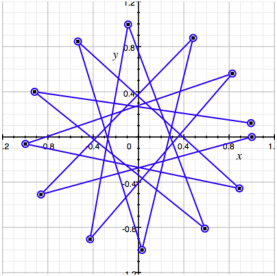
It’s all Greek to me
– a modern look at ancient logic

Peter M. Hines — University of York

York Semigroups Seminar

York, March 8th 2023



A puzzle from antiquity . . .

The Logician/Philosopher vs. the Astronomer/Mathematician :

"Chrysippus says that the number of conjunctions[a] [constructible] from only ten assertibles exceeds one hundred myriads [i.e. 10^6]. However, Hipparchus refuted this, demonstrating that the affirmative encompasses 103049 conjoined assertibles and the negative 310952."

— *Plutarch,* **Quæstiones Convivales** *(2nd C. AD)*

[a] 'combinations' in some documents . . .

This was reported as *common knowledge*,

"Chrysippus is refuted by *all the arithmeticians*, among them Hipparchus himself who proves that his error in calculation is enormous".

— *Plutarch,* **De Stoicorum Repugnantiis** *(2nd C. AD)*

but the precise meaning was lost.

"Since the exact terms of the problem are not stated, it is difficult to interpret the numerical answers . . . The Greeks took no interest in these matters".

— *N. L. Biggs* **The Roots of Combinatorics** *1979*

Interpretation and Composition

The significance of 103049 was realised in January 1994 by a graduate student (D. Hough) at George Washington University :

> Hipparchus, Plutarch, Schröder and Hough
> — *R. Stanley, American Mathematical Monthly (1997)*

It is simply the 10^{th} **little Schröder number**, counting (amongst other things[1]) the number of distinct **Rooted Planar Trees** with ten leaves.

A (too easy?) interpretation

It is tempting to interpret :

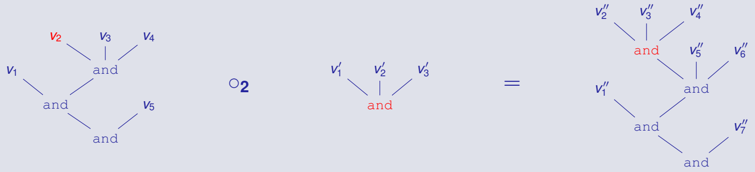
1. Each **branching** as a logical operation (conjunction?)
2. Each **leaf** as a simple assertible (variable?)

Building larger trees from smaller trees : **Substituting a tree for a given leaf.**

[1] e.g. the number of facets of the tenth associahedron

Replacing simple assertibles by non-simple composites

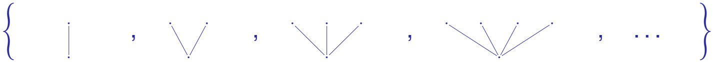
Operadic Composition :



Provided we

- Avoid clashes of free variable names, & identify α-equivalent trees,
- Identify up to (planar) topological equivalence,

we arrive at the non-symmetric operad $\mathbb{RPT}$ of **rooted planar trees**.
This is *freely generated* by one tree of each arity (number of leaves).



Formal definitions

An operad is an indexed family of disjoint sets $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \ldots\}$ of 'operations', together with composition functions

$$\circ_i : \mathcal{H}_n \times \mathcal{H}_m \rightarrow \mathcal{H}_{m+n-1} \quad , \quad i = 1 \ldots n$$

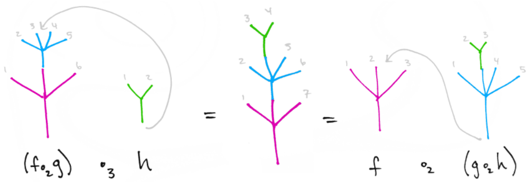
that include an identity in $\mathcal{H}_1$, and satisfy the following:

