

FREE IDEMPOTENT GENERATED SEMIGROUPS AND ENDOMORPHISM MONOIDS OF FREE G -ACTS

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Dedicated to the memory of Professor David Rees (1918–2013)

ABSTRACT. The study of the free idempotent generated semigroup $\text{IG}(E)$ over a biordered set E began with the seminal work of Nambooripad in the 1970s and has seen a recent revival with a number of new approaches, both geometric and combinatorial. Here we study $\text{IG}(E)$ in the case E is the biordered set of a wreath product $G \wr \mathcal{T}_n$, where G is a group and \mathcal{T}_n is the full transformation monoid on n elements. This wreath product is isomorphic to the endomorphism monoid of the free G -act $\text{End } F_n(G)$ on n generators, and this provides us with a convenient approach.

We say that the *rank* of an element of $\text{End } F_n(G)$ is the minimal number of (free) generators in its image. Let $\varepsilon = \varepsilon^2 \in \text{End } F_n(G)$. For rather straightforward reasons it is known that if $\text{rank } \varepsilon = n - 1$ (respectively, n), then the maximal subgroup of $\text{IG}(E)$ containing ε is free (respectively, trivial). We show that if $\text{rank } \varepsilon = r$ where $1 \leq r \leq n - 2$, then the maximal subgroup of $\text{IG}(E)$ containing ε is isomorphic to that in $\text{End } F_n(G)$ and hence to $G \wr \mathcal{S}_r$, where \mathcal{S}_r is the symmetric group on r elements. We have previously shown this result in the case $r = 1$; however, for higher rank, a more sophisticated approach is needed. Our current proof subsumes the case $r = 1$ and thus provides another approach to showing that *any* group occurs as the maximal subgroup of some $\text{IG}(E)$. On the other hand, varying r again and taking G to be trivial, we obtain an alternative proof of the recent result of Gray and Ruškuc for the biordered set of idempotents of \mathcal{T}_n .

1. INTRODUCTION

Let S be a semigroup and denote by $\langle E \rangle$ the subsemigroup of S generated by the set of idempotents $E = E(S)$ of S . If $S = \langle E \rangle$, then we say that S is *idempotent generated*. The significance of such semigroups was evident at an early stage: in 1966 Howie [15] showed that every semigroup may be embedded into one that is idempotent generated. To do so, he investigated the idempotent generated subsemigroups of transformation monoids, showing in particular that for the full transformation monoid \mathcal{T}_n on n generators (where n is finite), the subsemigroup of singular transformations is idempotent generated. Erdos [8] proved a corresponding ‘linearised’ result, showing that the multiplicative semigroup of singular square matrices over a field is idempotent generated (see also [19]). Fountain and Lewin [10] subsumed these results into the wider context of endomorphism monoids of independence algebras. We

2010 *Mathematics Subject Classification*. Primary 20M05; Secondary 20F05, 20M30.

Key words and phrases. G -act, idempotent, biordered set, wreath product.

Research supported by Grant No. EP/I032312/1 of EPSRC, Grant No. 174019 of the Ministry of Education, Science, and Technological Development of the Republic of Serbia, and by Grant No. 3606/2013 of the Secretariat of Science and Technological Development of the Autonomous Province of Vojvodina.

note here that sets and vector spaces over division rings are examples of independence algebras, as are free (left) G -acts over a group G .

For any set of idempotents $E = E(S)$ there is a free object $\text{IG}(E)$ in the category of semigroups that are generated by E , given by the presentation

$$\text{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef}, e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle,$$

where here $\overline{E} = \{\bar{e} : e \in E\}$. We say that $\text{IG}(E)$ is the *free idempotent generated semigroup over E* . The relations in the presentation for $\text{IG}(E)$ correspond to taking *basic products* in E , that is, products between $e, f \in E$ where e and f are comparable under one of the quasi-orders $\leq_{\mathcal{L}}$ or $\leq_{\mathcal{R}}$ defined on S . In fact, E has an abstract characterisation as a *biordered set*, that is, a partial algebra equipped with two quasi-orders satisfying certain axioms. Biordered sets were introduced in 1979 by Nambooripad [21] in his seminal work on the structure of regular semigroups, as was the notion of free idempotent generated semigroups $\text{IG}(E)$. A celebrated result of Easdown [6] shows every biordered set E occurs as $E(S)$ for some semigroup S , hence we lose nothing by assuming that our set of idempotents is of the form $E(S)$ for a semigroup S .

For any semigroup S and any idempotent $e \in E(S)$, there is a maximal subgroup of S (that is, a subsemigroup that is a group) having identity e ; standard semigroup theory, briefly outlined in Section 2, tells us that this group is the equivalence class of e under Green's relation \mathcal{H} , usually denoted by H_e . The study of maximal subgroups of $\text{IG}(E)$ has a somewhat curious history. It was thought from the 1970s that all such groups would be free (see, for example, [20, 22, 23]), but this conjecture was false. The first published example of a non-free group arising in this context appeared in 2009 [1]; an unpublished example of McElwee from the earlier part of that decade was announced by Easdown in 2011 [7]. Also, the paper [1] exhibited a strong relationship between maximal subgroups of $\text{IG}(E)$ and algebraic topology: namely, it was shown that these groups are precisely fundamental groups of a complex naturally arising from S (called the *Graham-Houghton complex* of S). The 2012 paper of Gray and Ruškuc [13] showed that *any* group occurs as a maximal subgroup of some $\text{IG}(E)$. Their approach is to use existing machinery which affords presentations of maximal subgroups of semigroups, itself developed by Ruškuc [24], refine this to give presentations of $\text{IG}(E)$, and then, given a group G , to carefully choose a biordered set E . Their techniques are significant and powerful, and have other consequences in [13]; we use their presentation in this article. However, to show that *any* group occurs as a maximal subgroup of $\text{IG}(E)$, a simple approach suffices [12]. We also note here that any group occurs as $\text{IG}(E)$ for some *band*, that is, a semigroup of idempotents [5].

The approach of [12] is to consider the biordered set E of non-identity idempotents of a wreath product $G \wr \mathcal{T}_n$ or, equivalently, of the endomorphism monoid $\text{End } F_n(G)$ of a free (left) G -act on n generators $\{x_1, \dots, x_n\}$ (see, for example, [18, Theorem 6.8]). It is known that for a rank r idempotent $\varepsilon \in \text{End } F_n(G)$ we have $H_\varepsilon \cong G \wr \mathcal{S}_r$. For a rank 1 idempotent $\varepsilon \in E$, the maximal subgroup H_ε is isomorphic to H_e and hence to G [12]. This followed a pattern established in [2] and [14] showing (respectively) that the multiplicative group of non-zero elements of any division ring Q occurs as a maximal subgroup of a rank 1 idempotent in

$\text{IG}(E)$, where E is the biordered set of idempotents of $M_n(Q)$ for $n \geq 3$, and that any \mathcal{S}_r occurs as a maximal subgroup of a rank r idempotent in $\text{IG}(F)$, where F is the biordered set of idempotents of a full transformation monoid \mathcal{T}_n for some $n \geq r + 2$. Another way of saying this is that in both these cases, $H_{\bar{\varepsilon}} \cong H_\varepsilon$ for the idempotent in question.

The aim of this current article is to extend the results of both [12] and [14] to show that for a rank r idempotent $\varepsilon \in \text{End } F_n(G)$, with $1 \leq r \leq n - 2$, we have that $H_{\bar{\varepsilon}}$ is isomorphic to H_ε and hence to $G \wr \mathcal{S}_r$. We proceed as follows. In Section 2 we recall some basics of Green's relations on semigroups, and specific details concerning the structure of $\text{End } F_n(G)$. In Section 3 we show how to use the generic presentation for maximal subgroups given in [13] (restated here as Theorem 3.3) to obtain a presentation of $H_{\bar{\varepsilon}}$; once these technicalities are in place we sketch the strategy employed in the rest of the paper, and work our way through this in subsequent sections. By the end of Section 6 we are able to show that for $1 \leq r \leq n/3$, $H_{\bar{\varepsilon}} \cong H_\varepsilon$ (Theorem 6.3), a result corresponding to that in [4] for full linear monoids. To proceed further, we need more sophisticated analysis of the generators of $H_{\bar{\varepsilon}}$. Finally, in Section 9, we make use of the presentation of $G \wr \mathcal{S}_r$ given in [17] to show that we have the required result, namely that $H_{\bar{\varepsilon}} \cong H_\varepsilon$, for any rank r with $1 \leq r \leq n - 2$ (Theorem 9.13). It is worth remarking that if G is trivial, then $F_n(G)$ is essentially a set, so that $\text{End } F_n(G) \cong \mathcal{T}_n$. We are therefore able to recover, via a rather different strategy, the main result of [14].

2. PRELIMINARIES: GREEN'S RELATIONS, AND ENDOMORPHISM MONOIDS OF FREE G -ACTS

In the course of studying the general structural features of semigroups, amongst the most basic tools are the five equivalence relations that capture the ideal structure of a given semigroup S , called *Green's relations*. We define for $a, b \in S$:

$$a \mathcal{R} b \Leftrightarrow aS^1 = bS^1, \quad a \mathcal{L} b \Leftrightarrow S^1a = S^1b, \quad a \mathcal{J} b \Leftrightarrow S^1aS^1 = S^1bS^1,$$

where S^1 denotes S with an identity element adjoined (unless S already has one); hence, these three relations record when two elements of S generate the same right, left, and two-sided principal ideals, respectively. Furthermore, we let $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, while $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is the join of the equivalences \mathcal{R} and \mathcal{L} . As is well known, for finite semigroups we always have $\mathcal{D} = \mathcal{J}$, while in general the inclusions $\mathcal{H} \subseteq \mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ hold. The \mathcal{R} -class of a is usually denoted by R_a , and in a similar fashion we use the notation L_a, J_a, H_a and D_a .

It is also well known that a single \mathcal{D} -class consists either entirely of regular elements, or of non-regular ones, see [16, Proposition 2.3.1]. If $a \in S$ is regular, that is, $a = aba$ for some $b \in S$, then, for any such b , it is clear that $ab, ba \in E(S)$ and $ab \mathcal{R} a \mathcal{L} ba$. Therefore, regular \mathcal{D} -classes are precisely those containing idempotents, and for each idempotent e , the \mathcal{H} -class H_e is a group with identity e . In fact, this is a maximal subgroup of the semigroup under consideration and all maximal subgroups arise in this way.

There are natural orders on the set of \mathcal{R} - and \mathcal{L} -classes of S , respectively, defined by $R_a \leq R_b$ if and only if $aS^1 \subseteq bS^1$, and $L_a \leq L_b$ if and only if $S^1a \subseteq S^1b$. In turn, these orders induce quasi-orders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ on S (mentioned in the introduction), given by $a \leq_{\mathcal{R}} b$ if and only

if $R_a \leq R_b$, and $a \leq_{\mathcal{L}} b$ if and only if $L_a \leq L_b$. Further details of Green's relations and other standard semigroup techniques may be found in [16].

Let S be a semigroup with $E = E(S)$. The semigroup $\text{IG}(E)$ defined in the introduction has some pleasant properties, particularly with respect to Green's relations. It follows from the definition that the natural map $\phi : \text{IG}(E) \rightarrow S$, given by $\bar{e}\phi = e$, is a morphism onto $S' = \langle E \rangle$. Since any morphism preserves \mathcal{L} -classes and \mathcal{R} -classes, certainly so does ϕ . In fact, the structure of the regular \mathcal{D} -classes of $\text{IG}(E)$ is closely related to that in S , as the following result, taken from [9, 21, 6, 2, 13], illustrates.

Proposition 2.1. *Let $S, S', E = E(S), \text{IG}(E)$ and ϕ be as above, and let $e \in E$.*

- (i) *The restriction of ϕ to the set of idempotents of $\text{IG}(E)$ is a bijection onto E (and an isomorphism of biordered sets).*
- (ii) *The morphism ϕ induces a bijection between the set of all \mathcal{R} -classes (respectively \mathcal{L} -classes) in the \mathcal{D} -class of \bar{e} in $\text{IG}(E)$ and the corresponding set in $\langle E \rangle$.*
- (iii) *The restriction of ϕ to $H_{\bar{e}}$ is a morphism onto H_e .*

We now turn our attention to $F_n(G)$ and the structure of its endomorphism monoid. The following notational convention will be useful: for any $u, v \in \mathbb{N}$ with $u \leq v$ we will denote $\{u, u+1, \dots, v-1, v\}$ and $\{u+1, \dots, v-1\}$ by $[u, v]$ and (u, v) , respectively.

Let G be a group, $n \in \mathbb{N}, n \geq 3$, and let $F_n(G) = \bigcup_{i=1}^n Gx_i$ be a rank n free left G -act. We recall that, as a set, $F_n(G)$ consists of the set of formal symbols $\{gx_i : g \in G, i \in [1, n]\}$, and we identify x_i with $1x_i$, where 1 is the identity of G . For any $g, h \in G$ and $1 \leq i, j \leq n$ we have that $gx_i = hx_j$ if and only if $g = h$ and $i = j$; the action of G is given by $g(hx_i) = (gh)x_i$. Let $\text{End } F_n(G)$ denote the endomorphism monoid of $F_n(G)$ (with composition left-to-right). The image of $\alpha \in \text{End } F_n(G)$ being a (free) G -subact, we can define the *rank* of α to be the rank of $\text{im } \alpha$.

Since $F_n(G)$ is an independence algebra, a direct application of Corollary 4.6 [11] gives a useful characterisation of Green's relations on $\text{End } F_n(G)$.

Lemma 2.2. [11] *For any $\alpha, \beta \in \text{End } F_n(G)$, we have the following:*

- (i) *$\text{im } \alpha = \text{im } \beta$ if and only if $\alpha \mathcal{L} \beta$;*
- (ii) *$\ker \alpha = \ker \beta$ if and only if $\alpha \mathcal{R} \beta$;*
- (iii) *$\text{rank } \alpha = \text{rank } \beta$ if and only if $\alpha \mathcal{D} \beta$ if and only if $\alpha \mathcal{J} \beta$.*

Each $\alpha \in \text{End } F_n(G)$ depends only on its action on the free generators $\{x_i : i \in [1, n]\}$ and it is therefore convenient to write

$$x_j \alpha = w_j^\alpha x_{j\bar{\alpha}}$$

for $j \in [1, n]$. This determines a function $\bar{\alpha} : [1, n] \rightarrow [1, n]$ and an element $\alpha_G = (w_1^\alpha, \dots, w_n^\alpha) \in G^n$. It will frequently be convenient to express α as above as

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ w_1^\alpha x_{1\bar{\alpha}} & w_2^\alpha x_{2\bar{\alpha}} & \dots & w_n^\alpha x_{n\bar{\alpha}} \end{pmatrix}.$$

Theorem 2.3. [25, 18] *The function*

$$\psi : \text{End } F_n(G) \mapsto G \wr T_n, \quad \alpha \mapsto (\alpha_G, \bar{\alpha})$$

is an isomorphism.

Let $1 \leq r \leq n$ and set $D_r = \{\alpha \in \text{End } F_n(G) \mid \text{rank } \alpha = r\}$, that is, D_r is the \mathcal{D} -class in $\text{End } F_n(G)$ of any rank r element. We let I and Λ denote the set of \mathcal{R} -classes and the set of \mathcal{L} -classes of D_r , respectively. Thus, I is in bijective correspondence with the set of kernels, and Λ with the set of images, of rank r endomorphisms, respectively. It is convenient to assume I is the set of kernels of rank r endomorphisms, and that

$$\Lambda = \{(u_1, u_2, \dots, u_r) : 1 \leq u_1 < u_2 < \dots < u_r \leq n\} \subseteq [1, n]^r.$$

Thus $\alpha \in R_i$ if and only if $\ker \alpha = i$ and $\alpha \in L_{(u_1, \dots, u_r)}$ if and only if

$$\text{im } \alpha = Gx_{u_1} \cup Gx_{u_2} \cup \dots \cup Gx_{u_r}.$$

For every $i \in I$ and $\lambda \in \Lambda$, we put $H_{i\lambda} = R_i \cap L_\lambda$ so that $H_{i\lambda}$ is an \mathcal{H} -class of D_r . Where $H_{i\lambda}$ is a subgroup, we denote its identity by $\varepsilon_{i\lambda}$. It is notationally standard to use the same symbol 1 to denote a selected element from both I and Λ . Here we let $1 = \langle (x_1, x_i) : r+1 \leq i \leq n \rangle \in I$, that is, the congruence generated by $\{(x_1, x_i) : r+1 \leq i \leq n\}$, and $1 = (1, 2, \dots, r) \in \Lambda$. Then $H = H_{11}$ is a group \mathcal{H} -class in D_r , with identity ε_{11} .

A typical element of H looks like

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r & x_{r+1} & \dots & x_n \\ w_1^\alpha x_{1\bar{\alpha}} & w_2^\alpha x_{2\bar{\alpha}} & \dots & w_r^\alpha x_{r\bar{\alpha}} & w_1^\alpha x_{1\bar{\alpha}} & \dots & w_1^\alpha x_{1\bar{\alpha}} \end{pmatrix}$$

which in view of the following lemma we may abbreviate without further remark to:

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ w_1^\alpha x_{1\bar{\alpha}} & w_2^\alpha x_{2\bar{\alpha}} & \dots & w_r^\alpha x_{r\bar{\alpha}} \end{pmatrix},$$

where here we are regarding $\bar{\alpha}$ as an element of \mathcal{S}_r .

