Zero-divisor graphs of idealizations with respect to semimodules over inclines

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Contents

Background

- Using graphs to study algebraic structures has become an exciting research topic in the last thirty years.
- Many mathematicians tend to assign a graph to a ring or other algebraic structures and then study the algebraic properties of these objects via the associated graphs.
- Beck (I. Beck, *Coloring of commutative rings*, J. Algebra 116 (1988), 208-226) introduced the concept of a zero-divisor graph of a commutative ring, but this work was mostly concerned with colorings of rings, and zero was taken to be a vertex of the graph.
- The most common definition of a zero-divisor graphs of a ring was firstly introduced by D. F. Anderson and P. S. Livingston (*The zero-divisor graph of a commutative ring*, J. Algebra 217 (1999) 434-447). This definition, unlike the earlier work of Anderson and Naseer (1993) and Beck (1988), does not take zero to be a vertex.

Background

Definition 1 (Anderson and Livingston, 1999)

Let R be a commutative ring with nonzero identity. The zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$.

The zero-divisor graph has been extended to other algebraic structures, for instance:

1. **semigroups** by DeMeyer et al. in 2002,
2. **semirings** by Dolzan and Oblak in 2010,
3. **po-semirings** by Yu and Wu in 2011, and
4. **MV-algebras** by Gan and Yang in 2018, etc.

Background

Definition 2 (Anderson and Winders, 2009)

Let R be a commutative ring with nonzero identity, and let N be an unitary R-module. The idealization of N in R, denoted by $R \ltimes N$, is the commutative ring $(R \times N, +, \cdot, (0,0), (1,0))$ with coordinate-wise addition, i.e. $(r,x) + (s,y) = (r+s, x+y)$ and multiplication $(r,x) \cdot (s,y) = (rs, ry + sx)$, where $r, s \in R$ and $x, y \in N$.

- Axtell and Stickles in 2006 completely characterize the girth of the zero-divisor graph $\Gamma(R \ltimes N)$ of $R \ltimes N$, and discuss when $\Gamma(R \ltimes N)$ will be complete and provide some conditions when $\Gamma(R \ltimes N)$ will have diameter 2.
- After replacing the commutative ring R with a commutative semiring and substituting the module N for a semimodule, Farzalipour and Ghiasvand in 2011 studied the zero-divisor graphs of idealizations with respect to prime semimodules.

Basic notions

We will investigate the zero-divisor graphs of idealizations with respect to semimodules over inclines.

Definition 3

A commutative semiring $(S, +, \cdot, 0, 1)$ is called an incline if $r + 1 = 1$ for any $r \in S$.

- By the above definition we have $1 + 1 = 1$, and so $r + r = r \cdot (1 + 1) = r$ for any $r \in S$. Hence S is additive idempotent. Consequently, S has a natural partial order: $r \leq s \Leftrightarrow r + s = s$ for all $r, s \in S$.
- If $r, s, t \in S$ and $r \leq s$, then $rt \leq st$ since $rt + st = (r + s)t = st$, in particular, $0 \leq rs \leq s$.

Definition 4

Let S be an incline. An S-semimodule M is an abelian monoid with scalar multiplication by S satisfying: for all $r, s \in S$ and all $x, y \in M$,

(1) $r(x+y) = rx + ry$;

(2) $(r+s)x = rx + sx$;

(3) $(r \cdot s)x = r(sx)$;

(4) $1_S x = x$;

(5) $0_S x = 0_M = r0_M$.

- By the definition 4, we have $x + x = (1+1)x = x$ for all $x \in M$. So M is also idempotent and hence M has a natural order: $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in M$. With respect to this order $\leq$, M is a $\vee$-semilattice with the minimum element 0_M.

Example

Every $\vee$-semilattice with the minimum element 0 is naturally a $\mathbb{B}$-semimodule, where $\mathbb{B} = (\{0,1\}, +, \cdot, 0, 1)$ is the simplest nontrivial incline with $0 + 0 = 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 \cdot 1 = 1 + 0 = 0 + 1 = 1 + 1 = 1$.

Basic notions

Now we recall some notions in graph theory.