For all $f \in \mathcal{H}_n$, $g \in \mathcal{H}_m$, and $h \in \mathcal{H}_p$,

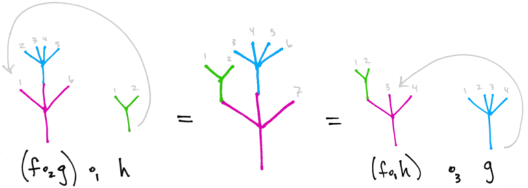
$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{j+p-1} g & \text{if } 1 \leqslant i \leqslant j-1 \\ f \circ_j (g \circ_{i-j+1} h) & \text{if } j \leqslant i \leqslant m+j-1 \\ (f \circ_{i-m+1} h) \circ_j g & \text{if } i \geqslant m+j \end{cases}$$

The axioms, graphically :

1. There exists an identity $Id \circ_1 T = T$ and $T \circ_k Id = T$.
2. "Composition is associative"



3. "Parallel composites commute"



Diagrams 'borrowed' from Tai-Danae Bradley's Math3ma blog

How did Hipparchus (and *"all the arithmeticians"*) calculate Schröder numbers??

> On the Shoulders of Hipparchus:
> A Reappraisal of Ancient Greek Combinatorics.
> — **F. Acerbi** *(2004)*

Why should we be interested?

> *Chrysippus' main achievement is the development of a propositional logic & deductive system[2]. He was innovative in topics central to contemporary formal and philosophical logic. The many close similarities with Gottlob Frege are especially striking.*
> — **Stanford Encyclopedia of Philosophy**

How & why did Chrysippus & Hipparchus come up with such different values?

> Combinatorics for Stoic Conjunction:
> Chrysippus Vindicated, Hipparchus Refuted.
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[2] best understood as a substructural backwards-working Gentzen-style natural-deduction system — **S.E.P.**

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The How and Why of counting Stoic conjunctions

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A misunderstanding of logic?

Bobzien's claim is that, "*Hipparchus, it seems, got his mathematics right. What I suggest in this paper is that he got his Stoic logic wrong.*"

Where it starts going wrong :

"He counts the *same sequence* of conjuncts but with *different bracketing* as different conjunctions . . . He counts

$$[P \wedge Q] \wedge R \qquad \underline{\qquad\qquad P \wedge Q \wedge R \qquad\qquad} \qquad P \wedge [Q \wedge R]$$

as different assertibles. Unlike modern propositional logic, Hipparchus assumes that a [elementary] conjunction can consist of two or more conjuncts.

In order to get to [the little Schröder number 103049], Hipparchus also had to take the order of the ten atomic assertibles as fixed.'" – S. B.

A synthesis via category theory

Between 'equal' and 'not equal' lies a compromise :

$$\text{The Same} \quad \frac{\textit{equal up to}}{\textit{unique isomorphism}} \quad \text{Different}$$

This should be understood at the level of **semantic models**.

What might we need ?

- A family of k-ary **elementary conjunctions**

$$(\_ \star \_) \quad , \quad (\_ \star \_ \star \_) \quad , \quad (\_ \star \_ \star \_ \star \_) \quad , \quad \ldots$$

- The operad they generate under substitution should be free (i.e. $\simeq \mathbb{RPT}$).
- A notion of 'equivalent up to isomorphism' that uniquely relates any two composites of the same arity.

An additional concern

We *cannot* assume idempotency of conjunction

"Non-simple [assertibles] are those that are, as it were, double $(\delta\iota\pi\lambda\alpha)$ – put together by means of a connecting particle from an assertible that is <u>taken twice</u> $(\delta\iota\zeta)$, or from two different assertibles." — S. B.

We need to reconsider the structural rules of **contraction** and **weakening**.

We find what we need in *models of Linear Logic.*

(As a bonus!) Everything is based on **Euclidean division**.

“Everything is *[in the endomorphism monoid of]* Numbers”

In J.-Y. Girard’s Geometry of Interaction system :

Propositions are modelled by operations on $\mathbb{N}$

For M.L.L. Elements of the symmetric group $\mathcal{S}(\mathbb{N})$
For M.E.L.L. Elements of the symmetric inverse monoid $\mathcal{I}(\mathbb{N})$

Conjunction is modelled by the following operation :

$$
\begin{aligned}
(f \star g)(2n) &= 2.f(n) \\
(f \star g)(2n+1) &= 2.g(n) + 1
\end{aligned}
$$

A simple description, based on Hilbert’s Grand Hotel

This “writes two functions as a single function”, by

replicating their behaviour on the **even** *and* **odd** *numbers respectively.*

This is an injective homomorphism $\mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$ of inverse monoids.