Lemma 2.4. *The groups H and $\text{Aut } F_r(G)$ are isomorphic under the map*

$$\begin{pmatrix} x_1 & x_2 & \dots & x_r & x_{r+1} & \dots & x_n \\ w_1^\alpha x_{1\bar{\alpha}} & w_2^\alpha x_{2\bar{\alpha}} & \dots & w_r^\alpha x_{r\bar{\alpha}} & w_1^\alpha x_{1\bar{\alpha}} & \dots & w_1^\alpha x_{1\bar{\alpha}} \end{pmatrix} \mapsto \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ w_1^\alpha x_{1\bar{\alpha}} & w_2^\alpha x_{2\bar{\alpha}} & \dots & w_r^\alpha x_{r\bar{\alpha}} \end{pmatrix}.$$

Under this convention, the identity $\varepsilon = \varepsilon_{11}$ of H becomes

$$\varepsilon = \begin{pmatrix} x_1 & \dots & x_r \\ x_1 & \dots & x_r \end{pmatrix}.$$

With the aim of specialising the presentation given in Theorem 3.3, we locate and distinguish elements in $H_{1\lambda}$ and H_{i1} for each $\lambda \in \Lambda$ and $i \in I$. For any equivalence relation τ on $[1, n]$ with r classes, we write $\tau = \{B_1^\tau, \dots, B_r^\tau\}$ (that is, we identify τ with the partition on $[1, n]$ that it induces). Let $l_1^\tau, \dots, l_r^\tau$ be the minimum elements of $B_1^\tau, \dots, B_r^\tau$, respectively. Without loss of generality we suppose that $l_1^\tau < \dots < l_r^\tau$. Then $l_1^\tau = 1$ and $l_j^\tau \geq j$, for any $j \in [2, r]$. Suppose now that $\alpha \in \text{End } F_n(G)$ and $\text{rank } \alpha = r$, that is, $\alpha \in D_r$. Then $\ker \bar{\alpha}$ has r equivalence classes. Where $\tau = \ker \bar{\alpha}$ we simplify our notation by writing $B_j^{\ker \bar{\alpha}} = B_j^\alpha$ and $l_j^{\ker \bar{\alpha}} = l_j^\alpha$. If there is no ambiguity over the choice of α we may simplify further to B_j and l_j .

Lemma 2.5. *Let $\alpha, \beta \in D_r$. Then $\ker \alpha = \ker \beta$ if and only if $\ker \bar{\alpha} = \ker \bar{\beta}$ and for any $j \in [1, r]$ there exists $g_j \in G$ such that for any $k \in B_j^\alpha = B_j = B_j^\beta$, we have $w_k^\alpha = w_k^\beta g_j$. Moreover, we can take $g_j = (w_{l_j}^\beta)^{-1} w_{l_j}^\alpha$ for $j \in [1, r]$.*

Proof. If $\ker \alpha = \ker \beta$, then clearly $\ker \bar{\alpha} = \ker \bar{\beta}$. Now for any $j \in [1, r]$ and $k \in B_j^\alpha = B_j^\beta$, we have that $((w_{l_j}^\alpha)^{-1} x_{l_j})\alpha = ((w_k^\alpha)^{-1} x_k)\alpha$ and so $((w_{l_j}^\alpha)^{-1} x_{l_j})\beta = ((w_k^\alpha)^{-1} x_k)\beta$, giving that $w_k^\alpha = w_k^\beta ((w_{l_j}^\beta)^{-1} w_{l_j}^\alpha)$. We may thus take $g_j = (w_{l_j}^\beta)^{-1} w_{l_j}^\alpha$.

Conversely, suppose that $\ker \bar{\alpha} = \ker \bar{\beta}$ (and has blocks $\{B_1, \dots, B_r\}$) and for any $j \in [1, r]$ there exists $g_j \in G$ satisfying the given condition. Let $ux_h, vx_k \in F_n(G)$. Then

$$\begin{aligned} (ux_h)\alpha = (vx_k)\alpha &\Leftrightarrow h, k \in B_j \text{ for some } j \in [1, r] \text{ and } uw_h^\alpha = vw_k^\alpha \\ &\Leftrightarrow h, k \in B_j \text{ for some } j \in [1, r] \text{ and } uw_h^\beta g_j = vw_k^\beta g_j \\ &\Leftrightarrow h, k \in B_j \text{ for some } j \in [1, r] \text{ and } uw_h^\beta = vw_k^\beta \\ &\Leftrightarrow (ux_h)\beta = (vx_k)\beta \end{aligned}$$

so that $\ker \alpha = \ker \beta$ as required. \square

For the following, we denote by $P(n, r)$ the set of equivalence relations on $[1, n]$ having r classes. Of course, $|P(n, r)| = S(n, r)$, where $S(n, r)$ is a Stirling number of the second kind, but we shall not need that fact here.

Corollary 2.6. *The map $\tau : I \rightarrow G^{n-r} \times P(n, r)$ given by*

$$i\tau = ((w_2^\alpha, \dots, w_{l_2-1}^\alpha, w_{l_2+1}^\alpha, \dots, w_{l_r-1}^\alpha, w_{l_r+1}^\alpha, \dots, w_n^\alpha), \ker \bar{\alpha})$$

where $\alpha \in R_i$ and $w_{l_j}^\alpha = 1_G$, for all $j \in [1, r]$, is a bijection.

Proof. For $i \in I$ choose $\beta \in R_i$ and then define $\alpha \in \text{End } F_n(G)$ by $x_k \alpha = w_k^\beta (w_{l_j}^\beta)^{-1} x_j$, where $k \in B_j^\beta$. It is clear from Lemma 2.5 that $\ker \alpha = \ker \beta$ and so $\alpha \in R_i$. Now $x_{l_j} \alpha = w_{l_j}^\beta (w_{l_j}^\beta)^{-1} x_j = x_j$, so that $i\tau$ is defined. An easy argument, again from Lemma 2.5, gives that τ is well defined and one-one.

For $\mu \in P(n, r)$ let $\nu_\mu : [1, n] \rightarrow [1, r]$ be given by $k\nu_\mu = j$ where $k \in B_j^\mu$. Now for $((h_1, \dots, h_{n-r}), \mu) \in G^{n-r} \times P(n, r)$, define

$$\alpha = ((1_G, h_1, \dots, h_{l_2-2}^\mu, 1_G, h_{l_2-1}^\mu, \dots, h_{l_r-1}^\mu, 1_G, h_{l_r-1}^\mu, \dots, h_{n-r}), \nu_\mu) \psi^{-1}.$$

It is clear that if $\alpha \in R_i$, then $i\tau = ((h_1, \dots, h_{n-r}), \mu)$. Thus τ is a bijection as required. \square

Corollary 2.7. *Let Θ be the set defined by*

$$\Theta = \{\alpha \in D_r : x_{l_j}^\alpha \alpha = x_j, j \in [1, r]\}.$$

Then Θ is a transversal of the \mathcal{H} -classes of L_1 .

Proof. Clearly, $\text{im } \alpha = Gx_1 \cup \dots \cup Gx_r$, for any $\alpha \in \Theta$, and so that Θ is a subset of L_1 .

Next, we show that for each $i \in I$, $|H_{i1} \cap \Theta| = 1$. Suppose that $\alpha, \beta \in \Theta$ and $\ker \alpha = \ker \beta$. Clearly $\ker \bar{\alpha} = \ker \bar{\beta}$ and so $B_j^\alpha = B_j = B_j^\beta$ for any $j \in [1, r]$, and by definition of Θ , $w_{l_j}^\alpha = w_{l_j}^\beta = 1_G$. It is then clear from Lemma 2.5 that for any $k \in B_j$ we have

$$x_k \alpha = w_k^\alpha x_j = w_k^\beta x_j = x_k \beta,$$

so that $\alpha = \beta$.

It only remains to show that for any $i \in I$ we have $|H_{i1} \cap \Theta| \neq \emptyset$. By Corollary 2.6, for $i \in I$ we can find $\alpha \in R_i$ such that $w_{l_j}^\alpha = 1_G$ for all $j \in [1, r]$. Composing α with $\beta \in \text{End } F_n(G)$ where $x_{l_j \bar{\alpha}} \beta = x_j$ for all $j \in [1, r]$ and $x_k \beta = x_1$ else, we clearly have that $\alpha \beta \in H_{i1} \cap \Theta$. \square

For each $i \in I$, we denote the unique element in $H_{i1} \cap \Theta$ by \mathbf{r}_i . Notice that $\mathbf{r}_1 = \varepsilon$.

On the other hand, for $\lambda = (u_1, u_2, \dots, u_r) \in \Lambda$, we define

$$\mathbf{q}_\lambda = \mathbf{q}_{(u_1, \dots, u_r)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ x_{u_1} & x_{u_2} & \cdots & x_{u_r} & x_{u_1} & \cdots & x_{u_1} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_{u_1} & x_{u_2} & \cdots & x_{u_r} \end{pmatrix}.$$

It is easy to see that $\mathbf{q}_\lambda \in H_{1\lambda}$, as $\ker \mathbf{q}_\lambda = \langle (x_1, x_i) : r+1 \leq i \leq n \rangle$. In particular, we have

$$\mathbf{q}_1 = \mathbf{q}_{(1, 2, \dots, r)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_1 & x_2 & \cdots & x_r \end{pmatrix} = \varepsilon.$$

At this point we invoke once more a little standard semigroup theory. Let K be a group, let J, Γ be non-empty sets and let $M = (m_{\gamma j})$ be a $\Gamma \times J$ matrix with entries from $K \cup \{0\}$ (where 0 is a new symbol), with the property that every row and column of M contains at least one entry from K . A *Rees matrix semigroup* $\mathcal{M}^0 = \mathcal{M}^0(K; J, \Gamma; M)$ has underlying set

$$(J \times K \times \Gamma) \cup \{0\}$$

with binary operation given by

$$(j, a, \lambda)(k, b, \mu) = (j, am_{\lambda k}b, \mu) \text{ if } m_{\lambda k} \neq 0,$$

all other products being 0.

By [11, Theorem 4.9] if we put

$$D_r^0 = D_r \cup \{0\}$$

and define a binary operation by

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \alpha, \beta \in D_r \text{ and } \text{rank } \alpha\beta = r \\ 0 & \text{else} \end{cases},$$

then D_r^0 is a semigroup under \cdot which is completely 0-simple. We do not need to give the specifics of what the latter property entails, since, by the Rees Theorem (see [16, Chapter III]), D_r^0 is isomorphic to $\mathcal{M}^0 = \mathcal{M}^0(H; I, \Lambda; P)$, where $P = (\mathbf{p}_{\lambda i})$ and $\mathbf{p}_{\lambda i} = (\mathbf{q}_\lambda \mathbf{r}_i)$ if $\text{rank } \mathbf{q}_\lambda \mathbf{r}_i = r$, and is 0 else. Our choice of P will allow us at crucial points to modify the presentation given in Theorem 3.3.

3. PRESENTATION OF MAXIMAL SUBGROUPS OF $\text{IG}(E)$

Let E be a biordered set; from [6] we can assume that $E = E(S)$ for some semigroup S . An *E-square* is a sequence (e, f, g, h, e) of elements of E with $e \mathcal{R} f \mathcal{L} g \mathcal{R} h \mathcal{L} e$. We draw such an *E-square* as $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$. The following results are folklore (cf. [12]).

Lemma 3.1. *The elements of an E -square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ form a rectangular band (within S) if and only if one (equivalently, all) of the following four equalities holds: $eg = f$, $ge = h$, $fh = e$ or $hf = g$.*

Lemma 3.2. *Let $\mathcal{M}^0 = \mathcal{M}^0(K; J, \Gamma; M)$ be a Rees matrix semigroup over a group K with sandwich matrix $M = (m_{\lambda j})$. For any $j \in J, \lambda \in \Gamma$ write $e_{j\lambda}$ for the idempotent $(j, m_{\lambda j}^{-1}, \lambda)$. Then an E -square $\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{j\lambda} & e_{j\mu} \end{bmatrix}$ is a rectangular band if and only if $m_{\lambda i}^{-1}m_{\lambda j} = m_{\mu i}^{-1}m_{\mu j}$.*

An E -square (e, f, g, h, e) is *singular* if, in addition, there exists $k \in E$ such that either:

$$\begin{cases} ek = e, fk = f, ke = h, kf = g \text{ or} \\ ke = e, kh = h, ek = f, hk = g. \end{cases}$$

We call a singular square for which the first condition holds an *up-down singular square*, and that satisfying the second condition a *left-right singular square*.

For $e \in E$ we let \bar{H} be the maximal subgroup of \bar{e} in $\text{IG}(E)$, (that is, $\bar{H} = H_{\bar{e}}$). We now recall the recipe for obtaining a presentation for \bar{H} obtained by Gray and Ruškuc [13]; for further details, we refer the reader to that article.

We use J and Γ to denote the set of \mathcal{R} -classes and the set of \mathcal{L} -classes, respectively, in the \mathcal{D} -class $\bar{D} = D_{\bar{e}}$ of \bar{e} in $\text{IG}(E)$. In view of Proposition 2.1, J and Γ also label the set of \mathcal{R} -classes and the set of \mathcal{L} -classes, respectively, in the \mathcal{D} -class $D = D_e$ of e in S . For every $i \in J$ and $\lambda \in \Gamma$, let $\bar{H}_{i\lambda}$ and $H_{i\lambda}$ denote, respectively, the \mathcal{H} -class corresponding to the intersection of the \mathcal{R} -class indexed by i and the \mathcal{L} -class indexed by λ in $\text{IG}(E)$, respectively S , so that $\bar{H}_{i\lambda}$ and $H_{i\lambda}$ are \mathcal{H} -classes of \bar{D} and D , respectively. Where $\bar{H}_{i\lambda}$ (equivalently, $H_{i\lambda}$) contains an idempotent, we denote it by $\bar{e}_{i\lambda}$ (respectively, $e_{i\lambda}$). Without loss of generality we assume $1 \in J \cap \Gamma$ and $\bar{e} = \bar{e}_{11} \in \bar{H}_{11} = \bar{H}$, so that $e = e_{11} \in H_{11} = H$. For each $\lambda \in \Gamma$, we abbreviate $\bar{H}_{1\lambda}$ by \bar{H}_λ , and $H_{1\lambda}$ by H_λ and so, $\bar{H}_1 = \bar{H}$ and $H_1 = H$.

Let \bar{h}_λ be an element in \bar{E}^* such that $\bar{H}_1 \bar{h}_\lambda = \bar{H}_\lambda$, for each $\lambda \in \Gamma$. The reader should be aware that this is a point where we are most certainly abusing notation: whereas \bar{h}_λ lies in the free monoid on \bar{E} , by writing $\bar{H}_1 \bar{h}_\lambda = \bar{H}_\lambda$ we mean that the image of \bar{h}_λ under the natural map that takes \bar{E}^* to (right translations in) the full transformation monoid on $\text{IG}(E)$ yields $\bar{H}_1 \bar{h}_\lambda = \bar{H}_\lambda$. In fact, it follows from Proposition 2.1 that the action of any generator $\bar{f} \in \bar{E}$ on an \mathcal{H} -class contained in the \mathcal{R} -class of \bar{e} in $\text{IG}(E)$ is equivalent to the action of f on the corresponding \mathcal{H} -class in the original semigroup S . Thus $\bar{H}_1 \bar{h}_\lambda = \bar{H}_\lambda$ in $\text{IG}(E)$ is equivalent to the corresponding statement $H_1 h_\lambda = H_\lambda$ for S , where h_λ is the image of \bar{h}_λ under the natural map to $\langle E \rangle^1$.

We say that $\{\bar{h}_\lambda \mid \lambda \in \Gamma\}$ forms a *Schreier system of representatives*, if every prefix of \bar{h}_λ (including the empty word) is equal to some \bar{h}_μ , where $\mu \in \Gamma$. Notice that the condition that $\bar{h}_\lambda \bar{e}_{i\mu} = \bar{h}_\mu$ is equivalent to saying that $\bar{h}_\lambda \bar{e}_{i\mu}$ lies in the Schreier system.

Define $K = \{(i, \lambda) \in J \times \Gamma : H_{i\lambda} \text{ is a group } \mathcal{H}\text{-class}\}$. Since D_e is regular, for each $i \in J$ we can find and fix an element $\omega(i) \in \Gamma$ such that $(i, \omega(i)) \in K$, so that $\omega : J \rightarrow \Gamma$ is a function. Again for convenience we take $\omega(1) = 1$.

Theorem 3.3. [13] *The maximal subgroup \overline{H} of \overline{e} in $\text{IG}(E)$ is defined by the presentation*

$$\mathcal{P} = \langle F : \Sigma \rangle$$

with generators:

$$F = \{f_{i,\lambda} : (i, \lambda) \in K\}$$

and defining relations Σ :

$$\begin{aligned} (R1) \quad & f_{i,\lambda} = f_{i,\mu} \quad (\overline{h}_\lambda \overline{e}_{i\mu} = \overline{h}_\mu); \\ (R2) \quad & f_{i,\omega(i)} = 1 \quad (i \in J); \\ (R3) \quad & f_{i,\lambda}^{-1} f_{i,\mu} = f_{k,\lambda}^{-1} f_{k,\mu} \quad \left(\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{k\lambda} & e_{k\mu} \end{bmatrix} \text{ is a singular square} \right). \end{aligned}$$

For the remainder of this paper, E will denote $E(\text{End } F_n(G))$. In addition, for the sake of notational convenience, we now observe the accepted convention of dropping the overline notation for elements of \overline{E}^* . In particular, idempotents of $\text{IG}(E)$ carry the same notation as those of $\text{End } F_n(G)$; the context should hopefully prevent confusion.

