subseteq \mathrm{Sing}_n$ and we are interested in $\langle D \rangle$.

Example 1

Let D be the transitive tournament on n vertices.

Then $\langle D \rangle = \mathrm{OI}_n = \{\alpha : v \leq v\alpha\}$.

E.g. $\alpha = (5, 2, 4, 5, 5) = (1 \to 5)(4 \to 5)(3 \to 4)$.



Example 2

Let D be the undirected path on n vertices.

Then $\langle D \rangle = \mathrm{O}_n = \{\alpha : u \leq v \Rightarrow u\alpha \leq v\alpha\}$.

$$1 \text{ ——— } 2 \text{ ——— } 3 \text{ ——— } 4 \text{ ——— } 5$$

Let D be the directed path on n vertices.

Then $\langle D \rangle = \mathrm{C}_n = \{\alpha : v \leq v\alpha, u \leq v \Rightarrow u\alpha \leq v\alpha\}$.

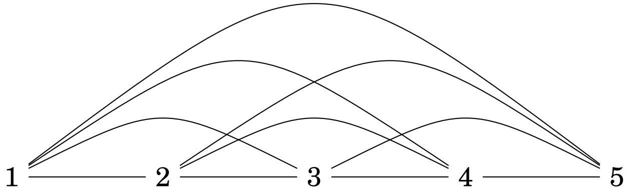
$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

Example 3

Let $D = K_n$ be the clique on n vertices.

Theorem (Howie 66)

$\langle K_n \rangle = \mathrm{Sing}_n$.



Previous results

(You, Yang 02; Yang, Yang 06; Yang, Yang 09)

Different properties of $\langle D \rangle$, such as:

- $\mathrm{Arcs}(\langle D \rangle) = D \cup \{(a \to b) : (b \to a)$ lies in a cycle of $D\}$.
- When $\langle D_1 \rangle = \langle D_2 \rangle$.
- When $\langle D_1 \rangle \cong \langle D_2 \rangle$ as semigroups.
- Classification of when $\langle D \rangle$ is regular.

More classifications

(East, G, Mitchell -in preparation)

Classification of when $\langle D \rangle$

- is inverse, or completely regular, or commutative, or simple, or 0-simple;
- is a semilattice, or a band, or a rectangular band;
- is $\mathscr{H}$-trivial, or $\mathscr{L}$-trivial, or $\mathscr{R}$-trivial, or $\mathscr{J}$-trivial;
- has a unique $\mathscr{L}$-class, or has a unique $\mathscr{R}$-class;
- has a left zero, a right zero, a zero.

Example of classification

Theorem (East, G, Mitchell)

Let D be a connected digraph. Then the following are equivalent:

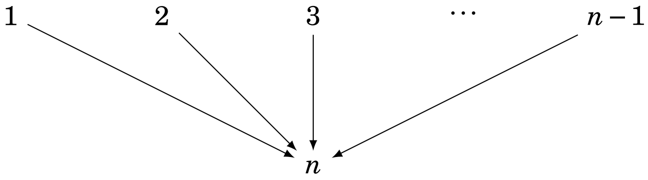
(i) $\langle D\rangle \cong (2^{[n-1]} \setminus \varnothing, \cup)$;

(ii) $\langle D\rangle$ *is a semilattice;*

(iii) $\langle D\rangle$ *is inverse;*

(iv) $\langle D\rangle$ *is commutative;*

(v) D *is a fan.*



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Length of words

We study the length of words $w \in D^*$ that express $\alpha \in \langle D \rangle$.