Potential vs. actual infinity

> *The Greeks feared infinity and tried to avoid it ... According to tradition, they were frightened off by the paradoxes of Zeno. ... Until the late C^{19th}, mathematicians were reluctant to accept infinity as more than "potential".*
>
> — *J. Stillwell,* **Mathematics and Its History** *2012*

This attitude persisted for a very long time :

> *"I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a figure of speech; an abridged form for the statement that limits exist.*
>
> *No contradictions will arise as long as Finite Man does not mistake the infinite for something fixed.*
>
> — Gauss, **Letter to Schumacher** (1831)

What about Greek proofs on infinite sets?

Not Euclid There exists an infinite number of primes.

Euclid The prime numbers are more numerous
than any proposed multitude of prime numbers.

Actual infinity was eventually forced by the requirements of medieval theology:

Duns Scotus on God (R. Cross, 2005)

John Duns Scotus (1266-1308) [ontological] argument may be summarised as,

"If God is composed of parts, then each part must be finite or infinite. . . . **If any given part is infinite, then it is equal to the whole**, *which is absurd"*

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Not strictly the same ...

In general : $(P \star Q) \star R \neq P \star (Q \star R)$

No injective homomorphism $M \times M \hookrightarrow M$ on a non-abelian monoid can be strictly associative.

Coherence & Strictification for Self-Similarity
Journal Homotopy & related Structures *(P.M.H. 2016)*

There is a non-trivial bijection

$$(a \star (b \star c)) = \alpha^{-1}((a \star b) \star c)\alpha \quad \forall \quad a, b, c \in \mathcal{I}(\mathbb{N})$$

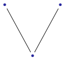
This unique **associator** is defined by $\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4, \end{cases}$

This is defined piece-wise linearly on modulo classes — a **congruential function**.

A Hipparchus-style generalisation

Girard gave a **binary** model of conjunction $(\_ \star \_) : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$.

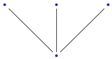
"$(a \star b)$ replicates a, b on the modulo classes $2\mathbb{N}$, $2\mathbb{N}+1$ respectively".

- We draw this as 

There is an obvious **ternary** analogue, $(\_ \star \_ \star \_) : \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \times \mathcal{I}(\mathbb{N}) \hookrightarrow \mathcal{I}(\mathbb{N})$

$$(a \star b \star c)(3n+i) \ = \begin{cases} 3.a(n) & i = 0 \\ 3.b(n)+1 & i = 1 \\ 3.c(n)+2 & i = 2 \end{cases}$$

"Replicate a, b, c on the modulo classes $3\mathbb{N}$, $3\mathbb{N}+1$, $3\mathbb{N}+2$ respectively".

- We draw this as 

The general case :

For any $k \geqslant 1$, we form the k^{th} **elementary conjunction** by :

$$(f_0 \star \dots f_{k-1})(kn+i) \;=\; k.f_i(n)+i \;\text{ where } i = 0,1,2,\dots,k-1$$

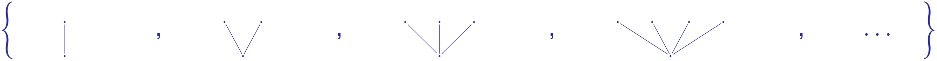
Alternatively & equivalently,

$$(f_0 \star \dots f_{k-1})(x) \;=\; k.f_i\left(\frac{x-i}{k}\right)+i \;\text{ where } x \equiv i \mod k$$

This gives, for any $k > 0$, an injective homomorphism $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$ that :

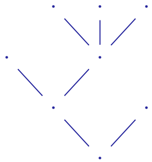
> *"replicates the action of f_0 , f_1 , $\dots$, f_{k-1} on the modulo classes $k\mathbb{N}$, $k\mathbb{N}+1$, $\dots$, $k\mathbb{N}+(k-1)$ respectively.*

For $k = 1,2,3,4,\dots$, we draw these as



Composing elementary conjunctions

These ‘compose by substitution’ to give an operad $\mathcal{H}ipp$ of **generalised conjunctions**. Each k-leaf tree determines an injective hom. $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$.



$$: \ \mathcal{I}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{I}(\mathbb{N})$$

$$(f_0, f_1, f_2, f_3, f_4) \quad \mapsto \quad ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$$

More formally :

We have one operation of each arity > 0 in the (non-symmetric) endomorphism operad of $\mathcal{I}(\mathbb{N})$ within the category $(\mathbf{Inv}, \times)$ of inverse monoids / homomorphisms with Cartesian product.