In order to specialise the above presentation to E , our first step is to identify the singular squares.

Lemma 3.4. *An E -square $\begin{bmatrix} \gamma & \delta \\ \xi & \nu \end{bmatrix}$ is singular if and only if $\{\gamma, \delta, \nu, \xi\}$ is a rectangular band.*

Proof. The proof of necessity is standard. We only need to show the sufficiency. Let $\{\gamma, \delta, \nu, \xi\}$ be a rectangular band so that $\gamma\nu = \delta, \nu\gamma = \xi, \delta\xi = \gamma$ and $\xi\delta = \nu$. Suppose $\text{im } \gamma = \text{im } \xi = \langle x_m \rangle_{m \in M}$ and $\text{im } \delta = \text{im } \nu = \langle x_n \rangle_{n \in N}$, where $|M| = |N| = r$. Put $L = M \cup N$. Define a mapping $\theta \in \text{End } F_n(G)$ by

$$x_i\theta = \begin{cases} x_i & \text{if } i \in L; \\ x_i\nu & \text{else.} \end{cases}$$

Since $\text{im } \theta = \langle x_l \rangle_{l \in L}$ and for each $l \in L$, $x_l\theta = x_l$, we see that θ is an idempotent. It is also clear that $\gamma\theta = \gamma$ and $\delta\theta = \delta$, as $\text{im } \gamma \cup \text{im } \delta \subseteq \text{im } \theta$.

Next, we will show $\theta\gamma = \xi$. If $i \in M$, then $x_i\theta\gamma = x_i\gamma = x_i = x_i\xi$. If $i \in N$, but $i \notin M$, then $x_i\theta\gamma = x_i\gamma = x_i\nu\gamma = x_i\xi$. If $i \notin L$, then $x_i\theta\gamma = x_i\nu\gamma = x_i\xi$. So, $\theta\gamma = \xi$. For the remaining equality $\theta\delta = \nu$ required in the definition of a singular square, observe that, for each $i \in N$, $x_i\theta\delta = x_i = x_i\nu$. On the other hand if $i \in M$, then $x_i\theta\delta = x_i\delta = x_i\gamma\nu = x_i\nu$, since $\delta = \gamma\nu$ by assumption. For $i \notin L$ we have $x_i\theta\delta = x_i\nu\delta = x_i\nu$, since $\delta \mathcal{L} \nu$.

We have proved that $\gamma\theta = \gamma$, $\delta\theta = \delta$, $\theta\gamma = \xi$ and $\theta\delta = \nu$, so that $\begin{bmatrix} \gamma & \delta \\ \xi & \nu \end{bmatrix}$ is an up-down singular square. \square

The proof of Lemma 3.4 shows the following:

Corollary 3.5. *An E -square is singular if and only if it is an up-down singular square.*

The next corollary is immediate from Lemmas 3.2 and 3.4.

Corollary 3.6. *Let $P = (\mathbf{p}_{\lambda_i})$ be the sandwich matrix of any completely 0-simple semigroup isomorphic to D_r^0 . Then (R3) in Theorem 3.3 can be restated as:*

$$(R3) \ f_{i,\lambda}^{-1} f_{i,\mu} = f_{k,\lambda}^{-1} f_{k,\mu} \quad (\mathbf{p}_{\lambda_i}^{-1} \mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1} \mathbf{p}_{\mu k}).$$

We focus on the idempotent $\varepsilon = \varepsilon_{11}$ of Section 2. For the presentation $\mathcal{P} = \langle F : \Sigma \rangle$ for our particular $\overline{H} = H_{\overline{\varepsilon}}$, we must define a Schreier system of words $\{\mathbf{h}_{\lambda} : \lambda \in \Lambda\}$. In this instance, we can do so inductively, using the restriction of the lexicographic order on $[1, n]^r$ to Λ . Recall that we are using the same notation for $\mathbf{h}_{\lambda} \in E^*$ and its image under the natural morphism to the set of right translations of $\text{IG}(E)$ and of $\text{End } F_n(G)$.

First, we define $\mathbf{h}_{(1,2,\dots,r)} = 1$, the empty word in E^* . Now let $(u_1, u_2, \dots, u_r) \in \Lambda$ with $(1, 2, \dots, r) < (u_1, u_2, \dots, u_r)$, and assume for induction that $\mathbf{h}_{(v_1, v_2, \dots, v_r)}$ has been defined for all $(v_1, v_2, \dots, v_r) < (u_1, u_2, \dots, u_r)$. Taking $u_0 = 0$ there must exist some $j \in [1, r]$ such that $u_j - u_{j-1} > 1$. Letting i be largest such that $u_i - u_{i-1} > 1$ observe that

$$(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r) < (u_1, u_2, \dots, u_r).$$

We now define

$$\mathbf{h}_{(u_1, \dots, u_r)} = \mathbf{h}_{(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r)} \alpha_{(u_1, \dots, u_r)},$$

where

$$\alpha_{(u_1, \dots, u_r)} = \begin{pmatrix} x_1 & \cdots & x_{u_1} & x_{u_1+1} & \cdots & x_{u_2} & \cdots & x_{u_{r-1}+1} & \cdots & x_{u_r} & x_{u_r+1} & \cdots & x_n \\ x_{u_1} & \cdots & x_{u_1} & x_{u_2} & \cdots & x_{u_2} & \cdots & x_{u_r} & \cdots & x_{u_r} & x_{u_r} & \cdots & x_{u_r} \end{pmatrix};$$

notice that $\alpha_{(u_1, \dots, u_r)} = \varepsilon_{l(u_1, \dots, u_r)}$ for some $l \in I$.

Lemma 3.7. *For all $(u_1, \dots, u_r) \in \Lambda$ we have $\varepsilon \mathbf{h}_{(u_1, \dots, u_r)} = \mathbf{q}_{(u_1, \dots, u_r)}$. Hence right translation by $\mathbf{h}_{(u_1, \dots, u_r)}$ induces a bijection from $L_{(1, \dots, r)}$ onto $L_{(u_1, \dots, u_r)}$ in both $\text{End } F_n(G)$ and $\text{IG}(E)$.*

Proof. We prove by induction on (u_1, \dots, u_r) that $\varepsilon \mathbf{h}_{(u_1, \dots, u_r)} = \mathbf{q}_{(u_1, \dots, u_r)}$. Clearly the statement is true for $(u_1, \dots, u_r) = (1, \dots, r)$. Suppose now result is true for all $(v_1, \dots, v_r) < (u_1, \dots, u_r)$, so that $\varepsilon \mathbf{h}_{(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r)} = \mathbf{q}_{(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r)}$. Since $x_{u_j} \alpha = x_{u_j}$ for all $j \in \{1, \dots, r\}$ and $x_{u_{i-1}} \alpha = x_{u_i}$, it follows that

$$\varepsilon \mathbf{h}_{(u_1, \dots, u_r)} = \varepsilon \mathbf{h}_{(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r)} \alpha_{(u_1, \dots, u_r)} = \mathbf{q}_{(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r)} \alpha_{(u_1, \dots, u_r)} = \mathbf{q}_{(u_1, \dots, u_r)}$$

as required.

Since by definition, $\mathbf{q}_{(u_1, \dots, u_r)} \in L_{(u_1, \dots, u_r)}$, the result for $\text{End } F_n(G)$ follows from Green's Lemma (see, for example, [16, Chapter II]), and that for $\text{IG}(E)$ by the comments in Section 3. \square

It is a consequence of Lemma 3.7 that $\{\mathbf{h}_{\lambda} : \lambda \in \Lambda\}$ forms the required Schreier system for a presentation \mathcal{P} for \overline{H} . It remains to define the function ω : we do so by setting $\omega(i) = (l_1^i, l_2^i, \dots, l_r^i) = (1, l_2^i, \dots, l_r^i)$ for each $i \in I$. Note that for any $i \in I$ we have $\mathbf{q}_{\omega(i)} \mathbf{r}_i = \varepsilon$, i.e. $\mathbf{p}_{\omega(i), i} = \varepsilon$.

Definition 3.8. Let $\mathcal{P} = \langle F : \Sigma \rangle$ be the presentation of \overline{H} as in Theorem 3.3, where ω and $\{\mathbf{h}_{\lambda} : \lambda \in \Lambda\}$ are given as above.

Without loss of generality, we assume that \overline{H} is the group with presentation \mathcal{P} .

In later parts of this work we will be considering for a non-zero entry $\phi \in P$, which $i \in I, \lambda \in \Lambda$ yield $\phi = \mathbf{p}_{\lambda i}$. For this and other purposes it is convenient to define the notion of district. For $i \in I$ we say that \mathbf{r}_i lies in *district* $(l_1^{\mathbf{r}_i}, l_2^{\mathbf{r}_i}, \dots, l_r^{\mathbf{r}_i})$ (of course, $1 = l_1^{\mathbf{r}_i}$). Note that lying in the same district induces a partition of Θ . The next lemma follows immediately from the definition of $\mathbf{r}_i, i \in I$.

Lemma 3.9. *For any $i \in I$, if \mathbf{r}_i lies in district $(1, l_2, \dots, l_r)$, then $l_s \geq s$ for all $s \in [1, r]$. Moreover, for $k \in [1, n]$, if $x_k \mathbf{r}_i = ax_j$, then $k \geq l_j$, with $k > l_j$ if $a \neq 1_G$.*

We pause to consider which elements of H can occur as an entry ϕ of P ; with abuse of terminology, we will say that $\phi \in P$. As indicated before Lemma 2.4, we can write $\phi \in H$ as

$$\phi = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ a_1 x_{1\bar{\phi}} & a_2 x_{2\bar{\phi}} & \dots & a_r x_{r\bar{\phi}} \end{pmatrix}$$

where $\bar{\phi} \in \mathcal{S}_r$ and $(a_1, \dots, a_r) \in G^r$. If $\phi = \mathbf{p}_{\lambda i} \in P$, where $\lambda = (u_1, \dots, u_r)$ and \mathbf{r}_i lies in district (l_1, \dots, l_r) , then the u_j s and l_k s are constrained by

$$1 = l_1 < l_2 < \dots < l_r, \quad u_1 < u_2 < \dots < u_r,$$

$$l_{j\bar{\phi}} \leq u_j \text{ for all } j \in [1, r] \text{ with } l_{j\bar{\phi}} < u_j \text{ if } a_j \neq 1_G,$$

and

$$l_k = u_j \text{ implies } k = j\bar{\phi} \text{ and } a_j = 1_G \text{ for all } k, j \in [1, r].$$

Conversely, if these constraints are satisfied by $l_1, \dots, l_r, u_1, \dots, u_r \in [1, n]$ with respect to some $\bar{\phi} \in \mathcal{S}_r$ and $(a_1, \dots, a_r) \in G^r$, then it is easy to see that if $\xi \in \text{End } F_n(G)$ is defined by

$$x_{l_k} \xi = x_k, \quad x_{u_k} \xi = a_k x_{k\bar{\phi}}, \quad k \in [1, r]$$

and

$$x_j \xi = x_1 \text{ for } j \notin \{l_1, \dots, l_r, u_1, \dots, u_r\},$$

then $\xi = r_i$ for some $i \in I$, where r_i lies in district (l_1, l_2, \dots, l_r) . Clearly, $\mathbf{p}_{\lambda i} = \phi$.

Lemma 3.10. *If $|G| > 1$ then every element of H occurs as an entry in P if and only if $2r \leq n$. If $|G| = 1$ then every element of H occurs as an entry in P if and only if $2r \leq n + 1$.*

Proof. Suppose first that $|G| > 1$. If $2r \leq n$, then given any $\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ a_1 x_{1\bar{\alpha}} & a_2 x_{2\bar{\alpha}} & \dots & a_r x_{r\bar{\alpha}} \end{pmatrix}$ in H , we can take $(l_1, \dots, l_r) = (1, 2, \dots, r)$ and $(u_1, \dots, u_r) = (r + 1, \dots, 2r)$. Conversely, if $2r > n$, then $\begin{pmatrix} x_1 & x_2 & \dots & x_r \\ a x_r & x_{r-1} & \dots & x_1 \end{pmatrix}$, where $a \neq 1_G$, cannot lie in P , since we would need $l_{1\bar{\alpha}} = l_r < u_1$.

Consider now the case where $|G| = 1$. If $2r \leq n + 1$, then given any $\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ x_{1\bar{\alpha}} & x_{2\bar{\alpha}} & \dots & x_{r\bar{\alpha}} \end{pmatrix}$ in H , let $1\bar{\alpha} = t$ and choose

$$(l_1, \dots, l_r) = (1, \dots, r) \text{ and } (u_1, \dots, u_r) = (t, r + 1, \dots, 2r - 1).$$

It follows from the discussion preceding the lemma that $\alpha \in P$. Conversely, if $2r > n + 1$, then $\begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_r & x_{r-1} & \cdots & x_1 \end{pmatrix}$ cannot lie in P , since now we would require $l_{1\bar{\alpha}} = l_r \leq u_1$. \square

We are now in a position to outline the proof of our main theorem, Theorem 9.13, which states that \overline{H} is isomorphic to H , and hence to $G \wr \mathcal{S}_r$.

We first claim that for any $i, j \in I$ and $\lambda, \mu \in \Lambda$, if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$, then $f_{i,\lambda} = f_{j,\mu}$. We verify our claim via a series of steps. We first deal with the case where $\mathbf{p}_{\lambda i} = \varepsilon$ and here show that $f_{i,\lambda}$ (and $f_{j,\mu}$) is the identity of \overline{H} (Lemma 4.1). Next, we verify the claim in the case where $\mu = \lambda$ (Lemma 5.1) or $i = j$ (Lemma 5.3). We then show that for $r \leq \frac{n}{2} - 1$, this is sufficient (via finite induction) to prove the claim holds in general (Lemma 6.1). However, a counterexample shows that for larger r this strategy will fail.

To overcome the above problem, we begin by showing that if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ is what we call a *simple form*, that is,

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-1} & x_{k+1} & x_{k+2} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \cdots & x_r \end{pmatrix},$$

for some $k \geq 1, m \geq 0, a \in G$, then $f_{i,\lambda} = f_{j,\mu}$. We then introduce the notion of *rising point* and verify by induction on the rising point, with the notion of simple form forming the basis of our induction, that our claim holds. As a consequence of our claim we denote a generator $f_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \phi$ by f_ϕ .

For $r \leq \frac{n}{2}$ it is easy to see that every element of H occurs as *some* $\mathbf{p}_{\lambda i}$ and for $r \leq \frac{n}{3}$ we have enough room for manoeuvre (the reader studying Sections 5 and 6 will come to an understanding of what this means) to show that $f_\phi f_\varphi = f_{\varphi\phi}$ and it is then easy to see that $\overline{H} \cong H$ (Theorem 6.3).

To deal with the general case of $r \leq n - 2$ we face two problems. One is that for $r > \frac{n}{2}$, not every element of H occurs as some element of P and secondly, we need more sophisticated techniques to show that the multiplication in \overline{H} behaves as we would like. To this end we show that \overline{H} is generated by a restricted set of elements $f_{i,\lambda}$, such that the corresponding $\mathbf{p}_{\lambda i}$ form a standard set of generators of H (regarded as a wreath product). We then check that the corresponding identities to determine $G \wr \mathcal{S}_r$ are satisfied by these generators, and it is then a short step to obtain our goal, namely, that $\overline{H} \cong H$ (Theorem 9.13). We note, however, that even at this stage more care is required than, for example, in the corresponding situation for \mathcal{T}_n [14] or \mathcal{PT}_n [3], since we cannot assume that G is finite. Indeed our particular choice of Schreier system will be seen to be a useful tool.

4. IDENTITY GENERATORS

As stated at the end of Section 3, our first step is to show that if $(i, \lambda) \in K$ and $\mathbf{p}_{\lambda i} = \varepsilon$, then $f_{i,\lambda} = 1_{\overline{H}}$. To this end we make use of our particular choice of Schreier system and function ω . The proof is by induction on $\lambda \in \Lambda$, where we recall that Λ is ordered lexicographically.

Lemma 4.1. *For any $(i, \lambda) \in K$ with $\mathbf{p}_{\lambda i} = \varepsilon$, we have $f_{i,\lambda} = 1_{\overline{H}}$.*

Proof. If $\mathbf{p}_{(1,2,\dots,r)i} = \varepsilon$, that is, $\mathbf{q}_{(1,2,\dots,r)\mathbf{r}_i} = \varepsilon$, then by definition of $\mathbf{q}_{(1,2,\dots,r)}$ we have $x_1\mathbf{r}_i = x_1, \dots, x_r\mathbf{r}_i = x_r$. Hence \mathbf{r}_i lies in district $(1, 2, \dots, r)$, so that $\omega(i) = (1, 2, \dots, r)$. Condition (R2) of the presentation \mathcal{P} now gives that $f_{i,(1,2,\dots,r)} = f_{i,\omega(i)} = 1_{\overline{H}}$.