For any D and $\alpha \in \langle D \rangle$, let

$$\ell(D,\alpha) := \min\left\{\mathrm{length}(w) : w \in D^*, w = \alpha\right\}.$$

We are interested in the longest elements:

$$\ell(D) := \max\{\ell(D,\alpha) : \alpha \in \langle D \rangle\}.$$

Acyclic digraphs

Let Q_n be the following acyclic digraph.

$$
\begin{tikzcd}
 & & & & 5 \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[ur] \arrow[dr] & \\
 & & & & 6
\end{tikzcd}
$$

Theorem (Cameron, Castillo-Ramirez, G, Mitchell 16)

For any acyclic digraph A on $[n]$,

$$\ell(A) \leq \ell(Q_n) = \frac{1}{2}(n^2 - 3n + 4).$$

Results for the clique K_n

Theorem (Iwahori 77; Howie 80)

$$\ell(K_n, \alpha) = n - \text{fix}(\alpha) + \text{cycl}(\alpha),$$

where $\text{fix}(\alpha) = \{v : v\alpha = v\}$ *and* $\text{cycl}(\alpha)$ *is the number of cyclic components of* α.

Easy to maximise:

$$\ell(K_n) = \left\lfloor \frac{3n-3}{2} \right\rfloor.$$

In (Cameron, Castillo-Ramirez, G, Mitchell 16), we characterise the digraphs D such that $\ell(D, \alpha) = \ell(K_n, \alpha)$ for all $\alpha \in \langle D \rangle$.

Strong tournaments

Theorem (Howie 78)

For any $n \geq 3$, $\langle D \rangle = \mathrm{Sing}_n$ iff D contains a strong tournament.

Hence, strong tournaments are the "almighty" ones: the minimal arc generating sets of Sing_n. Let

$$\ell_{\min}(n) := \min\{\ell(D) : D \text{ is a strong tournament on } [n]\},$$
$$\ell_{\max}(n) := \max\{\ell(D) : D \text{ is a strong tournament on } [n]\}.$$

Theorem (Cameron, Castillo-Ramirez, G, Mitchell 16)

For any $n \geq 3$,

$$2n - 3 \leq \ell_{\min}(n) \leq 10n - 8,$$
$$\frac{1}{4}n^2 - 2 \leq \ell_{\max}(n) \leq 6n^2 - 15n + 1.$$

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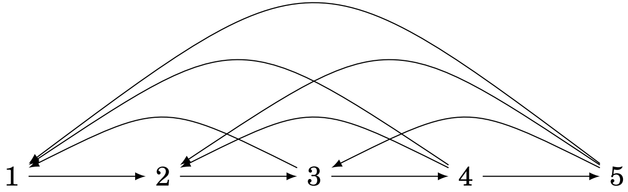
The “bad” tournament

Let π_n be the tournament below.



Conjecture (Cameron, Castillo-Ramirez, G, Mitchell 16)

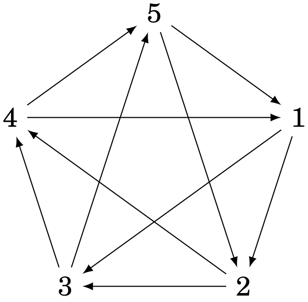
For any $n \geq 3$,

$$\ell_{\max}(n) = \ell(\pi_n, \alpha) = \frac{n^2 + 3n - 6}{2},$$

where $\alpha = (n, n-1, \ldots, 2, n)$.

The “good” tournament

Let $n = 2m + 1$ and κ_n be the circulant tournament $\{(v \to v + [m])\}$.



Conjecture (Cameron, Castillo-Ramirez, G, Mitchell 16)

For any $n \geq 3$, $\ell(\kappa_n) = \ell_{\min}(n)$.

More general conjectures and open problems

Conjecture (Cameron, Castillo-Ramirez, G, Mitchell 16)

The diameter of any semigroup generated by a digraph is polynomial.
More precisely, $\exists c$ such that for any D on n large enough, $\ell(D) \leq n^c$.

Question (G)

The diameter of any transformation semigroup generated by idempotents is polynomial.
For any set $T \subseteq E(\mathrm{Sing}_n)$ (n large enough), $\ell(T) \leq n^c$.

Question (Kornhauser, Miller, Spirakis 84)

The diameter of any permutation group is polynomial.