- A *graph* Γ is a pair $(V(\Gamma), E(\Gamma))$ of sets such that $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$; thus, the elements of $E(\Gamma)$ are 2-element subsets of $V(\Gamma)$. The elements of $V = V(\Gamma)$ are referred to as *vertices* and the elements of $E = E(\Gamma)$ are called *edges*.
- A vertex v is said to be *incident* with an edge e if $v \in e$. Two vertices u and v are *adjacent* if they are incident with a common edge e.
- An edge with identical ends is called a *loop*, and an edge with distinct ends a *link*. Two or more links with the same pair of ends are said to be *parallel edges*.
- A graph is *simple* if it has no loops or parallel edges.

Basic notions

- A *complete graph* is a simple graph in which any two vertices are adjacent, and we denote by K_n the n-vertex complete graph.
- An *empty graph* is a graph whose edge set is empty.
- The graph with no vertices (and hence no edges) is the *null graph*.
- A *path* is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. The *length* of a path is the number of its edges.

Basic notions

- The *distance* $d_\Gamma(a, b)$ between a pair of vertices a and b in Γ is the length of the shortest path between them.
- The *diameter* $diam(\Gamma)$ of a graph Γ is defined to be the supremum of the distances between any pair of vertices.
- A *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise.
- The *girth* of a graph Γ, denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ provided Γ contains a cycle; otherwise, $gr(\Gamma) = \infty$.

Basic notions

- A graph is said to be a *star graph* if the graph is connected with all edges sharing a common vertex.
- A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph, and X and Y its parts. We denote a bipartite graph Γ with bipartition (X, Y) by $\Gamma[X, Y]$.
- If $\Gamma[X, Y]$ is simple and every vertex in X is joined to every vertex in Y, then Γ is called a *complete bipartite graph*. When $|X| = m$ and $|Y| = n$, we denote the complete bipartite graph $\Gamma[X, Y]$ with bipartition $[X, Y]$ by $K_{m,n}$. Clearly, $K_{1,n}$ is a star graph, and $K_{1,1}$ is the 2-element complete graph K_2.

Basic notions

- A graph is called *double star* if it is the graph obtained by joining the center of two star graphs. If a double star graph Γ satisfies $V(\Gamma) = X \cup \{v_1, v_2\} \cup Y$ and $v_1 - v_2$, $x - v_1$ for any $x \in X$, and $v_2 - y$ for $y \in Y$, then we denote it by $X - v_1 - v_2 - Y$. In particular, if $|X| = m$ and $|Y| = n$, then we write $D_{m,n}$ for the double star graph $X - v_1 - v_2 - Y$. Note that $D_{0,n} = K_{1,n+1}$ is a star graph.

The zero-divisor graph of a semiring

Definition 5

The zero-divisor graph of a commutative semiring S, denoted by $\Gamma(S)$, is the simple graph whose vertices are the nonzero zero-divisors of S, and for distinct vertices r, s, there is an edge connecting r and s if and only if $r \cdot s = 0$.

By Theorem 2.1 in *D. Dolzan and P. Oblak, The zero-divisor graphs of semirings, https://www.researchgate.net/publication/48166265, 2010*, we have

Lemma

If S is a commutative semiring, then $\Gamma(S)$ is connected and $diam(\Gamma(S)) \leq 3$.

Definition 6

Let S be an incline and M be an S-semimodule. The idealization of M in S, denoted by $S \ltimes M$, is the commutative semiring $(S \times M, +, \cdot, (0,0), (1,0))$ with coordinate-wise addition and multiplication $(r,x)(s,y) = (r \cdot s, ry + sx)$ for all $(r,x), (s,y) \in S \times M$.

In the following, unless specified stated, we will always assume that neither the incline S nor the semimodule M is trivial. For the sake of convenience, we state some other notations used throughout.

- For a commutative semiring R, we denote the set of zero-divisors of R by $Z(R)$.
- For any sets X and Y, we denote the cardinality of X by $|X|$, the subset $\{x \in X \mid x \notin Y\}$ of X by $X \backslash Y$, and the set $X \backslash \{0\}$ by X^* when X contains an element 0.

The zero-divisor graph $\Gamma(S \ltimes M)$

(1) The vertex set of $\Gamma(S \ltimes M)$

Proposition 1

Let $A = \{(0, b) \mid b \in M^*\}$, $B = \{(r, x) \mid r \in Z(S)^*, x \in M\}$ and $C = \{(r, x) \mid r \in S \backslash Z(S), x \in M \text{ and } rc = 0 \text{ for some } c \in M^*\}$. Then A, B and C are mutually disjoint, and $V(\Gamma(S \ltimes M)) = Z(S \ltimes M)^* = A \cup B \cup C$.