Suppose now that $\mathbf{p}_{(u_1,u_2,\dots,u_r)i} = \varepsilon$ where $(1, 2, \dots, r) < (u_1, u_2, \dots, u_r)$. We make the inductive assumption that for any $(v_1, v_2, \dots, v_r) < (u_1, u_2, \dots, u_r)$, if $\mathbf{p}_{(v_1,v_2,\dots,v_r)l} = \varepsilon$, for any $l \in I$, then $f_{l,(v_1,v_2,\dots,v_r)} = 1_{\overline{H}}$.

With $u_0 = 0$, pick the largest number, say j , such that $u_j - u_{j-1} > 1$. By our choice of Schreier words, we have

$$\mathbf{h}_{(u_1,u_2,\dots,u_r)} = \mathbf{h}_{(u_1,u_2,\dots,u_{j-1},u_j-1,u_{j+1},\dots,u_r)}\alpha_{(u_1,u_2,\dots,u_r)},$$

where $\alpha_{(u_1,u_2,\dots,u_r)} = \varepsilon l_{(u_1,u_2,\dots,u_r)}$.

By definition,

$$\mathbf{r}_l = \begin{pmatrix} x_1 & \dots & x_{u_1} & x_{u_1+1} & \dots & x_{u_2} & \dots & x_{u_{r-1}+1} & \dots & x_{u_r} & x_{u_r+1} & \dots & x_n \\ x_1 & \dots & x_1 & x_2 & \dots & x_2 & \dots & x_r & \dots & x_r & x_r & \dots & x_r \end{pmatrix}.$$

By choice of j we have $u_{j-1} < u_j - 1 < u_j$ so that $x_{u_{j-1}}\mathbf{r}_l = x_j$, giving

$$\mathbf{p}_{(u_1,u_2,\dots,u_{j-1},u_j-1,u_{j+1},\dots,u_r)l} = \varepsilon.$$

Since

$$(u_1, u_2, \dots, u_{j-1}, u_j - 1, u_{j+1}, \dots, u_r) < (u_1, u_2, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_r),$$

we call upon our inductive hypothesis to obtain $f_{l,(u_1,u_2,\dots,u_{j-1},u_j-1,u_{j+1},\dots,u_r)} = 1_{\overline{H}}$. On the other hand, we have $f_{l,(u_1,u_2,\dots,u_r)} = f_{l,(u_1,u_2,\dots,u_{j-1},u_j-1,u_{j+1},\dots,u_r)}$ by (R1), and so we conclude that $f_{l,(u_1,u_2,\dots,u_r)} = 1_{\overline{H}}$.

Suppose that \mathbf{r}_i lies in district (l_1, l_2, \dots, l_r) . Since $\mathbf{q}_{(u_1,u_2,\dots,u_r)\mathbf{r}_i} = \varepsilon$, we have $x_{u_k}\mathbf{r}_i = x_k$, so that $l_k \leq u_k$ by the definition of districts, for all $k \in [1, r]$. If $l_k = u_k$ for all $k \in [1, r]$, then $f_{i,(u_1,\dots,u_r)} = f_{i,\omega(i)} = 1_{\overline{H}}$ by \mathcal{P} . Otherwise, we let m be smallest such that $l_m < u_m$ and so (putting $u_0 = l_0 = 0$) we have $u_{m-1} = l_{m-1} < l_m < u_m$. Clearly $(u_1, u_2, \dots, u_{m-1}, l_m, u_{m+1}, \dots, u_r) \in \Lambda$ and as $u_{m-1} < l_m < u_m$, we have $x_{l_m}\mathbf{r}_l = x_m$ by the definition of \mathbf{r}_l . We thus have the matrix equality

$$\begin{pmatrix} \mathbf{q}_{(u_1,u_2,\dots,u_r)\mathbf{r}_l} & \mathbf{q}_{(u_1,u_2,\dots,u_r)\mathbf{r}_i} \\ \mathbf{q}_{(u_1,u_2,\dots,u_{m-1},l_m,u_{m+1},\dots,u_r)\mathbf{r}_l} & \mathbf{q}_{(u_1,u_2,\dots,u_{m-1},l_m,u_{m+1},\dots,u_r)\mathbf{r}_i} \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Remember that we have already proven $f_{l,(u_1,u_2,\dots,u_r)} = 1_{\overline{H}}$. Furthermore, as $l_m < u_m$ by assumption,

$$(u_1, u_2, \dots, u_{m-1}, l_m, u_{m+1}, \dots, u_r) < (u_1, u_2, \dots, u_{m-1}, u_m, u_{m+1}, \dots, u_r),$$

so that induction gives that $f_{l,(u_1,u_2,\dots,u_{m-1},l_m,u_{m+1},\dots,u_r)} = f_{l,(u_1,u_2,\dots,u_{m-1},u_m,u_{m+1},\dots,u_r)} = 1_{\overline{H}}$. From (R3) we deduce that $f_{i,(u_1,u_2,\dots,u_r)} = 1_{\overline{H}}$ and the proof is completed. \square

5. GENERATORS CORRESPONDING TO THE SAME COLUMNS OR ROWS, AND CONNECTIVITY

The first aim of this section is to show that if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} \neq 0$ where $\lambda = \mu$ or $i = j$, then $f_{i,\lambda} = f_{j,\mu}$. We begin with the more straightforward case, where $i = j$.

Lemma 5.1. *If $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu i}$, then $f_{i,\lambda} = f_{i,\mu}$.*

Proof. Let $\lambda = (u_1, \dots, u_r)$ and $\mu = (v_1, \dots, v_r)$. By hypothesis we have that $\mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_i = \mathbf{q}_{(v_1, \dots, v_r)} \mathbf{r}_i = \psi \in H$. By definition of the $\mathbf{q}_{\lambda s}$ we have $x_{u_j} \mathbf{r}_i = x_{v_j} \mathbf{r}_i$ for $1 \leq j \leq r$, and as $\text{rank } \mathbf{r}_i = r$ it follows that $u_j, v_j \in B_{j'}^{\mathbf{r}_i}$ where $j \mapsto j'$ is a bijection of $[1, r]$. We now define $\alpha \in \text{End } F_n(G)$ by setting $x_{u_j} \alpha = x_j = x_{v_j} \alpha$ for all $j \in [1, r]$ and $x_p \alpha = x_1$ for all $p \in [1, n] \setminus \{u_1, \dots, u_r, v_1, \dots, v_r\}$.

Clearly $\alpha \in D_r$, indeed $\alpha \in L_1$. Since $w_m^\alpha = 1_G$ for all $m \in [1, n]$ and $\min\{u_j, v_j\} < \min\{u_k, v_k\}$ for $1 \leq j < k \leq r$, we certainly have that $\alpha = \mathbf{r}_l$ for some $l \in I$. By our choice of \mathbf{r}_l we have the matrix equality

$$\begin{pmatrix} \mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_i & \mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_l \\ \mathbf{q}_{(v_1, \dots, v_r)} \mathbf{r}_i & \mathbf{q}_{(v_1, \dots, v_r)} \mathbf{r}_l \end{pmatrix} = \begin{pmatrix} \psi & \varepsilon \\ \psi & \varepsilon \end{pmatrix}.$$

Using Lemma 4.1 and (R3) of the presentation \mathcal{P} , we obtain $f_{i,(u_1, \dots, u_r)} = f_{i,(v_1, \dots, v_r)}$ as required. \square

We need more effort for the case $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$. For this purpose we introduce the following notions of ‘bad’ and ‘good’ elements.

For any $i, j \in I$, suppose that \mathbf{r}_i and \mathbf{r}_j lie in districts $(1, k_2, \dots, k_r)$ and $(1, l_2, \dots, l_r)$, respectively. We call $u \in [1, n]$ a mutually *bad* element of \mathbf{r}_i with respect to \mathbf{r}_j , if there exist $m, s \in [1, r]$ such that $u = k_m = l_s$, but $m \neq s$; all other elements are said to be mutually *good* with respect to \mathbf{r}_i and \mathbf{r}_j . We call u a bad element of \mathbf{r}_i with respect to \mathbf{r}_j because, from the definition of districts, \mathbf{r}_i maps x_{k_m} to x_m , and similarly, \mathbf{r}_j maps x_{l_s} to x_s . Hence, if $u = k_m = l_s$ is bad, then it is impossible for us to find some \mathbf{r}_t to make both \mathbf{r}_i and \mathbf{r}_j ‘happy’ in the point x_u , that is, for \mathbf{r}_t (or, indeed, any other element of $\text{End } F_n(G)$) to agree with both \mathbf{r}_i and \mathbf{r}_j on x_u .

Notice that if m is the minimum subscript such that $u = k_m$ is a bad element of \mathbf{r}_i with respect to \mathbf{r}_j and $k_m = l_s$, then s is also the minimum subscript such that l_s is a bad element of \mathbf{r}_j with respect to \mathbf{r}_i . For, if $l_{s'} < l_s$ is a bad element of \mathbf{r}_j with respect to \mathbf{r}_i , then by definition we have some $k_{m'}$ such that $l_{s'} = k_{m'}$ where $s' \neq m'$. By the minimality of m , we have $m' > m$ and so $l_{s'} = k_{m'} > k_m = l_s$, a contradiction. We also remark that since $l_1 = k_1 = 1$, the maximum possible number of bad elements is $r - 1$.

Let us run a simple example. Let $n = 7$ and $r = 4$, and suppose \mathbf{r}_i lies in district $(1, 3, 4, 6)$ and \mathbf{r}_j lies in district $(1, 4, 6, 7)$. By definition, $x_4 \mathbf{r}_i = x_3$ and $x_6 \mathbf{r}_i = x_4$, while $x_4 \mathbf{r}_j = x_2$ and $x_6 \mathbf{r}_j = x_3$. Therefore, \mathbf{r}_i and \mathbf{r}_j differ on x_4 and x_6 , so that we say 4 and 6 are bad elements of \mathbf{r}_i with respect to \mathbf{r}_j .

Lemma 5.2. *For any $i, j \in I$, suppose that \mathbf{r}_i and \mathbf{r}_j lie in districts $(1, k_2, \dots, k_r)$ and $(1, l_2, \dots, l_r)$, respectively. Let $\mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_i = \mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_j = \psi \in H$. Suppose $\{1, l_2, \dots, l_s\}$ is a set good elements of \mathbf{r}_i with respect to \mathbf{r}_j such that $1 < l_2 < \dots < l_s < k_{s+1} < \dots < k_r$. Then*

there exists $p \in I$ such that \mathbf{r}_p lies in district $(1, l_2, \dots, l_s, k_{s+1}, \dots, k_r)$, $\mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_p = \psi$ and $f_{p, (u_1, \dots, u_r)} = f_{i, (u_1, \dots, u_r)}$. Further, if $s = r$ then we can take $p = j$.

Proof. We begin by defining $\alpha \in D_r$, starting by setting $x_{k_m} \alpha = x_m$, $m \in [1, r]$. Now for $m \in [1, s]$ we put $x_{l_m} \alpha = x_m$. Notice that for $1 \leq m \leq s$, if $k_{m'} = l_m$ for $m' \in [1, r]$, then by the goodness of $\{1, l_2, \dots, l_s\}$ we have that $m' = m$. We now set $x_{u_m} \alpha = x_{u_m} \mathbf{r}_i$ for $m \in [1, r]$. Again, we need to check we are not violating well-definedness. Clearly we need only check the case where $u_m = l_{m'}$ for some $m' \in [1, s]$, since here we have already defined $x_{l_{m'}} \alpha = x_{m'}$. We now use the fact that by our hypothesis, $x_{u_m} \mathbf{r}_i = x_{u_m} \mathbf{r}_j$ for all $m \in [1, r]$, so that $x_{u_m} \mathbf{r}_i = x_{u_m} \mathbf{r}_j = x_{l_{m'}} \mathbf{r}_j = x_{m'}$. Finally, we set $x_m \alpha = x_1$, for all $m \in [1, n] \setminus \{1, l_1, \dots, l_s, k_2, \dots, k_r, u_1, \dots, u_r\}$.

We claim that $\alpha = \mathbf{r}_t$ for some $t \in I$. First, it is clear from the definition that $\alpha \in D_r$, indeed, $\alpha \in L_1$. We also have that for $1 \leq m \leq s$, $x_{l_m} \alpha = x_{k_m} \alpha = x_m$ and also for $s+1 \leq m \leq r$, $x_{k_m} \alpha = x_m$. We claim that for $m \in [1, s]$ we have $l_m^\alpha = v_m$ where $v_m = \min \{k_m, l_m\}$ and for $m \in [s+1, n]$ we have $l_m^\alpha = k_m$. It is clear that $1 = l_1^\alpha$. Suppose that for $m \in [2, r]$ we have $x_{u_q} \alpha = ax_m$. By definition, $x_{u_q} \mathbf{r}_i = ax_m = x_{u_q} \mathbf{r}_j$, so that $k_m, l_m \leq u_q$ and our claim holds. It is now clear that $\alpha = \mathbf{r}_t$ for some $t \in I$ and lies in district $(v_1, \dots, v_s, k_{s+1}, \dots, k_r)$.

Having constructed \mathbf{r}_t , it is immediate that

$$\begin{pmatrix} \mathbf{q}_{(1, k_2, \dots, k_r)} \mathbf{r}_i & \mathbf{q}_{(1, k_2, \dots, k_r)} \mathbf{r}_t \\ \mathbf{q}_{(u_1, u_2, \dots, u_r)} \mathbf{r}_i & \mathbf{q}_{(u_1, u_2, \dots, u_r)} \mathbf{r}_t \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \psi & \psi \end{pmatrix},$$

so that in view of Corollary 3.6 and (R3) we deduce that $f_{i, (u_1, u_2, \dots, u_r)} = f_{t, (u_1, u_2, \dots, u_r)}$.

Notice now that if $s = r$ then

$$\begin{pmatrix} \mathbf{q}_{(1, l_2, \dots, l_r)} \mathbf{r}_t & \mathbf{q}_{(1, l_2, \dots, l_r)} \mathbf{r}_j \\ \mathbf{q}_{(u_1, u_2, \dots, u_r)} \mathbf{r}_t & \mathbf{q}_{(u_1, u_2, \dots, u_r)} \mathbf{r}_j \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \psi & \psi \end{pmatrix},$$

which leads to $f_{t, (u_1, u_2, \dots, u_r)} = f_{j, (u_1, u_2, \dots, u_r)}$, and so that $f_{j, (u_1, \dots, u_r)} = f_{i, (u_1, \dots, u_r)}$ as required.

Without the assumption that $s = r$, we now define \mathbf{r}_p in a similar, but slightly more straightforward way, to \mathbf{r}_t . Namely, we first define $\beta \in \text{End } F_n(G)$ by putting $x_{l_m} \beta = x_m$ for $m \in [1, s]$, $x_{k_m} \beta = x_m$ for $m \in [s+1, r]$, $x_{u_m} \beta = x_{u_m} \mathbf{r}_i$ for $m \in [1, r]$ and $x_m \beta = x_1$ for $m \in [1, n] \setminus \{1, l_1, \dots, l_s, k_{s+1}, \dots, k_r, u_1, \dots, u_r\}$. It is easy to check that $\beta = \mathbf{r}_p$ where \mathbf{r}_p lies in district $(1, l_2, \dots, l_s, k_{s+1}, \dots, k_r)$. Moreover, we have

$$\begin{pmatrix} \mathbf{q}_{(1, l_2, \dots, l_s, k_{s+1}, \dots, k_r)} \mathbf{r}_t & \mathbf{q}_{(1, l_2, \dots, l_s, k_{s+1}, \dots, k_r)} \mathbf{r}_p \\ \mathbf{q}_{(u_1, u_2, \dots, u_r)} \mathbf{r}_t & \mathbf{q}_{(u_1, u_2, \dots, u_r)} \mathbf{r}_p \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \psi & \psi \end{pmatrix},$$

which leads to $f_{t, (u_1, u_2, \dots, u_r)} = f_{p, (u_1, u_2, \dots, u_r)}$, and so to $f_{p, (u_1, \dots, u_r)} = f_{i, (u_1, \dots, u_r)}$ as required. \square

Lemma 5.3. *If $\mathbf{p}_{\lambda_i} = \mathbf{p}_{\lambda_j}$, then $f_{i, \lambda} = f_{j, \lambda}$.*

Proof. Suppose that \mathbf{r}_i and \mathbf{r}_j lie in districts $(1, k_2, \dots, k_r)$ and $(1, l_2, \dots, l_r)$, respectively. Let $\lambda = (u_1, \dots, u_r)$ so that $\mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_i = \mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_j = \psi \in H$ say. We proceed by induction on the number of mutually bad elements. If this is 0, then the result holds by Lemma 5.2. We make the inductive assumption that if $\mathbf{p}_{\lambda_l} = \mathbf{p}_{\lambda_t}$ and $\mathbf{r}_l, \mathbf{r}_t$ have $k-1$ bad elements, where $0 < k \leq r-1$, then $f_{l, \lambda} = f_{t, \lambda}$.