These generate the sub-operad $\mathcal{H}ipp$.

Claim :

The operad $\mathcal{H}ipp$ of generalised conjunctions extends Girard's operation from the Geometry of Interaction, to provide a semantic model for Hipparchus' (mis-)understanding of Chrysippus' Stoic Logic.

More concisely(!)

$\mathcal{H}ipp \cong \mathbb{RPT}$, so each tree determines a *distinct* homomorphism $\mathcal{I}(\mathbb{N})^{\times k} \hookrightarrow \mathcal{I}(\mathbb{N})$.

Proof?

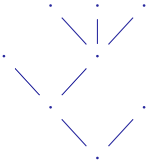
Proving this requires a concept the Greeks (notoriously) did not have[3]:

> *The greatest calamity in the history of science was the failure of Archimedes to invent positional notation.* – C. F. Gauss

Gauss was referring to Archimedes' "Sand Reckoner" $(\Psi\alpha\mu\mu\iota\tau\eta\zeta)$.

[3] ...but may (occasionally) have borrowed from their neighbours

Let me see you counting like they do in Babylon



defines a homomorphism : $\mathcal{I}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{I}(\mathbb{N})$

In the operadic composite $(f_0, f_1, f_2, f_3, f_4) \mapsto ((f_0 \star (f_1 \star f_2 \star f_3)) \star f_4)$, the action of each f_j is mapped :

from The whole of the natural numbers $\mathbb{N}$

to Some modulo class $A_j\mathbb{N} + B_j$.

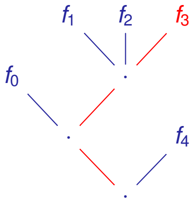
For example : f_3 is translated onto $12\mathbb{N} + 10$.

Question:

How do we derive these coefficients *from the tree*?

A root and branch approach

Deriving $12\mathbb{N} + 10$, from the leaf-to-root path :



Branch number 2 of 3

Branch number 1 of 2

Branch number 0 of 2

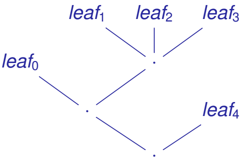
Multiplicative coefficient : $12 = 3 \times 2 \times 2$

Additive coefficient :	(Decimal)		Base 3	Base 2	Base 2
	10	=	2	1	0

Positional mixed-radix number systems

First formal study by G. Cantor, *Über einfache Zahlensysteme* (1869)

Rooted Planar Trees are uniquely determined by the addresses of their leaves



$leaf_0$		$(0,2)$	$(0,2)$	$4\mathbb{N}$
$leaf_1$	$(0,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+2$
$leaf_2$	$(1,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+6$
$leaf_3$	$(2,3)$	$(1,2)$	$(0,2)$	$12\mathbb{N}+10$
$leaf_4$			$(1,2)$	$2\mathbb{N}+1$

which uniquely determine ordered **exact covering systems**, such as

$$4\mathbb{N}\ ,\ 12\mathbb{N}+2\ ,\ 12\mathbb{N}+6\ ,\ 12\mathbb{N}+10\ ,\ 2\mathbb{N}+1$$

Heavily studied by P. Erdös (1950s)

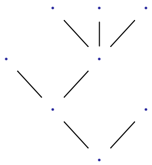
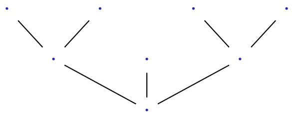
Ordered sets of pairwise-disjoint modulo classes, whose union is the whole of $\mathbb{N}$

(& hence, by PMH / MVL 1998, embeddings of polycyclic monoids into $\mathcal{I}(\mathbb{N})$).

Corollary: Distinct trees determine distinct homomorphisms.