Proof.

Firstly, $A \subseteq Z(S \ltimes M)^*$ since $(0, b)(0, c) = (0, 0)$ for all $b, c \in M^*$. Secondly, let $r \in Z(S)^*$ and $x \in M$. Then $r \cdot s = 0$ for some $s \in Z(S)^*$. If $sM = \{0\}$, then $(r, x)(s, 0) = (0, 0)$. If $sM \neq \{0\}$, then there exists $c \in M^*$ such that $sc \neq 0$. It follows that $(r, x)(0, sc) = (0, 0)$. Hence $B \subseteq Z(S \ltimes M)^*$. Finally, if $(r, x) \in Z(S \ltimes M)^*$ and $r \in S \backslash Z(S)$, then we must have $(r, x)(0, c) = (0, 0)$ for some $c \in M^*$. Hence $rc = 0$, and consequently $(r, x) \in C$. □

(2) The girth of $\Gamma(S \ltimes M)$

When looking at the girth of the zero-divisor graph $\Gamma(S \ltimes M)$, things are very simple if the semimodule M is large enough.

- If $|M| \geq 4$, then $gr(\Gamma(S \ltimes M)) = 3$, since $(0, x) - (0, y) - (0, z) - (0, x)$ is a cycle of length 3 (where x, y, and z are distinct nonzero elements of M).
- So, we only need to consider when the semimodule M has two or three elements.
- First let's consider the case of $|M| = 3$. Since M is a $\vee$-semilattice with the minimum element 0, we can assume that $M = \mathbf{C}_3$, where $\mathbf{C}_3 = \{0, u, v\}$ is the 3-element chain with $0 < u < v$.

Theorem 1

$gr(\Gamma(S \ltimes \mathbf{C}_3)) = \{3, \infty\}$. Moreover, $gr(\Gamma(S \ltimes \mathbf{C}_3)) = 3$ if and only if $ann(\mathbf{C}_3) \neq \{0\}$ or $r^2 = 0$ for some $r \in ann(u)^*$, where $ann(\mathbf{C}_3) = \{r \in S \mid rv = ru = 0\}$ and $ann(u)^* = \{r \in S^* \mid ru = 0\}$.

From the proof of Theorem 1, we immediately get

Corollary 1

$gr(\Gamma(S \ltimes \mathbf{C}_3)) = \infty$ if and only if $\Gamma(S \ltimes \mathbf{C}_3)$ is a star graph with center $(0, u)$. Moreover, if $\Gamma(S \ltimes \mathbf{C}_3)$ is a finite graph, and $gr(\Gamma(S \ltimes \mathbf{C}_3)) = \infty$, then $\Gamma(S \ltimes \mathbf{C}_3) \cong K_{1,3n+1}$ for some nonnegative integer number n.

We now consider the case of $|M| = 2$. Since M is a $\vee$-semilattice with the minimum element 0, we can assume that $M = \mathbf{C}_2$, where $\mathbf{C}_2 = \{0, a\}$ is the 2-element chain with $0 < a$.

Theorem 2

$gr(\Gamma(S \ltimes \mathbf{C}_2)) = 3$ if and only if one of the following conditions holds: (i) $gr(\Gamma(S)) = 3$; (ii) $r^2 = 0$ for some $r \in S^*$; or (iii) there exist distinct $r, s \in Z(S)^*$ such that $rs = 0$ and $ra = sa = 0$.

Theorem 3

$gr(\Gamma(S \ltimes \mathbf{C}_2)) \in \{3, 4, \infty\}$. Moreover, $gr(\Gamma(S \ltimes \mathbf{C}_2)) = \infty$ if and only if $Z(S)^* = \emptyset$ or the following conditions hold:

(*i*) $|Z(S)^*| \geq 2$;

(*ii*) $\Gamma(S)$ is a star graph with center s such that $sa = 0$;

(*iii*) $r^2 \neq 0$ for any $r \in S^*$;

(*iv*) $ta = a$ for any $t \in Z(S)^* \backslash \{s\}$.

(3) The diameter of $\Gamma(S \ltimes M)$

- In studying the diameter of $\Gamma(S \ltimes M)$, we firstly know by Lemma 1 that $diam(\Gamma(S \ltimes M)) \leq 3$.
- In this section, we provide necessary and sufficient conditions to ensure that $\Gamma(S \ltimes M)$ is complete, and provide some results concerning when $diam(\Gamma(S \ltimes M))$ is equal to 2.