Suppose now that \mathbf{r}_j has k bad elements with respect to \mathbf{r}_i . Let s be the smallest subscript such that l_s is bad element of \mathbf{r}_j with respect to \mathbf{r}_i . Then there exists some m such that $l_s = k_m$. Note, m is also the smallest subscript such that k_m is bad, as we explained before. Certainly $s, m > 1$; without loss of generality, assume $s > m$. Then $1 = l_1, l_2, \dots, l_{s-1}$ are all good elements and $1 < l_2 < \dots < l_{s-1} < k_s < \dots < k_r$. By Lemma 5.2, there exists $p \in I$ such that \mathbf{r}_p lies in district $(1, l_2, \dots, l_{s-1}, k_s, \dots, k_r)$, $\mathbf{q}_{(u_1, \dots, u_r)} \mathbf{r}_p = \psi$ and $f_{p, (u_1, \dots, u_r)} = f_{i, (u_1, \dots, u_r)}$.

We consider the sets B and C of mutually bad elements of \mathbf{r}_j and \mathbf{r}_p , and of \mathbf{r}_j and \mathbf{r}_i , respectively. Clearly $B \subseteq \{l_s, l_{s+1}, \dots, l_r\}$. We have $l_s = k_m < k_s$, so that $l_s \notin B$. On the other hand if $l_q \in B$ where $s+1 \leq q \leq r$, then we must have $l_q = k_{q'}$ for some $q' \geq s$ with $q' \neq q$, so that $l_q \in C$. Thus $|B| < |C|$. Our inductive hypothesis now gives that $f_{p, (u_1, \dots, u_r)} = f_{j, (u_1, \dots, u_r)}$ and we deduce that $f_{i, (u_1, \dots, u_r)} = f_{j, (u_1, \dots, u_r)}$ as required. \square

Definition 5.4. Let $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$. We say that $(i, \lambda), (j, \mu)$ are *connected* if there exist

$$i = i_0, i_1, \dots, i_m = j \in I \text{ and } \lambda = \lambda_0, \lambda_1, \dots, \lambda_m = \mu \in \Lambda$$

such that for $0 \leq k < m$ we have $\mathbf{p}_{\lambda_k i_k} = \mathbf{p}_{\lambda_k, i_{k+1}} = \mathbf{p}_{\lambda_{k+1} i_{k+1}}$.

The following picture illustrates that $(i, \lambda) = (i_0, \lambda_0)$ is connected to $(j, \mu) = (i_m, \lambda_m)$:

$$\begin{array}{ccccccc} \mathbf{p}_{\lambda_0 i_0} & \text{---} & \mathbf{p}_{\lambda_1 i_1} & & & & \\ & & | & & & & \\ & & \mathbf{p}_{\lambda_1 i_1} & \text{---} & \mathbf{p}_{\lambda_1 i_2} & & \\ & & & & | & & \\ & & & & \text{---} & & \\ & & & & | & & \\ & & & & \mathbf{p}_{\lambda_{m-1} i_{m-1}} & \text{---} & \mathbf{p}_{\lambda_{m-1} i_m} \\ & & & & & & | \\ & & & & & & \mathbf{p}_{\lambda_m i_m} \end{array}$$

Lemmas 5.1 and 5.3 now yield:

Corollary 5.5. Let $i, j \in I$ and $\lambda, \mu \in \Lambda$ be such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ where $(i, \lambda), (j, \mu)$ are connected. Then $f_{i, \lambda} = f_{j, \mu}$.

6. THE RESULT FOR RESTRICTED r

We are now in a position to finish the proof of our first main result, Theorem 6.3, in a relatively straightforward way. Of course, in view of Theorem 9.13, it is not strictly necessary to provide such a proof here. However, the techniques used will be useful in the remainder of the paper.

Let $\alpha = \mathbf{p}_{\lambda i} \in P$ and suppose that $\lambda = (u_1, \dots, u_r)$ and \mathbf{r}_i lies in district (l_1, \dots, l_r) . Define $U(\lambda, i) = \{l_1, \dots, l_r, u_1, \dots, u_r\}$ and $S(\lambda, i) = [1, n] \setminus U$.

Step D: moving l_s down: Suppose that $l_j < t < l_{j+1}$ and $t \in S(\lambda, i)$. Define \mathbf{r}_k by

$$x_t \mathbf{r}_k = x_{j+1} \text{ and } x_s \mathbf{r}_j = x_s \mathbf{r}_i \text{ for } s \neq t.$$

It is easy to see that $\mathbf{r}_k \in \Theta$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda k}$ and \mathbf{r}_k lies in district $(l_1, \dots, l_j, t, l_{j+2}, \dots, l_r)$. Clearly, (i, λ) is connected to (k, λ) .

Step U: moving us up: Suppose that $u_j < t < u_{j+1}$ or $u_r < t$, where $t \in S(\lambda, i)$. Define \mathbf{r}_m by

$$x_t \mathbf{r}_m = x_{u_j} \mathbf{r}_i \text{ and } x_s \mathbf{r}_m = x_s \mathbf{r}_i \text{ for } s \neq t.$$

It is easy to see that $\mathbf{r}_m \in \Theta$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda m}$ and \mathbf{r}_m lies in district (l_1, l_2, \dots, l_r) . Let $\mu = (u_1, \dots, u_{j-1}, t, u_{j+1}, \dots, u_r)$. Clearly, $\mathbf{p}_{\lambda m} = \mathbf{p}_{\mu m}$ so that (i, λ) is connected to (m, μ) .

Step U': moving us down: Suppose that $t < u_{j+1}$ and $[t, u_{j+1}) \subseteq S(\lambda, i)$. Define \mathbf{r}_l by

$$x_t \mathbf{r}_l = x_{u_{j+1}} \mathbf{r}_i \text{ and } x_s \mathbf{r}_l = x_s \mathbf{r}_i \text{ for } s \neq t.$$

It is easy to see that $\mathbf{r}_l \in \Theta$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda l}$. Further, \mathbf{r}_l lies in district (l_1, l_2, \dots, l_r) unless $u_{j+1} = l_{(j+1)\bar{\alpha}}$, in which case $l_{(j+1)\bar{\alpha}}$ is replaced by t . Let $\mu = (u_1, \dots, u_j, t, u_{j+2}, \dots, u_r)$; clearly, $\mathbf{p}_{\lambda l} = \mathbf{p}_{\mu l}$, so that (i, λ) is connected to (l, μ) .

Lemma 6.1. *Suppose that $n \geq 2r + 1$. Let $\lambda = (u_1, \dots, u_r) \in \Lambda$, and $i \in I$ with $\mathbf{p}_{\lambda i} \in H$. Then (i, λ) is connected to (j, μ) for some $j \in I$ and $\mu = (n - r + 1, \dots, n)$. Consequently, if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\nu k}$ for any $i, k \in I$ and $\lambda, \nu \in \Lambda$, then $f_{i, \lambda} = f_{k, \nu}$.*

Proof. Suppose that \mathbf{r}_i lies in district (l_1, \dots, l_r) . For the purposes of this proof, let $W(\lambda, i) = \sum_{k=1}^r (u_k - l_k)$; clearly $W(\lambda, i)$ takes greatest value T where $(l_1, \dots, l_r) = (1, \dots, r)$ and $(u_1, \dots, u_r) = (n - r + 1, \dots, n)$. We verify our claim by finite induction, with starting point T , under the reverse of the usual ordering on \mathbb{N} . We have remarked that our result holds if $W(\lambda, i) = T$.

Suppose now that $W(\lambda, i) < T$ and the result is true for all pairs (ν, l) where $W(\lambda, i) < W(\nu, l) \leq T$.

If $u_r < n$, then as certainly $l_r \leq u_r \overline{\mathbf{p}_{\lambda i}^{-1}} \leq u_r$, we can apply Step U to show that (i, λ) is connected to (l, ν) where $\nu = (u_1, \dots, u_{r-1}, u_r + 1)$ and \mathbf{r}_l lies in district (l_1, \dots, l_r) . Clearly $W(\lambda, i) < W(\nu, l)$.

Suppose that $u_r = n$. We know that $l_1 = 1$, and by our hypothesis that $2r + 1 \leq n$, certainly $S(\lambda, i) \neq \emptyset$. If there exists $t \in S(\lambda, i)$ with $t < l_w$ for some $w \in [1, r]$, then choosing k with $l_k < t < l_{k+1}$, we have by Step D that (i, λ) is connected to (l, λ) , where \mathbf{r}_l lies in district $(l_1, \dots, l_k, t, l_{k+2}, \dots, l_r)$; clearly then $W(\lambda, i) < W(\lambda, l)$. On the other hand, if there exists $t \in S(\lambda, i)$ with $u_w < t$ for some $w \in [1, r]$, then now choosing $k \in [1, r]$ with $u_k < t < u_{k+1}$, we use Step U to show that (i, λ) is connected to (m, ν) where $\nu = (u_1, \dots, u_{k-1}, t, u_{k+1}, \dots, u_r)$, and \mathbf{r}_m lies in district (l_1, \dots, l_r) . Again, $W(\lambda, i) < W(\nu, m)$.

The only other possibility is that $S(\lambda, i) \subseteq (l_r, u_1)$, in which case, $W(\lambda, i) = T$, a contradiction. \square

In view of Lemma 6.1 we may define, for $r \leq \frac{n-1}{2}$ and $\phi \in H$, an element $f_\phi \in \overline{H}$, where $f_\phi = f_{\lambda, i}$ for some (any) $(i, \lambda) \in K$ with $\mathbf{p}_{\lambda i} = \phi$.

Lemma 6.2. *Let $r \leq n/3$. Then for any $\phi, \theta \in H$, we have $f_{\phi\theta} = f_\theta f_\phi$ and $f_{\phi^{-1}} = f_\phi^{-1}$.*

Proof. Since $n \geq 3$ and $r \leq n/3$ we deduce that $2r + 1 \leq n$. Define \mathbf{r}_i by

$$x_j \mathbf{r}_i = x_j, j \in [1, r]; x_j \mathbf{r}_i = x_{j-r} \phi \theta, j \in [r+1, 2r]; x_j \mathbf{r}_i = x_{j-2r} \theta, j \in [2r+1, 3r]$$

and

$$x_j \mathbf{r}_i = x_1, j \in [3r+1, n].$$

Clearly, $\mathbf{r}_i \in \Theta$ and \mathbf{r}_i lies in district $(1, \dots, r)$. Next we define \mathbf{r}_l by

$$x_j \mathbf{r}_l = x_j, j \in [1, r]; x_j \mathbf{r}_l = x_{j-r} \phi, j \in [r+1, 2r]; x_j \mathbf{r}_l = x_{j-2r}, j \in [2r+1, 3r];$$

and

$$x_j \mathbf{r}_l = x_1, j \in [3r+1, n].$$

Again, \mathbf{r}_l is well defined and lies in district $(1, \dots, r)$. By considering the submatrix

$$\begin{pmatrix} \mathbf{q}_{(r+1, \dots, 2r)} \mathbf{r}_l & \mathbf{q}_{(r+1, \dots, 2r)} \mathbf{r}_i \\ \mathbf{q}_{(2r+1, \dots, 3r)} \mathbf{r}_l & \mathbf{q}_{(2r+1, \dots, 3r)} \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \phi & \phi \theta \\ \varepsilon & \theta \end{pmatrix},$$

of P , Corollary 3.6 gives that $f_{i, (r+1, \dots, 2r)} = f_{i, (2r+1, \dots, 3r)} f_{l, (r+1, \dots, 2r)}$, which in our new notation says $f_{\phi \theta} = f_{\theta} f_{\phi}$, as required.

Finally, since $1_{\overline{H}} = f_{\varepsilon} = f_{\phi \phi^{-1}} = f_{\phi^{-1}} f_{\phi}$, we have $f_{\phi^{-1}} = f_{\phi}^{-1}$. \square

Theorem 6.3. *Let $r \leq n/3$. Then \overline{H} is isomorphic to H under ψ , where $f_{\phi} \psi = \phi^{-1}$.*

Proof. We have that $\overline{H} = \{f_{\phi} : \phi \in H\}$ by Lemma 6.2 and ψ is well defined, by Lemma 6.1. By Lemma 3.10, ψ is onto and it is a homomorphism by Lemma 6.2. Now $f_{\phi} \psi = \varepsilon$ means that $\phi = \varepsilon$, so that $f_{\phi} = 1_{\overline{H}}$ by Lemma 4.1. Consequently, ψ is an isomorphism as required. \square

7. NON-IDENTITY GENERATORS WITH SIMPLE FORM

First we explain the motivation for this section. It follows from Section 6 that for any r and n with $n \geq 2r + 1$, all entries in the sandwich matrix P are connected. However, this connectivity will fail for higher ranks. Hence, the aim here is to identify the connected entries in P in the case of higher rank. It turns out that entries with *simple form* are always connected. For the reason given in abstract, from now on we may assume that $1 \leq r \leq n - 2$.

We run an easy example to explain the lack of connectivity for $r \geq n/2$.

Let $n = 4$, $r = 2$, and

$$\alpha = \begin{pmatrix} x_1 & x_2 \\ ax_1 & bx_2 \end{pmatrix},$$

with $a, b \neq 1_G \in G$. It is clear from Lemma 3.10 that there exists $i \in I$, $\lambda \in \Lambda$ such that $\alpha = \mathbf{p}_{\lambda i} \in P$, in fact we can take

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & ax_1 & bx_2 \end{pmatrix}$$

and $\lambda = (3, 4)$.

How many copies of α occur in the sandwich matrix P ? Suppose that $\alpha = \mathbf{p}_{\mu j}$ where \mathbf{r}_j lies in district (l_1, l_2) and $\mu = (u_1, u_2)$. Since $\bar{\alpha}$ is the identity of \mathcal{S}_2 , and $a, b \neq 1_G$, we must have

$1 = l_1 < l_2$, $u_1 < u_2$, $l_1 < u_1$, $l_2 < u_2$ and $\{l_1, l_2\} \cap \{u_1, u_2\} = \emptyset$. Thus the only possibilities are

$$(l_1, l_2) = (1, 2), (u_1, u_2) = (3, 4) = \lambda$$

and

$$(l_1, l_2) = (1, 3), (u_1, u_2) = (2, 4) = \mu.$$

In the first case, $\alpha = \mathbf{p}_{\lambda i}$ and in the second, $\alpha = \mathbf{p}_{\mu j}$ where $\mathbf{r}_j = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & ax_1 & x_2 & bx_2 \end{pmatrix}$. Clearly then, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} \in H$ but (i, λ) is *not* connected to (j, μ) .

We know from Lemma 3.10, that in case $r \geq n/2$, not every element of H lies in P . However, we are guaranteed that for $r \leq n - 2$ certainly all elements with simple form

$$\phi = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-1} & x_{k+1} & x_{k+2} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \cdots & x_r \end{pmatrix},$$

where $k \geq 1, m \geq 0, a \in G$, lie in P . In particular, we can choose

$$\mathbf{r}_l = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & \cdots & x_{r+1} & x_{r+2} & \cdots & x_n \\ x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}$$

and $\mu = (1, \dots, k-1, k+1, \dots, r+1)$ to give $\mathbf{p}_{\mu l} = \mathbf{q}_{\mu} \mathbf{r}_l = \phi$. We now proceed to show that if $\mathbf{p}_{\lambda i} = \phi \neq \varepsilon$, then (i, λ) is connected to (j, μ) for some $j \in I$ and hence to (l, μ) .

Lemma 7.1. *Let $\varepsilon \neq \phi$ be as above and suppose that $\phi = \mathbf{p}_{\lambda i}$ where $\lambda = (u_1, \dots, u_r)$ and \mathbf{r}_i lies in district (l_1, \dots, l_r) . Then (i, λ) is connected to (j, μ) for some $j \in I$.*

Proof. Notice that as $\phi = \mathbf{p}_{\lambda i}$, we have $x_{u_k} r_i = x_k \phi$, so that $x_{u_k} r_i = x_{k+1}$ if $m > 0$, and so $u_k \geq l_{k+1} > l_k$ by Lemma 3.9; or if $m = 0$ and $a \neq 1_G$, $x_{u_k} r_i = ax_k$ so that $u_k > l_k$ by Lemma 3.9 again. Further, from the constraints on (l_1, \dots, l_r) it follows that

$$l_1 < l_2 < \cdots < l_{k-1} < l_k < u_k.$$

We first ensure that (i, λ) is connected to some (j, κ) where $\kappa = (1, \dots, k-1, u_k, \dots, u_r)$, by induction on $(l_1, \dots, l_{k-1}) \in [1, n]^r$ under the lexicographic order.

If $(l_1, \dots, l_{k-1}) = (1, \dots, k-1)$, then clearly $(i, \lambda) = (i, \kappa)$. Suppose now that $(l_1, \dots, l_{k-1}) > (1, \dots, k-1)$ and the result is true for all $(l'_1, \dots, l'_{k-1}) \in [1, n]^r$ where $(l'_1, \dots, l'_{k-1}) < (l_1, \dots, l_{k-1})$, namely, if $\mathbf{p}_{\eta l} = \phi$ with \mathbf{r}_l in district (l'_1, \dots, l'_r) , then (l, η) is connected to some (j, κ) .