Mappings between between gen. conjunctions

Consider the generalised conjunctions $T, U : \mathcal{I}(\mathbb{N})^{\times 5} \hookrightarrow \mathcal{I}(\mathbb{N})$

$T =$  $U =$ 

We build a bijection $\eta_{T,U} : T \Rightarrow U$ by monotonically mapping between their respective ordered covering systems :

leaf 0	$4\mathbb{N}$	$\mapsto$	$6\mathbb{N}$
leaf 1	$12\mathbb{N}+2$	$\mapsto$	$6\mathbb{N}+3$
leaf 2	$12\mathbb{N}+6$	$\mapsto$	$3\mathbb{N}+1$
leaf 3	$12\mathbb{N}+10$	$\mapsto$	$6\mathbb{N}+2$
leaf 4	$2\mathbb{N}+1$	$\mapsto$	$6\mathbb{N}+4$

This gives, as desired,

$$((a \star (b \star c \star d)) \star e) \; = \; \eta_{T,U}^{-1}((a \star b) \star c \star (d \star e))\eta_{T,U}$$

Formulæ for the general case

Given any two k-ary generalised conjunctions T, U, we derive two ordered exact covering systems

$$\begin{array}{lccc} \textbf{leaf } 0 & A_0\mathbb{N} + B_0 & \mapsto & C_0\mathbb{N} + D_0 \\ \textbf{leaf } 1 & A_1\mathbb{N} + B_1 & \mapsto & C_1\mathbb{N} + D_1 \\ \vdots & \vdots & & \vdots \\ \textbf{leaf } k-1 & A_{k-1}\mathbb{N} + B_{k-1} & \mapsto & C_{k-1}\mathbb{N} + D_{k-1} \end{array}$$

The **natural isomorphism** $\eta_{T,U}$ is the bijection

$$\eta_{T,U}(x) = \frac{1}{A_j}\left(C_j x + \begin{vmatrix} A_j & B_j \\ C_j & D_j \end{vmatrix}\right) \text{ where } x \equiv B_j \mod A_j$$

We arrive at John Conway's **congruential functions**

"Unpredictable Iterations" — J. Conway (1972)

as used to demonstrate undecidability in elementary arithmetic.

A category for Chrysippus

Observe that :

- $\eta_{T,T} = Id \in \mathcal{I}(\mathbb{N})$
- $\eta_{T,U}\eta_{S,T} = \eta_{S,U}$
- $\eta^{-1}_{T,U} = \eta_{U,T}$

We may define a **(posetal) groupoid** $\mathcal{C}hrys$ of functors / natural iso.s, given by :

Objects Generalised conjunctions (operations of $\mathcal{H}ipp$)

Arrows $\mathcal{C}hrys(T,U) = \begin{cases} \{\eta_{T,U}\} & T, U \text{ have the same arity,} \\ \emptyset & \text{otherwise.} \end{cases}$

There is at most one arrow between any two objects

- This means that 'all diagrams commute' — a *big deal* in category theory.
- By forgetting sources / targets, we derive commuting diagrams (i.e. identities) between composites of congruential functions.

Concrete examples : back to the Greeks

Recall the *'three simple assertibles'* example of S. Bobzien :

$$\begin{tikzcd}
((_ \star _) \star _) & & (_ \star (_ \star _)) \arrow[ll, Leftarrow, "\alpha"'] \\
& (_ \star _ \star _) \arrow[ul, Rightarrow, bend left, "\gamma_\flat"] \arrow[ur, Rightarrow, bend right, "\gamma"'] &
\end{tikzcd}$$

Natural isomorphisms between them have already been named :

The associator

$$\alpha(n) = \begin{cases} 2n & n \equiv 0 \mod 2, \\ n+1 & n \equiv 1 \mod 4, \\ \frac{n-1}{2} & n \equiv 3 \mod 4. \end{cases}$$

The 'amusical permutation'

$$\gamma(n) = \begin{cases} \frac{2n}{3} & n \equiv 0 \mod 3, \\ \frac{4n-1}{3} & n \equiv 1 \mod 3, \\ \frac{4n+1}{3} & n \equiv 2 \mod 3. \end{cases}$$

The 'flattened permutation'

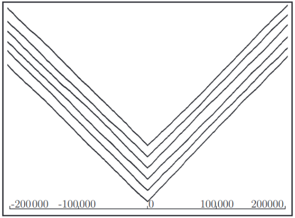
$$\gamma_\flat(n) = \begin{cases} \frac{4n}{3} & n \equiv 0 \mod 3, \\ \frac{4n+2}{3} & n \equiv 1 \mod 3, \\ \frac{2n-1}{3} & n \equiv 2 \mod 3. \end{cases}$$

Musical permutations?