For our purpose, we state three properties that will be used in the sequel:

(a) $(Z(S))^2 = \{0\}$.

(b) For every $r \in S \backslash Z(S)$, $rx \neq 0$ for all $x \in M^*$.

(c) If $r \in Z(S)^*$, then $rM = \{0\}$.

The zero-divisor graph $\Gamma(S \ltimes M)$

Theorem 4

$\Gamma(S \ltimes M)$ is a complete graph if and only if $S \ltimes M$ satisfies properties $(a), (b)$, and (c).

Proposition 1

Let $S \ltimes M$ satisfy the property (b), but but both of properties $(a), (c)$. Then, $diam(\Gamma(S \ltimes M)) = 2$ if and only if $S \ltimes M$ satisfies the property

(d) for all $r, s \in Z(S)^*$, either there exists $z \in M^*$ such that $rz = sz = 0$, or there exists $k \in Z(S)^*$ such that $rk = sk = 0$.

The zero-divisor graph $\Gamma(S \ltimes M)$

Proposition 2

Let $S \ltimes M$ satisfy the property (c) but not the property (b). Then, $diam(\Gamma(S \ltimes M)) = 2$ if and only if $S \ltimes M$ satisfies the property

- (e) for all $r, s \in S \backslash Z(S)$, if there exists $p, q \in M^*$ such that $rp = sq = 0$, then there exists a common element $z \in M^*$ such that $rz = sz = 0$.

Theorem 5

Let $\Gamma(S \ltimes M)$ be not complete. Then, $diam(\Gamma(S \ltimes M)) = 2$ if and only if $S \ltimes M$ satisfies properties $(d), (e)$ and

- (f) for any $s \in S \backslash Z(S)$, if there exists $p \in M^*$ such that $sp = 0$, then for every $r \in Z(S)^*$, there exists $z \in M^*$ such that $rz = sz = 0$.

(4) Some realizable graphs for idealization

- A graph is called realizable (for idealization) if it is isomorphic to $\Gamma(S \ltimes M)$ for some incline S and some S-semimodule M.
- For convenience, we denote the set of nonnegative integer numbers by $\mathbf{N}$.

In this section, some realizable graph for idealization will be given.

Proposition 3

Any complete graph is realizable.

The zero-divisor graph $\Gamma(S \ltimes M)$

Proposition 4

Let $m, n \in \mathbf{N}$ with $m \geq 2$ and $n \geq 2$. Then the complete bipartite graph $K_{m,n}$ is not realizable.

Proposition 5

Let $m \in \mathbf{N}$. Then, the star graph $K_{1,m}$ is realizable if and only if $m = 2n$ or $m = 3n + 1$ for some $n \in \mathbf{N}$.

Proposition 6

Let $m \in \mathbf{N}$. Then, the double star graph $D_{1,m}$ is realizable if and only if $m = 2n$ for some $n \geq 0$.

The zero-divisor graph $\Gamma(S \ltimes M)$

Proposition 7

Let $m, n \in \mathbf{N}$ with $m \geq 2$ and $n \geq 2$. Then the double star graph $D_{m,n}$ is not realizable.

Corollary 2

Let G be a finite tree. Then G is realizable if and only if G is one of the following graphs: the complete graph K_1, K_2, the star graph $K_{1,2n}$, $K_{1,3n+1}$, and the double star graph $D_{1,2n}$, where $n \geq 1$.

References

1 I. Beck, *Coloring of commutative rings*, J. Algebra 116 (1988) 208-226.

2 D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra 159 (1993) 500-514.

3 D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra 217 (1999) 434-447.

4 F. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum 65(2) (2002) 206-214.

References

5 F. DeMeyer and L. DeMeyer, *Zero divisor graphs of semigroups*, J. Algebra, 283(1) (2005), 190-198.

6 D. Dolzan and P. Oblak, *The zero-divisor graphs of semirings*, https://www.researchgate.net/publication/48166265, 2010.

7 A.P. Gan and Y.C. Yang, *Zero-divisor graphs of MV-algebras*, submitted to Soft Computing.

8 M. Axtell and J. Stickles, *Zero-divisor graphs of idealization*, Journal of Pure and Applied Algebra 204 (2006), 235-243.

9 Y.Q. Bai and Y.C. Yang, *Structure and representation of semimodules over inclines*, to appear.

Thanks

Thank you for your attention!