By putting $\nu = (l_1, \dots, l_{k-1}, u_k, \dots, u_r)$ we have $\mathbf{p}_{\nu i} = \mathbf{p}_{\lambda i}$. Since we have $(l_1, \dots, l_{k-1}) > (1, \dots, k-1)$, there must be a $t \in (l_s, l_{s+1}) \cap S(\nu, i)$ for some $s \in [0, k-2]$, where $l_0 = 0$. We can use Step D to move l_{s+1} down to t , obtaining \mathbf{r}_p in district $(l_1, \dots, l_s, t, l_{s+2}, \dots, l_r)$ such that $\mathbf{p}_{\nu i} = \mathbf{p}_{\nu p}$. Clearly $(l_1, \dots, l_s, t, l_{s+2}, \dots, l_{k-1}) < (l_1, \dots, l_{k-1})$, so that by induction (p, ν) (and hence (i, λ)) is connected to some (j, κ) .

We now proceed via induction on $(u_k, \dots, u_r) \in [k+1, n]^r$ under the lexicographic order to show that (j, κ) is connected to some (l, μ) where $\mu = (1, \dots, k-1, k+1, \dots, r+1)$. Clearly, this is true for $(u_k, \dots, u_r) = (k+1, \dots, r+1)$.

Suppose that $(u_k, \dots, u_r) > (k+1, \dots, r+1)$, and the result is true for all $(v_k, \dots, v_r) \in [k+1, n]^r$ where $(v_k, \dots, v_r) < (u_1, \dots, u_r)$. Then we define \mathbf{r}_w by:

$$x_l \mathbf{r}_w = x_l, l \in [1, k], x_{u_l} \mathbf{r}_w = x_{u_l} \mathbf{r}_j, l \in [k, r] \text{ and } x_v \mathbf{r}_w = x_1 \text{ for all other } x_v.$$

It is easy to see that $\mathbf{r}_w \in \Theta$, \mathbf{r}_w lies in district $(1, 2, \dots, k, u_k, \dots, u_{k+m-1}, u_{k+m+1}, \dots, u_r)$ and $\mathbf{p}_{\kappa j} = \mathbf{p}_{\kappa w}$. There must be a $t < u_h$ for some $h \in [k, r]$ with $[t, u_h] \subseteq S(\kappa, w)$. By Step U', we have that (w, κ) is connected to (v, ρ) where $\rho = (1, \dots, k-1, u_k, \dots, u_{h-1}, t, u_{h+1}, \dots, u_r)$. Clearly, $(u_k, \dots, u_{h-1}, t, u_{h+1}, \dots, u_r) < (u_k, \dots, u_{h-1}, u_h, u_{h+1}, \dots, u_r)$, so that by induction (v, ρ) is connected to (l, μ) . The proof is completed. \square

The following corollary is immediate from Lemma 4.1, Corollary 5.5 and Lemma 7.1.

Corollary 7.2. *Let $\mathbf{p}_{\lambda i} = \mathbf{p}_{\nu k}$ have simple form. Then $f_{i,\lambda} = f_{k,\nu}$.*

8. NON-IDENTITY GENERATORS WITH ARBITRARY FORM

Our aim here is to prove that for any $\alpha \in H$, if $i, j \in I$ and $\lambda, \mu \in \Lambda$ with $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} = \alpha \in H$, then $f_{i,\lambda} = f_{j,\mu}$. This property of α is called *consistency*. Notice that Corollary 7.2 tells us that all elements with simple form are consistent.

Before we explain the strategy in this section, we run the following example by the reader, which shows that if $|G| > 1$, we cannot immediately separate an element $\alpha \in \overline{H}$ into a product, $\beta\gamma$ or $\gamma\beta$, where β is essentially an element of \mathcal{S}_r , and $\overline{\gamma}$ is the identity in \mathcal{S}_r .

Let $1_G \neq a$, $n = 6$ and $r = 4$, so that $\alpha = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & ax_2 & x_4 & x_1 \end{pmatrix} \in H$. By putting

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & ax_2 & x_4 & x_1 \end{pmatrix}$$

and $\lambda = (3, 4, 5, 6)$, clearly we have $\mathbf{p}_{\lambda i} = \alpha$.

Next we argue that $i \in I$ and $\lambda \in \Lambda$ are unique such that $\mathbf{p}_{\lambda i} = \alpha$. Let $\mu = (u_1, u_2, u_3, u_4)$ and \mathbf{r}_j lie in district (l_1, l_2, l_3, l_4) with $\mathbf{p}_{\mu j} = \alpha$; we show that $\mathbf{r}_j = \mathbf{r}_i$ and $\mu = \lambda$. Since $x_{u_1} \mathbf{r}_j = x_1 \alpha = x_3$ by assumption, we must have $l_1 < l_2 < l_3 \leq u_1$, so that $u_1 \geq 3$. As $3 \leq u_1 < u_2 < u_3 < u_4 \leq n = 6$, we have $\mu = (u_1, u_2, u_3, u_4) = (3, 4, 5, 6) = \lambda$, and $(l_1, l_2) = (1, 2)$. Clearly then $\mathbf{r}_j = \mathbf{r}_i$.

Certainly $\alpha = \gamma\beta = \beta\gamma$ with

$$\gamma = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_2 & x_4 & x_1 \end{pmatrix}, \beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & ax_2 & x_3 & x_4 \end{pmatrix}.$$

Our question is, can we find a sub-matrix of P with one of the following forms:

$$\begin{pmatrix} \gamma & \alpha \\ \varepsilon & \beta \end{pmatrix} \text{ or } \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}.$$

Clearly, here the answer is in the negative, as it is easy to see from the definition \mathbf{r}_i that there does not exist $\nu \in \Lambda$ with $\mathbf{p}_{\nu i} = \beta$ or $\mathbf{p}_{\nu i} = \gamma$.

Now it is time for us to explain our trick of how to split an arbitrary element α in H into a product of elements with simple form (defined in the previous section), and moreover, how this splitting matches the products of generators $f_{i,\lambda}$ in \overline{H} .

Our main strategy is as follows. We introduce a notion of ‘rising point’ of $\alpha \in H$. Now, given $\mathbf{p}_{\lambda_i} = \alpha$, we decompose α as a product $\alpha = \beta\gamma$ depending only on α such that γ is an element with simple form, $\beta = \mathbf{p}_{\lambda_j}$ has a lower rising point than α , $\gamma = \mathbf{p}_{\mu_i}$ for some $j \in I, \mu \in \Lambda$ such that our presentation gives $f_{i,\lambda} = f_{i,\mu}f_{j,\lambda}$.

Definition 8.1. Let $\alpha \in H$. We say that α has *rising point* $r + 1$ if $x_m\alpha = ax_r$ for some $m \in [1, r]$ and $a \neq 1_G$; otherwise, the rising point is $k \leq r$ if there exists a sequence

$$1 \leq i < j_1 < j_2 < \cdots < j_{r-k} \leq r$$

with

$$x_i\alpha = x_k, x_{j_1}\alpha = x_{k+1}, x_{j_2}\alpha = x_{k+2}, \cdots, x_{j_{r-k}}\alpha = x_r$$

and such that if $l \in [1, r]$ with $x_l\alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$.

It is easy to see that the only element with rising point 1 is the identity of H , and elements with rising point 2 have either of the following two forms:

- (i) $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ ax_1 & x_2 & \cdots & x_r \end{pmatrix}$, where $a \neq 1_G$;
- (ii) $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_r \\ x_2 & x_3 & \cdots & x_k & ax_1 & x_{k+1} & \cdots & x_r \end{pmatrix}$, where $k \geq 2$.

Note that both of the above two forms are the so called simple forms; however, elements with simple form can certainly have rising point greater than 2, indeed, it can be $r + 1$. From Lemma 4.1 and Corollary 7.2 we immediately deduce:

Corollary 8.2. *Let $\alpha \in H$ have rising point 1 or 2. Then α is consistent.*

Next, we will see how to decompose an element with a rising point at least 3 into a product of an element with a lower rising point and an element with simple form.

Lemma 8.3. *Let $\alpha \in H$ have rising point $k \geq 3$. Then α can be expressed as a product of some $\beta \in H$ with rising point no more than $k - 1$ and some $\gamma \in H$ with simple form.*

Proof. *Case (0)* By definition of rising point, if $k = r + 1$, then we have $x_m\alpha = ax_r$ for some $a \neq 1_G$ and $m \in [1, r]$. We define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{r-1} & x_r \\ x_1 & x_2 & \cdots & x_{r-1} & ax_r \end{pmatrix}$$

and β by $x_m\beta = x_r$ and for other $j \in [1, r]$, $x_j\beta = x_j\alpha$. Clearly, $\alpha = \beta\gamma$, γ is a simple form, and β has rising point no greater than r .

On the other hand, if $k \leq r$ there exists a sequence

$$1 \leq i < j_1 < j_2 \cdots < j_{r-k} \leq r$$

with

$$x_i\alpha = x_k, x_{j_1}\alpha = x_{k+1}, x_{j_2}\alpha = x_{k+2}, \cdots, x_{j_{r-k}}\alpha = x_r$$

such that if $l \in [1, r]$ with $x_l\alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$. We proceed by considering the following cases:

Case (i) If $l < i$, so that $a \neq 1_G$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & ax_{k-1} & x_k & \cdots & x_r \end{pmatrix}$$

and put $\beta = \alpha\gamma^{-1}$. It is easy to check that $x_l\beta = x_l\alpha\gamma^{-1} = x_{k-1}$ and $x_p\beta = x_p\alpha$, for other $p \in [1, r]$.

Case (ii) If $i < l < j_1$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & x_{k+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & x_k & ax_{k-1} & x_{k+1} & \cdots & x_r \end{pmatrix}$$

and again, we put $\beta = \alpha\gamma^{-1}$. By easy calculation we have

$$x_i\beta = x_{k-1}, x_l\beta = x_k, x_{j_1}\beta = x_{k+1}, \dots, x_{j_{r-k}}\beta = x_r$$

and for other $p \in [1, r]$, $x_p\beta = x_p\alpha$.

Case (iii) If $j_{r-k} < l$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & x_{k+1} & \cdots & x_{r-1} & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & x_k & x_{k+1} & x_{k+2} & \cdots & x_r & ax_{k-1} \end{pmatrix}$$

and again, we define $\beta = \alpha\gamma^{-1}$. It is easy to see that

$$x_i\beta = x_{k-1}, x_{j_1}\beta = x_k, x_{j_2}\beta = x_{k+1}, \dots, x_{j_{r-k}}\beta = x_{r-1}, x_l\beta = x_r$$

and for other $p \in [1, r]$, $x_p\beta = x_p\alpha$.

Case (iv) If $j_u < l < j_{u+1}$ for some $u \in [1, r-k-1]$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & \cdots & x_{k+u-1} & x_{k+u} & x_{k+u+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & x_k & x_{k+1} & \cdots & x_{k+u} & ax_{k-1} & x_{k+u+1} & \cdots & x_r \end{pmatrix}$$

and again, we put $\beta = \alpha\gamma^{-1}$. Then we have

$$x_i\beta = x_{k-1}, x_{j_1}\beta = x_k, \dots, x_{j_u}\beta = x_{k+u-1}, x_l\beta = x_{k+u}, x_{j_{u+1}}\beta = x_{k+u+1}, \dots, x_{j_{r-k}}\beta = x_r$$

and for other $p \in [1, r]$, $x_p\beta = x_p\alpha$.

In each of Cases (i) – (iv) it is clear that γ has simple form, $\alpha = \beta\gamma$ and β has a rising point no more than $k-1$. The proof is completed. \square

Note that in each of Cases (ii) – (iv) of Lemma 8.3, that is, where $i < l$, we have $x_p\beta = x_p\alpha$ for all $p < i$.

Lemma 8.4. *Let $\alpha, \beta, \gamma \in H$ with $\alpha = \beta\gamma$ and β, γ consistent. Suppose that whenever $\alpha = \mathbf{p}_{\lambda j}$, we can find $(t, \lambda), (j, \mu) \in K$ with $\beta = \mathbf{p}_{\lambda t}, \gamma = \mathbf{p}_{\mu j}$ and $f_{j, \lambda} = f_{j, \mu} f_{t, \lambda}$. Then α is consistent.*

Proof. Let α, β, γ satisfy the hypotheses of the lemma. If $\alpha = \mathbf{p}_{\lambda j} = \mathbf{p}_{\lambda' j'}$, then by assumption we can find $(t, \lambda), (j, \mu), (t', \lambda'), (j', \mu') \in K$ with $\beta = \mathbf{p}_{\lambda t} = \mathbf{p}_{\lambda' t'}, \gamma = \mathbf{p}_{\mu j} = \mathbf{p}_{\mu' j'}, f_{j, \lambda} = f_{j, \mu} f_{t, \lambda}$ and $f_{j', \lambda'} = f_{j', \mu'} f_{t', \lambda'}$. The result now follows from the consistency of β and γ . \square

Proposition 8.5. *Every $\alpha \in P$ is consistent. Further, if $\alpha = \mathbf{p}_{\lambda_j}$ then $f_{j,\lambda}$ is equal in \overline{H} to a product $f_{i_1,\lambda_1} \cdots f_{i_k,\lambda_k}$, where $\mathbf{p}_{\lambda_t, i_t}$ is an element with simple form, $t \in [1, k]$.*

Proof. We proceed by induction on the rising point of α . If α has rising point 1 or 2, and $\mathbf{p}_{\lambda_i} = \alpha$, then the result is true by Corollary 8.2 and the comments preceding it. Suppose for induction that the rising point of α is $k \geq 3$, and the result is true for all $\beta \in H$ with rising point strictly less than k and all $f_{i,\mu} \in F$ where $\mathbf{p}_{\mu i} = \beta$.

We proceed on a case by case basis, using γ and β as defined in Lemma 8.3. Since γ has simple form, it is consistent by Corollary 7.2 and as β has rising point strictly less than k , β is consistent by our inductive hypothesis.

Suppose that $\alpha = \mathbf{p}_{\lambda_j}$ where $\lambda = (u_1, \dots, u_r)$ and \mathbf{r}_j lies in district (l_1, \dots, l_r) .

Case (0) If $k = r + 1$, then we have $x_m \alpha = ax_r$ for some $a \neq 1_G$. We now define \mathbf{r}_t by $x_{u_m} \mathbf{r}_t = x_r$ and $x_s \mathbf{r}_t = x_s \mathbf{r}_j$, for other $s \in [1, n]$. As $x_{u_m} \mathbf{r}_j = ax_r$, it is easy to see that $\mathbf{r}_t \in \Theta$. Notice that $l_{r-1} < l_r = l_{m\bar{\alpha}} < u_m$. Then by setting $\mu = (1, l_2, \dots, l_{r-1}, u_m)$ we have

$$\begin{pmatrix} \mathbf{p}_{\lambda t} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu t} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

and our presentation gives $f_{j,\lambda} = f_{j,\mu} f_{t,\lambda}$.

We now suppose that $k \leq r$. By definition of rising point there exists a sequence

$$1 \leq i < j_1 < j_2 \cdots < j_{r-k} \leq r$$

such that

$$x_i \alpha = x_k, x_{j_1} \alpha = x_{k+1}, x_{j_2} \alpha = x_{k+2}, \dots, x_{j_{r-k}} \alpha = x_r$$

such that if $l \in [1, r]$ with $x_l \alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$.

We consider the following cases:

Case (i) If $l < i$ we define \mathbf{r}_t by $x_{u_l} \mathbf{r}_t = x_{k-1}$ and for other $p \in [1, n]$, $x_p \mathbf{r}_t = x_p \mathbf{r}_j$. As by assumption $x_{u_l} \mathbf{r}_j = x_l \alpha = ax_{k-1}$, clearly $\mathbf{r}_t \in \Theta$. Then by putting

$$\mu = (1, l_2, \dots, l_{k-2}, u_l, u_i, u_{j_1}, \dots, u_{j_{r-k}})$$

we have

$$\begin{pmatrix} \mathbf{p}_{\lambda t} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu t} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

which implies $f_{j,\lambda} = f_{j,\mu} f_{t,\lambda}$.

Case (ii) If $i < l < j_1$ we define \mathbf{r}_s by

$$x_p \mathbf{r}_s = x_p \mathbf{r}_j \text{ for } p < u_i, x_{u_w} \mathbf{r}_s = x_w \beta \text{ for } i \leq w \leq r \text{ and } x_v \mathbf{r}_s = x_1 \text{ for all other } v \in [1, n].$$

We must argue that $\mathbf{r}_s \in \Theta$. Note that from the comment following Lemma 8.3, for any $v < i$ we have that

$$x_{u_v} \mathbf{r}_s = x_{u_v} \mathbf{r}_j = x_v \alpha = x_v \beta,$$

so that in particular, $\text{rank } \mathbf{r}_s = r$. Further,

$$x_{u_i} \mathbf{r}_s = x_i \beta = x_{k-1}, x_{u_l} \mathbf{r}_s = x_l \beta = x_k, x_{u_{j_1}} \mathbf{r}_s = x_{j_1} \beta = x_{k+1}, \dots, x_{u_{j_{r-k}}} \mathbf{r}_s = x_{j_{r-k}} \beta = x_r$$

so that

$$\langle x_{u_i}, x_{u_l}, x_{u_{j_1}}, \dots, x_{u_{j_{r-k}}} \rangle_{\mathbf{r}_s} = \langle x_{k-1}, x_k, \dots, x_r \rangle.$$

Thus for any $v \neq \{i, l, j_1, \dots, j_{r-k}\}$, $x_{u_v} \mathbf{r}_s = x_v \beta \in \langle x_1, \dots, x_{k-2} \rangle$.