The *amusical permutation* $\gamma(n) = \begin{cases} \frac{2n}{3} & n \equiv 0 \mod 3, \\ \frac{4n-1}{3} & n \equiv 1 \mod 3, \\ \frac{4n+1}{3} & n \equiv 2 \mod 3. \end{cases}$

Introduced by L. Collatz (2nd July 1932), who conjectured that the orbit of 8 under γ is infinite.

A plot of $log_2 (\gamma^n(k))$, for $k = 8, 14, 40, 64, 80, 82$



$$\gamma^{200000}(8) \approx 10^{5000}$$

Conway described this as *'obviously' both true, and undecidable* — he claimed it as the motivation for his proof of undecidability in arithmetic.

The name **amusical permutation** is due to J. Conway

> *"There are twelve notes per octave, which represents a doubling of frequency, just as twelve steps [of γ or γ^{-1}] approximately doubles a number, on average."*
>
> – On unsettleable arithmetic problems (2013)

This expected doubling is not exact :

- [The amusical permutation] a factor of $\frac{3^{12}}{2^{18}} \approx 2$
- [Its inverse] a factor of $\frac{2^{20}}{3^{12}} \approx 2$

"A frequency ratio of $\frac{3^{12}}{2^{19}}$ *is called the* **Pythagorean comma**[4] *and is that between* $B^\flat$ *and* $A^\sharp$*. So there really is a connection with music."*

The amusing musical permutation

"Since the series always ascends by a fifth, modulo octaves, it does not sound very musical. It has always amused me to call it amusical."

[4] First written down by Euclid

What about the flattened permutation?

The other natural isomorphism

$$\gamma_\flat(n) = \begin{cases} \frac{4n}{3} & n \equiv 0 \mod 3, \\ \frac{4n+2}{3} & n \equiv 1 \mod 3, \\ \frac{2n-1}{3} & n \equiv 2 \mod 3. \end{cases}$$

is simply the amusical permutation, 'one step down'. Precisely, $1 + \gamma_\flat(n) = \gamma(n+1)$.

The 'flattened permutation' replicates the behaviour of the 'amusical permutation', *one semitone lower*.

Mathematically / categorically :

The successor function is the *natural partial isomorphism* between γ and $\gamma_\flat$.

$$\gamma_\flat^K = Succ^{-1}\gamma^K Succ \quad \forall K \in \mathbb{N}$$

Finally, some semigroups ...

Structure from *'three simple assertibles'* :

$$\begin{tikzcd}
((_ \star _) \star _) & & (_ \star (_ \star _)) \arrow[ll, Leftarrow, "\alpha"'] \\
& (_ \star _ \star _) \arrow[ul, Rightarrow, bend left, "\gamma_\flat"] \arrow[ur, Rightarrow, bend right, "\gamma"'] &
\end{tikzcd}$$

Elementary properties :

- $\alpha = \gamma_\flat \gamma^{-1}$
- $\mathit{Succ}.\gamma_\flat = \gamma.\mathit{Succ}$
- $\alpha = \left[\mathit{Succ}, \gamma^{-1}\right]$

Claim : The inverse submonoid of $\mathcal{I}(\mathbb{N})$ generated by

$$\{\, \gamma \;,\; \gamma_\flat \;,\; \alpha \;,\; \mathit{Succ} \,\} \;\subseteq \mathcal{I}(\mathbb{N})$$

is related to a wide range of topics : *Girard's conjunction, Thompson's* $\mathcal{F}$*, MacLane's pentagon, bicyclic / polycyclic monoids, Conjectures of Collatz, associahedra, categorical coherence, computational universality / undecidability, the circle of fifths, the Pythagorean comma . . .*