As $x_{u_i} \mathbf{r}_j = x_k$, we have $1 = l_1 < l_2 < \dots < l_{k-1} < l_k \leq u_i$. Let h be the largest number with

$$1 = l_1 < l_2 < \dots < l_{k-1} < l_k < l_{k+1} < \dots < l_{(k-1)+h} < u_i.$$

Clearly here we have $h \in [0, r - k + 1]$. Now we claim that $\mathbf{r}_s \in \Theta$ and lies in district

$$(l_1, l_2, \dots, l_{(k-1)+h}, u_{j_h}, u_{j_{h+1}}, \dots, u_{j_{r-k}}).$$

To simplify our notation we put

$$(l_1, l_2, \dots, l_{(k-1)+h}, u_{j_h}, u_{j_{h+1}}, \dots, u_{j_{r-k}}) = (z_1, z_2, \dots, z_{(k-1)+h}, z_{k+h}, \dots, z_r),$$

where $j_0 = l$. Clearly, by the definition of \mathbf{r}_s , we have $x_{z_v} \mathbf{r}_s = x_v$ for all $v \in [1, r]$. Hence, to show $\mathbf{r}_s \in \Theta$, by the definition we only need to argue that for any $m \in [1, n]$ and $b \in G$, $x_m \mathbf{r}_s = bx_t$ implies $m \geq z_t$.

Suppose that $t \in [1, (k-1) + h]$, so that $z_t = l_t < u_i$. If $m < z_t$, then from the definition of \mathbf{r}_s we have $x_m \mathbf{r}_s = x_m \mathbf{r}_j$, so that $x_m \mathbf{r}_j = bx_t$. As $\mathbf{r}_j \in \Theta$ and $x_{l_t} \mathbf{r}_j = x_t$, we have $z_t = l_t \leq m$, a contradiction, and we deduce that $m \geq z_t$.

Suppose now that $t \in [k+h, r]$. Note that $m \geq u_i$; because, if $m < u_i$, then $x_m \mathbf{r}_j = x_m \mathbf{r}_s = bx_t$. As $\mathbf{r}_j \in \Theta$, $l_t \leq m < u_i$ and so $t \leq (k-1) + h$, a contradiction. Thus $m \geq u_i$. Now, by the definition of \mathbf{r}_s , we know there is exactly one possibility that $x_m \mathbf{r}_s = bx_t$ with $t \in [k+h, r]$, that is, $x_{z_t} \mathbf{r}_s = x_t$, so that $m = z_t$ and $b = 1$. Thus $\mathbf{r}_s \in \Theta$.

Now set

$$\eta = (1, l_2, \dots, l_{k-2}, u_i, u_l, u_{j_1}, \dots, u_{j_{r-k}})$$

then we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\eta s} & \mathbf{p}_{\eta j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix},$$

which implies $f_{j,\lambda} = f_{j,\eta} f_{s,\lambda}$.

Case (iii) If $j_{r-k} < l$, then, defining \mathbf{r}_s as in *Case (ii)*, a similar argument gives that $\mathbf{r}_s \in \Theta$ and $x_{u_v} \mathbf{r}_s = x_v \beta$ for all $v \in [1, r]$ (of course here β is defined differently to that given in *Case (ii)* and the district of \mathbf{r}_s will have a different appearance.). Moreover, by setting

$$\delta = (1, l_2, \dots, l_{k-2}, u_i, u_{j_1}, \dots, u_{j_{r-k}}, u_l)$$

we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\delta s} & \mathbf{p}_{\delta j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}.$$

implying $f_{j,\lambda} = f_{j,\delta} f_{s,\lambda}$.

Case (iv) If $j_u < l < j_{u+1}$ for some $u \in [1, r - k - 1]$, then again by defining \mathbf{r}_s as in *Case (ii)*, we have $\mathbf{r}_s \in \Theta$ and $x_{u_v} \mathbf{r}_s = x_v \beta$ for all $v \in [1, r]$. Take

$$\sigma = (1, l_2, \dots, l_{k-2}, u_i, u_{j_1}, u_{j_u}, u_l, u_{j_{u+1}}, \dots, u_{j_{r-k}}).$$

Then we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\sigma s} & \mathbf{p}_{\sigma j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

so that $f_{j,\lambda} = f_{j,\sigma}f_{s,\lambda}$.

In each of the cases above, the consistency of α follows from Lemma 8.4. The result now follows by induction. \square

In view of Lemma 8.4, we can now denote all generators $f_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \alpha$ by f_α , where $(i, \lambda) \in K$.

9. THE MAIN THEOREM

Our eventual aim is to show that $\overline{\overline{H}}$ is isomorphic to H and hence to the wreath product $G \wr \mathcal{S}_r$. With this in mind, given the knowledge we have gathered concerning the generators $f_{i,\lambda}$, we first specialise the general presentation given in Theorem 3.3 to our specific situation.

We will say that for $\phi, \varphi, \psi, \sigma \in P$ the quadruple $(\phi, \varphi, \psi, \sigma)$ is *singular* if $\phi^{-1}\psi = \varphi^{-1}\sigma$ and we can find $i, j \in I, \lambda, \mu \in \Lambda$ with $\phi = \mathbf{p}_{\lambda i}, \varphi = \mathbf{p}_{\mu i}, \psi = \mathbf{p}_{\lambda j}$ and $\sigma = \mathbf{p}_{\mu j}$.

In the sequel, we denote the free group on a set X by \tilde{X} . For convenience, we use, for example, the same symbol $f_{i,\lambda}$ for an element of \tilde{F} and \overline{H} . We hope that the context will prevent ambiguities from arising.

Lemma 9.1. *Let $\overline{\overline{H}}$ be the group given by the presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ with generators:*

$$S = \{f_\phi : \phi \in P\}$$

and with the defining relations Γ :

(P1) $f_\phi^{-1}f_\varphi = f_\psi^{-1}f_\sigma$ where $(\phi, \varphi, \psi, \sigma)$ is singular;

(P2) $f_\varepsilon = 1$.

Then $\overline{\overline{H}}$ is isomorphic to \overline{H} .

Proof. From Theorem 3.3, we know that \overline{H} is given by the presentation $\mathcal{P} = \langle F : \Sigma \rangle$, where $F = \{f_{i,\lambda} : (i, \lambda) \in K\}$ and Σ is the set of relations as defined in (R1) – (R3), and where the function ω and the Schreier system $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ are fixed as in Section 3. Note that (R3) is reformulated in Corollary 3.6.

By freeness of the generators we may define a morphism $\theta : \tilde{F} \rightarrow \overline{\overline{H}}$ by $f_{i,\lambda}\theta = f_\phi$, where $\phi = \mathbf{p}_{\lambda i}$. We show that $\Sigma \subseteq \ker \theta$. It is clear from (P1) that relations of the form (R3) lie in $\ker \theta$.

Suppose first that $\mathbf{h}_\lambda \varepsilon_{i\mu} = \mathbf{h}_\mu$ in \overline{E}^* . Then $\varepsilon \mathbf{h}_\lambda \varepsilon_{i\mu} = \varepsilon \mathbf{h}_\mu$ in $\text{End } F_n(G)$, so that from Lemma 3.7, $\mathbf{q}_\lambda \varepsilon_{i\mu} = \mathbf{q}_\mu$. Hence $\mathbf{q}_\mu \mathbf{r}_i = \mathbf{q}_\lambda \varepsilon_{i\mu} \mathbf{r}_i = \mathbf{q}_\lambda \mathbf{r}_i$, so that $\mathbf{p}_{\mu i} = \mathbf{p}_{\lambda i}$ and $f_{i,\lambda}\theta = f_{i,\mu}\theta$. Now suppose that $i \in I$; we have remarked that $\mathbf{p}_{\omega(i)i} = \varepsilon$, so that $f_{i,\omega(i)}\theta = f_\varepsilon = 1\theta$.

We have shown that $\Sigma \subseteq \ker \theta$ and so there exists a morphism $\overline{\theta} : \overline{H} \rightarrow \overline{\overline{H}}$ such that $f_{i,\lambda}\overline{\theta} = f_\phi$ where $\phi = \mathbf{p}_{\lambda i}$.

Conversely, we define a map $\psi : \tilde{S} \rightarrow \overline{H}$ by $f_\phi\psi = f_{i,\lambda}$, where $\phi = \mathbf{p}_{\lambda i}$. By Lemma 8.4, ψ is well defined. Since $f_\varepsilon\psi = f_{i,\lambda}$ where $\mathbf{p}_{\lambda i} = \varepsilon$, we have $f_\varepsilon\psi = 1_{\overline{H}}$ by Lemma 4.1. Clearly

relations (P1) lie in $\ker \psi$, so that $\Gamma \subseteq \ker \psi$. Consequently, there is a morphism $\overline{\psi} : \overline{H} \rightarrow \overline{H}$ such that $f_\phi \overline{\psi} = f_{i,\lambda}$, where $\phi = \mathbf{p}\lambda_i$.

It is clear that $\overline{\theta} \overline{\psi}$ and $\overline{\psi} \overline{\theta}$ are, respectively, the identity maps on the generators of \overline{H} and $\overline{\overline{H}}$, respectively. It follows immediately that they are mutually inverse isomorphisms. \square

We now recall the presentation of $G \wr \mathcal{S}_r$ obtained by Lavers [17]. In fact, we translate his presentation to one for our group H .

We begin by defining the following elements of H : for $a \in G$ and for $1 \leq i \leq r$ we put

$$\iota_{a,i} = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_i & x_{i+1} & \cdots & x_r \end{pmatrix};$$

for $1 \leq k \leq r-1$ we put

$$(k \ k+1 \ \cdots \ k+m) = \begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & \cdots & x_{k-1} & x_{k+1} & \cdots & x_{k+m} & x_k & x_{k+m+1} & \cdots & x_r \end{pmatrix}$$

and we denote $(k \ k+1)$ by τ_k .

It is clear that G^r has presentation $\mathcal{V} = \langle Z : \Pi \rangle$, with generators

$$Z = \{\iota_{a,i} : i \in [1, r], a \in G\}$$

and defining relations Π consisting of (W4) and (W5) below. Using a standard presentation for \mathcal{S}_r , we employ the recipe of [17] to obtain:

Lemma 9.2. *The group H has a presentation $\mathcal{U} = \langle Y : \Upsilon \rangle$, with generators*

$$Y = \{\tau_i, \iota_{a,j} : 1 \leq i \leq r-1, 1 \leq j \leq r, a \in G\}$$

and defining relations Υ :

- (W1) $\tau_i \tau_i = 1, 1 \leq i \leq r-1$;
- (W2) $\tau_i \tau_j = \tau_j \tau_i, j \pm 1 \neq i \neq j$;
- (W3) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, 1 \leq i \leq r-2$;
- (W4) $\iota_{a,i} \iota_{b,j} = \iota_{b,j} \iota_{a,i}, a, b \in G$ and $1 \leq i \neq j \leq r$;
- (W5) $\iota_{a,i} \iota_{b,i} = \iota_{ab,i}, 1 \leq i \leq r$ and $a, b \in G$;
- (W6) $\iota_{a,i} \tau_j = \tau_j \iota_{a,i}, 1 \leq i \neq j, j+1 \leq r$;
- (W7) $\iota_{a,i} \tau_i = \tau_i \iota_{a,i+1}, 1 \leq i \leq r-1$ and $a \in G$.

Now we turn to our maximal subgroup \overline{H} . From Lemma 9.1, we know that \overline{H} is isomorphic to $\overline{\overline{H}}$, and it follows from the definition of the isomorphism and Proposition 8.5 that

$$\overline{\overline{H}} = \langle f_\alpha : \alpha \text{ has simple form} \rangle.$$

We now simplify our generators further. For ease in the remainder of the paper, it is convenient to use the following convention: for $u, v \in [1, r+2]$ with $u < v$, we denote by $\neg(u, v)$ the r -tuple

$$(1, \dots, u-1, u+1, \dots, v-1, v+1, \dots, r+2).$$

Lemma 9.3. *Consider the element*

$$\alpha = \begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & \cdots & x_{k-1} & x_{k+1} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \cdots & x_r \end{pmatrix}$$

in simple form, where $m \geq 1$. Then $f_\alpha = f_\gamma f_\beta$ in $\overline{\overline{H}}$, where $\beta = \iota_{a, k+m}$ and $\gamma = (k \ k + 1 \ \cdots \ k + m)$.

Proof. Define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_k & ax_k & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(k, k + m + 1)$ and $\mu = \neg(k, k + m + 2)$. Then $\mathbf{p}_{\lambda t} = \alpha$ and $\mathbf{p}_{\mu t} = \gamma$.

Next we define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k-1} & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & ax_{k+m} & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \beta$ and $\mathbf{p}_{\mu s} = \varepsilon$. Notice that $\alpha = \beta\gamma$ and

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

which implies $f_\alpha = f_\gamma f_\beta$. □

Lemma 9.4. *Let $\alpha = (k \ k + 1 \ \cdots \ k + m)$, where $m \geq 1$. Then $f_\alpha = f_{\tau_k} f_{\tau_{k+1}} \cdots f_{\tau_{k+m-1}}$ in $\overline{\overline{H}}$.*

Proof. We proceed by induction on m : clearly the result is true for $m = 1$. Assume now that $m \geq 2$, $\alpha = (k \ k + 1 \ \cdots \ k + m)$ and that $f_{(k \ k + 1 \ \cdots \ k + s)} = f_{\tau_k} f_{\tau_{k+1}} \cdots f_{\tau_{k+s-1}}$ for any $s < m$. It is easy to check that $\alpha = \tau_{k+m-1} \gamma$, where

$$\gamma = (k \ k + 1 \ \cdots \ k + m - 1).$$

Now we define \mathbf{r}_j by

$$\begin{pmatrix} x_1 & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k+m-1} & x_k & x_{k+m} & x_k & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(k, k + m)$ and $\mu = \neg(k, k + m + 2)$. Then $\mathbf{p}_{\lambda j} = \alpha$ and $\mathbf{p}_{\mu j} = \gamma$.

Next we define \mathbf{r}_l by

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k-1} & x_k & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m-1} & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_{k+m-1}$ and $\mathbf{p}_{\mu l} = \varepsilon$. Thus we have

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu l} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \tau_{k+m-1} & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

implying $f_\alpha = f_\gamma f_{\tau_{k+m-1}}$ and so $f_\alpha = f_{\tau_k} \cdots f_{\tau_{k+m-1}}$, using our inductive hypothesis applied to γ . □

It follows from Lemmas 9.3 and 9.4 that

$$\overline{H} = \langle f_{\tau_i}, f_{l_{a,j}} : 1 \leq i \leq r-1, 1 \leq j \leq r, a \in G \rangle.$$

Now it is time for us to find a series of relations satisfied by these generators. These correspond to those in Lemma 9.2, with the exception of a twist in (W5).

Lemma 9.5. *For all $i \in [1, r-1]$, $f_{\tau_i} f_{\tau_i} = 1$, and so $f_{\tau_i}^{-1} = f_{\tau_i}$.*

Proof. Notice that $\tau_i \tau_i = \varepsilon$. First we define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, i+1)$ and $\mu = \neg(i, i+3)$. Then $\mathbf{p}_{\lambda s} = \tau_i$ and $\mathbf{p}_{\mu s} = \varepsilon$.

Next, we define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda t} = \varepsilon$ and $\mathbf{p}_{\mu t} = \tau_i$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \tau_i & \varepsilon \\ \varepsilon & \tau_i \end{pmatrix}$$

which implies $f_{\tau_i} f_{\tau_i} = 1$. □

Lemma 9.6. *For any $j \pm 1 \neq i \neq j$ we have $f_{\tau_i} f_{\tau_j} = f_{\tau_j} f_{\tau_i}$.*

Proof. Without loss of generality, suppose that $i > j$ and $i \neq j+1$. First, define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{j+1} & x_{j+2} & x_{j+3} & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j+1} & x_j & x_{j+2} & \cdots & x_{i-1} & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Note that if $i = j+2$ then the section from $j+3$ to i is empty. Let $\lambda = \neg(j, i+1)$ and $\mu = \neg(j, i+3)$, so that $\mathbf{p}_{\lambda t} = \tau_i \tau_j$ and $\mathbf{p}_{\mu t} = \tau_j$. Next define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_j & x_{j+1} & x_{j+2} & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_j & x_j & x_{j+1} & \cdots & x_{i-1} & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \tau_i$ and $\mathbf{p}_{\mu s} = \varepsilon$. Thus we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \tau_i & \tau_i \tau_j \\ \varepsilon & \tau_j \end{pmatrix}$$

implying $f_{\tau_i} f_{\tau_j} = f_{\tau_j} f_{\tau_i}$.

To complete the proof, we define \mathbf{r}_l by

$$\begin{pmatrix} x_1 & \cdots & x_{j+1} & x_{j+2} & x_{j+3} & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j+1} & x_j & x_{j+2} & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_j$. Put $\eta = \neg(j+2, i+1)$. Then $\mathbf{p}_{\eta l} = \varepsilon$ and $\mathbf{p}_{\eta t} = \tau_i$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \tau_j & \tau_j \tau_i \\ \varepsilon & \tau_i \end{pmatrix}$$

which implies $f_{\tau_j} f_{\tau_i} = f_{\tau_i} f_{\tau_j}$, and hence $f_{\tau_j} f_{\tau_i} = f_{\tau_i} f_{\tau_j}$. □

Lemma 9.7. *For any $i \in [1, r-2]$ we have $f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}}$.*

Proof. Let $\rho = \tau_{i+1}\tau_i = (i \ i+1 \ i+2)$ so that $\rho^2 = (i \ i+2 \ i+1)$.

First, we show that $f_{\rho^2} = f_{\rho}f_{\rho}$. For this purpose, we define \mathbf{r}_j by

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+1} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, i+1)$ and $\mu = \neg(i, i+4)$, so that $\mathbf{p}_{\lambda j} = (i \ i+2 \ i+1) = \rho^2$ and $\mathbf{p}_{\mu j} = (i \ i+1 \ i+2) = \rho$. Next we define \mathbf{r}_l by

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_i & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = (i \ i+1 \ i+2) = \rho$ and $\mathbf{p}_{\mu l} = \varepsilon$, so here we have

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu l} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \rho & \rho^2 \\ \varepsilon & \rho \end{pmatrix}$$

Hence we have $f_{\rho^2} = f_{\rho}f_{\rho}$.

Secondly, we show that $f_{\rho} = f_{\tau_i} f_{\tau_{i+1}}$. Note that $\tau_{i+1}\rho = \tau_i$. Now we define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+2} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\nu = \neg(i, i+2)$ and $\xi = (i, i+4)$. Then $\mathbf{p}_{\nu s} = \tau_i$ and $\mathbf{p}_{\xi s} = \rho = (i \ i+1 \ i+2)$. Next, define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_{i+2} & x_{i+1} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\nu t} = \tau_{i+1}$ and $\mathbf{p}_{\xi t} = \varepsilon$, and so we have

$$\begin{pmatrix} \mathbf{p}_{\nu t} & \mathbf{p}_{\nu s} \\ \mathbf{p}_{\xi t} & \mathbf{p}_{\xi s} \end{pmatrix} = \begin{pmatrix} \tau_{i+1} & \tau_i \\ \varepsilon & \rho \end{pmatrix}$$

implying $f_{\tau_i} = f_{\rho}f_{\tau_{i+1}}$, so $f_{\rho} = f_{\tau_i}f_{\tau_{i+1}}$ by Lemma 9.5.

Finally, we show that $f_{\rho^2} = f_{\tau_{i+1}}f_{\tau_i}$. Note that $\rho^2 = (i \ i+2 \ i+1) = \tau_i\tau_{i+1}$. Define \mathbf{r}_u by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+1} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\tau = \neg(i, i+1)$ and $\delta = \neg(i+1, i+3)$. Then $\mathbf{p}_{\tau u} = \rho^2$ and $\mathbf{p}_{\delta u} = \tau_{i+1}$. Define \mathbf{r}_v by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\tau v} = \tau_i$ and $\mathbf{p}_{\delta v} = \varepsilon$, so we have

$$\begin{pmatrix} \mathbf{p}_{\tau v} & \mathbf{p}_{\tau u} \\ \mathbf{p}_{\delta v} & \mathbf{p}_{\delta u} \end{pmatrix} = \begin{pmatrix} \tau_i & \rho^2 \\ \varepsilon & \tau_{i+1} \end{pmatrix}$$

Hence $f_{\rho^2} = f_{\tau_{i+1}}f_{\tau_i}$. We now calculate:

$$f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_i} f_{\rho^2} = f_{\tau_i} f_{\rho} f_{\rho} = f_{\tau_i} f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}},$$

the final step using Lemma 9.5. □

We warn the reader that the relation we find below is a twist on that in (W5).

Lemma 9.8. *For all $i \in [1, r]$, $a, b \in G$, $f_{\iota_{b,i}} f_{\iota_{a,i}} = f_{\iota_{ab,i}}$, and so $f_{\iota_{a,i}}^{-1} = f_{\iota_{a^{-1},i}}$.*

Proof. Define \mathbf{r}_j by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & bx_i & abx_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, i+2)$ and $\mu = \neg(i, i+1)$, then $\mathbf{p}_{\lambda j} = \iota_{b,i}$ and $\mathbf{p}_{\mu j} = \iota_{ab,i}$. Next, we define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda t} = \varepsilon$ and $\mathbf{p}_{\mu t} = \iota_{a,i}$, so we have

$$\begin{pmatrix} \mathbf{p}_{\mu t} & \mathbf{p}_{\mu j} \\ \mathbf{p}_{\lambda t} & \mathbf{p}_{\lambda j} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{ab,i} \\ \varepsilon & \iota_{b,i} \end{pmatrix}$$

implying $f_{\iota_{ab,i}} = f_{\iota_{b,i}} f_{\iota_{a,i}}$. □

Lemma 9.9. *For all $i \neq j$ and $a, b \in G$ we have $f_{\iota_{a,i}} f_{\iota_{b,j}} = f_{\iota_{b,j}} f_{\iota_{a,i}}$.*

Proof. Without loss of generality, suppose that $i > j$. Recall that $\iota_{a,i} \iota_{b,j} = \iota_{b,j} \iota_{a,i}$. First define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j-1} & x_j & bx_j & x_{j+1} & \cdots & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(j, i+1)$ and $\mu = \neg(j, i+2)$. Then $\mathbf{p}_{\lambda t} = \iota_{a,i} \iota_{b,j}$ and $\mathbf{p}_{\mu t} = \iota_{b,j}$.

Next, we define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j-1} & x_j & x_j & x_{j+1} & \cdots & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \iota_{a,i}$ and $\mathbf{p}_{\mu s} = \varepsilon$. Thus we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{a,i} \iota_{b,j} \\ \varepsilon & \iota_{b,j} \end{pmatrix}$$

implying $f_{\iota_{b,j}} f_{\iota_{a,i}} = f_{\iota_{a,i} \iota_{b,j}}$.

Define \mathbf{r}_l by

$$\begin{pmatrix} x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j-1} & x_j & bx_j & x_{j+1} & \cdots & x_i & x_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \iota_{b,j}$. On the other hand, by putting $\eta = \neg(j+1, i+1)$ we have $\mathbf{p}_{\eta l} = \varepsilon$ and $\mathbf{p}_{\eta t} = \iota_{a,i}$, and so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \iota_{b,j} & \iota_{b,j} \iota_{a,i} \\ \varepsilon & \iota_{a,i} \end{pmatrix}$$

which implies $f_{\iota_{b,j} \iota_{a,i}} = f_{\iota_{a,i}} f_{\iota_{b,j}}$, and hence $f_{\iota_{a,i}} f_{\iota_{b,j}} = f_{\iota_{b,j}} f_{\iota_{a,i}}$. □

Lemma 9.10. *For any i, j with $i \neq j, j+1$ and $a \in G$ we have $f_{\iota_{a,i}} f_{\tau_j} = f_{\tau_j} f_{\iota_{a,i}}$.*

Proof. Suppose that $i < j$; the proof for $j < i$ is entirely similar. Then

$$\iota_{a,i}\tau_j = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_i & x_{i+1} & \cdots & x_{j-1} & x_{j+1} & x_j & x_{j+2} & \cdots & x_r \end{pmatrix}.$$

Define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_j & x_{j+1} & x_{j+2} & x_{j+3} & x_{j+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & ax_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_{j+1} & x_j & x_{j+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, j+1)$ and $\mu = \neg(i+1, j+1)$. Then $\mathbf{p}_{\lambda t} = \iota_{a,i}\tau_j$ and $\mathbf{p}_{\mu t} = \tau_j$.

Define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_j & x_{j+1} & x_{j+2} & x_{j+3} & x_{j+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & ax_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_j & x_{j+1} & x_{j+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \iota_{a,i}$ and $\mathbf{p}_{\mu s} = \varepsilon$. Hence we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{a,i}\tau_j \\ \varepsilon & \tau_j \end{pmatrix}$$

implying $f_{\iota_{a,i}\tau_j} = f_{\tau_j}f_{\iota_{a,i}}$.

Next we define $\eta = \neg(i, j+3)$, so that $\mathbf{p}_{\eta t} = \iota_{a,i}$. Now let \mathbf{r}_l be

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_j & x_{j+1} & x_{j+2} & x_{j+3} & x_{j+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_{j+1} & x_j & x_{j+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_j$ and $\mathbf{p}_{\eta l} = \varepsilon$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \tau_j & \tau_j\iota_{a,i} \\ \varepsilon & \iota_{a,i} \end{pmatrix}$$

implying $f_{\tau_j\iota_{a,i}} = f_{\iota_{a,i}}f_{\tau_j}$, so $f_{\tau_j}f_{\iota_{a,i}} = f_{\iota_{a,i}}f_{\tau_j}$. □

Lemma 9.11. *For any $i \in [1, r-1]$ and $a \in G$ we have $f_{\iota_{a,i}}f_{\tau_i} = f_{\tau_i}f_{\iota_{a,i+1}}$.*

Proof. We have

$$\iota_{a,i}\tau_i = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_{i+1} & x_i & x_{i+2} & \cdots & x_r \end{pmatrix} = \tau_i\iota_{a,i+1}.$$

Define \mathbf{r}_t by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & ax_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Define $\lambda = \neg(i, i+1)$ and $\mu = \neg(i, i+2)$. Then $\mathbf{p}_{\lambda t} = \iota_{a,i}\tau_i$ and $\mathbf{p}_{\mu t} = \tau_i$. Define \mathbf{r}_s by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \iota_{a,i}$ and $\mathbf{p}_{\mu s} = \varepsilon$, so we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{a,i}\tau_i \\ \varepsilon & \tau_i \end{pmatrix}$$

so $f_{\iota_{a,i}\tau_i} = f_{\tau_i}f_{\iota_{a,i}}$.

Now put $\eta = \neg(i+1, i+3)$, so that $\mathbf{p}_{\eta t} = \iota_{a, i+1}$. Define \mathbf{r}_l by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_i$ and $\mathbf{p}_{\eta l} = \varepsilon$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \tau_i & \tau_i \iota_{a, i+1} \\ \varepsilon & \iota_{a, i+1} \end{pmatrix}$$

so that $f_{\tau_i \iota_{a, i+1}} = f_{\iota_{a, i+1}} f_{\tau_i}$. Thus $f_{\tau_i} f_{\iota_{a, i}} = f_{\iota_{a, i+1}} f_{\tau_i}$ and so $f_{\iota_{a, i}} f_{\tau_i} = f_{\tau_i} f_{\iota_{a, i+1}}$, bearing in mind Lemmas 9.5 and 9.8. \square

We denote by Ω all the following relations we have obtained so far on the set of generators

$$T = \{f_{\tau_i}, f_{\iota_{a, j}} : 1 \leq i \leq r-1, 1 \leq j \leq r, a \in G\}$$

of $\overline{\overline{H}}$:

- (T1) $f_{\tau_i} f_{\tau_i} = 1, 1 \leq i \leq r-1.$
- (T2) $f_{\tau_i} f_{\tau_j} = f_{\tau_j} f_{\tau_i}, j \pm 1 \neq i \neq j.$
- (T3) $f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}}, 1 \leq i \leq r-2.$
- (T4) $f_{\iota_{a, i}} f_{\iota_{b, j}} = f_{\iota_{b, j}} f_{\iota_{a, i}}, a, b \in G$ and $1 \leq i \neq j \leq r.$
- (T5) $f_{\iota_{b, i}} f_{\iota_{a, i}} = f_{\iota_{ab, i}}, 1 \leq i \leq r$ and $a, b \in G.$
- (T6) $f_{\iota_{a, i}} f_{\tau_j} = f_{\tau_j} f_{\iota_{a, i}}, 1 \leq i \neq j, j+1 \leq r.$
- (T7) $f_{\iota_{a, i}} f_{\tau_i} = f_{\tau_i} f_{\iota_{a, i+1}}, 1 \leq i \leq r-1$ and $a \in G.$

We now have all the ingredients in place to prove the following.

Proposition 9.12. *The group $\overline{\overline{H}}$ with a presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ of Lemma 9.1 is isomorphic to the presentation $\mathcal{U} = \langle Y : \Upsilon \rangle$ of H given in Lemma 9.2, so that $\overline{\overline{H}} \cong H$.*

Proof. We define a map $\theta : \tilde{Y} \rightarrow \overline{\overline{H}}$ by

$$\tau_i \theta = f_{\tau_i}^{-1} (= f_{\tau_i}), \iota_{a, j} \theta = f_{\iota_{a, j}}^{-1} (= f_{\iota_{a-1, j}})$$

where $1 \leq i \leq r-1, 1 \leq j \leq r, a \in G$. Now we claim that $\Upsilon \subseteq \ker \theta$. Clearly, the relations corresponding to (W1) – (W4) and (W6) and (W7) lie in $\ker \theta$. Moreover, considering (W5)

$$(\iota_{a, i} \iota_{b, i}) \theta = \iota_{a, i} \theta \iota_{b, i} \theta = f_{\iota_{a, i}}^{-1} f_{\iota_{b, i}}^{-1} = f_{\iota_{a-1, i}} f_{\iota_{b-1, i}} = f_{\iota_{b-1, a-1, i}} = f_{\iota_{(ab)-1, i}} = \iota_{ab, i} \theta$$

so that $\Upsilon \subseteq \ker \theta$, and hence there exists a well defined morphism $\overline{\theta} : H \rightarrow \overline{\overline{H}}$ given by $\tau_i \overline{\theta} = f_{\tau_i}^{-1}$ and $\iota_{a, j} \overline{\theta} = f_{\iota_{a, j}}^{-1}$, where $1 \leq i \leq r-1, 1 \leq j \leq r, a \in G$.

Conversely, we define $\psi : \tilde{S} \rightarrow H$ by $f_\phi \psi = \phi^{-1}$. We show that $\Gamma \subseteq \ker \psi$. Clearly, $f_\varepsilon \psi = \varepsilon^{-1} = \varepsilon = 1 \psi$. Suppose that $(\phi, \varphi, \psi, \sigma)$ is singular, giving $\phi \varphi^{-1} = \psi \sigma^{-1}$. Then

$$(f_\phi^{-1} f_\varphi) \psi = (f_\phi \psi)^{-1} f_\varphi \psi = \phi \varphi^{-1} = \psi \sigma^{-1} = (f_\psi \psi)^{-1} f_\sigma \psi = (f_\psi^{-1} f_\sigma) \psi$$

so $\Gamma \subseteq \ker \psi$. Thus there exists a well defined morphism $\overline{\psi} : \overline{\overline{H}} \rightarrow H$ given by $f_\phi \overline{\psi} = \phi^{-1}$. Then

$$\tau_i \overline{\theta} \overline{\psi} = f_{\tau_i}^{-1} \overline{\psi} = (f_{\tau_i} \overline{\psi})^{-1} = \tau_i$$

and

$$\iota_{a,i} \bar{\theta} \bar{\psi} = f_{\iota_{a,i}}^{-1} \bar{\psi} = (f_{\iota_{a,i}} \bar{\psi})^{-1} = \iota_{a,i}$$

hence $\bar{\theta} \bar{\psi}$ is the identity mapping, and so $\bar{\theta}$ is one-one. Since T is a set of generators for $\overline{\overline{H}}$, it is clear that $\bar{\theta}$ is onto, and so

$$\overline{\overline{H}} \cong \overline{H} \cong H \cong G \wr S_r.$$

□

We can now state the main theorem of this paper.

Theorem 9.13. *Let $\text{End } F_n(G)$ be the endomorphism monoid of a free G -act $F_n(G)$ on n generators, where $n \in \mathbb{N}$ and $n \geq 3$, let E be the biordered set of idempotents of $\text{End } F_n(G)$, and let $\text{IG}(E)$ be the free idempotent generated semigroup over E .*

For any idempotent $\varepsilon \in E$ with rank r , where $1 \leq r \leq n - 2$, the maximal subgroup $\overline{\overline{H}}$ of $\text{IG}(E)$ containing $\bar{\varepsilon}$ is isomorphic to the maximal subgroup H of $\text{End } F_n(G)$ containing ε and hence to $G \wr S_r$.

Note that if ε is an idempotent with rank n , that is, the identity map, then $\overline{\overline{H}}$ is the trivial group, since it is generated (in $\text{IG}(E)$) by idempotents of the same rank. On the other hand, if the rank of ε is $n - 1$, then $\overline{\overline{H}}$ is the free group as there are no non-trivial singular squares in the \mathcal{D} -class of ε in $\text{End } F_n(G)$.

Finally, if G is trivial, then $\text{End } F_n(G)$ is essentially \mathcal{T}_n , so we deduce the following result from [14].

Corollary 9.14. [14] *Let $n \in \mathbb{N}$ with $n \geq 3$ and let $\text{IG}(E)$ be the free idempotent generated semigroup over the biordered set E of idempotents of \mathcal{T}_n .*

For any idempotent $\varepsilon \in E$ with rank r , where $1 \leq r \leq n - 2$, the maximal subgroup $\overline{\overline{H}}$ of $\text{IG}(E)$ containing $\bar{\varepsilon}$ is isomorphic to the maximal subgroup H of \mathcal{T}_n containing ε , and hence to S_r .

Acknowledgements. The authors would like to thank Robert D. Gray (University of East Anglia) and Nik Ruškuc (University of St Andrews) for some useful discussions, and John Fountain (University of York) for suggesting the problem that led to this work.

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