Revisiting automatic semigroups - change of generators

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Overview

- Notation and definitions we need
- Automatic groups and semigroups
- Change of generators
- Change of generators: monoids
- Change of generators: completely simple semigroups
- Change of generators: $S = SS$.

Finite state automata

- alphabet: A
- set of states: Σ
- partial function: $\mu : \Sigma \times A \to \Sigma$
- initial state $\sigma \in \Sigma$
- final states $T \subseteq \Sigma$

$$\mathcal{A} = (\Sigma, A, \sigma, \mu, T).$$

Notation

Let A be a finite set, and let \$ be a symbol not contained in A. Let

$$A(2,\$) = ((A \cup \$) \times (A \cup \$)) \setminus (\$,\$).$$

Define

$$\delta_A : A^* \times A^* \to A(2,\$)^*$$

by

$$(a_1 \ldots a_n, b_1 \ldots b_m)\delta_A = (a_1, b_1) \ldots (a_n, b_n) \quad \text{if } n = m$$

$$(a_1 \ldots a_n, b_1 \ldots b_m)\delta_A = (a_1, b_1) \ldots (a_n, b_n)(\$, b_{n+1}) \ldots (\$, b_m) \quad \text{if } n < m$$

$$(a_1 \ldots a_n, b_1 \ldots b_m)\delta_A = (a_1, b_1) \ldots (a_m, b_m)(a_{m+1}, \$) \ldots (a_n, \$) \quad \text{if } n > m$$

Automatic semigroups

Let S be a semigroup generated by a finite set A. Let L be a regular language over A and $\varphi : A^+ \to S$ a homomorphism. We say that (A, L) is an automatic structure for S if

- $L\varphi = S$,
- $L_= = \{(u, v) | u, v \in L, u = v\}\delta_A$ is a regular language,
- $L_a = \{(u, v) | u, v \in L, ua = v\}\delta_A$ is a regular language for all $a \in A$.

Examples

- Finite semigroups and groups
- Finitely generated free groups and free semigroups
- Finitely generated subgroups of free groups
- Finitely generated abelian groups

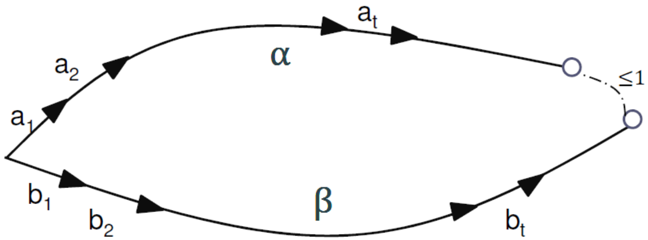
Properties of automatic groups

- Finitely presented
- Invariance under the change of generating set
- Characterized by the fellow traveller property

Fellow traveller property

Assume $G = \langle X \rangle$ and $L \in X^+$ is a regular language such that $L\varphi = G$.

$\alpha \equiv a_1 a_2 ... a_n \in L$ $\qquad\qquad$ $\beta \equiv b_1 b_2 ... b_m \in L$



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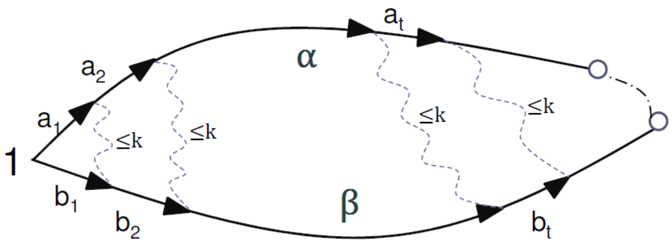

Properties of automatic semigroups

Campbell, Robertson, Ruškuc, Thomas: Automatic semigroups

- Not necessarily finitely presented
- Depends on the choice of the generating set
- Fellow traveller property does not characterize automaticity

Changing generators

Let F denote the free semigroup on $\{a, b, c\}$. Let

$$u = c, \quad v = ac, \quad w = ca, \quad x = ab, \quad y = baba$$

and let

$$S = \langle\ u, v, w, x, y\ \rangle = \langle\ A\ \rangle.$$

Then

$$S = \langle A \mid\ ux^{2^i}v = wy^iu \quad (i \geq 0) \rangle.$$

S cannot have an automatic structure (A, L).

Changing generators

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$$u = c, \quad v = ac, \quad w = ca, \quad x = ab, \quad y = baba, \quad z = abab$$

and let

$$S = \langle\ u, v, w, x, y, z\ \rangle = \langle\ B\ \rangle.$$

Then

- $S = \langle B \mid ux^{2i}v = wy^iu \quad (i \geq 0),\ z = x^2 \rangle.$
- $L = B^+ - (B^*\{zx\}B^* \cup B^*\{w\}\{y\}^*\{u\}B^* \cup B^*\{x^2\}B^*)$
- $L_=, \quad L_v, \quad L_w, \quad L_y, \quad L_z, \quad L_u, \quad L_x$ are regular languages.

Changing generators - Monoids

Theorem (Duncan, Ruškuc, Robertson): Let M be a monoid with automatic structure (A, L) and let B be a finite generating set for M. Then there exists an automatic structure (B, K) for M.

Changing generators - Monoids

Step 1:

- (A, L) is automatic structure for M;
- B also generates M;
- $B_1 = B \cup \{\iota\}$, where $\iota = \mathbf{1}_M$;
- (B_1, K) automatic structure for M.

Key idea in the proof:

$$a_i = c_1 c_2 \dots c_k$$

$$L\theta = K \quad \text{regular language}$$

Step 2:

- There exists an automatic structure (B, N) for M.

Key idea in the proof:

- $z = z_1 \dots z_n$ such that $\overline{z} = 1_M$.
- Given $w \in B_1^+$, form $w\psi$ by substituting z for every n^{th} occurrence of ι and deleting all other occurrences of ι in w.
- Set $N = K\psi$.

Theorem (Campbell, Robertson, Ruškuc, Thomas):
$\mathcal{M}[H, ; I, J, P]$ is automatic if and only if H is automatic.

Corollary: Existence of an automatic structure is independent of the choice of the generating set.

$S = SS$

Let S be a semigroup and $s \in S$. We say that s is **decomposable**, if there exist

$$s_1, s_2 \in S \qquad \text{such that} \qquad s = s_1 s_2.$$

We assume that every element of S is decomposable and hence

$$S = SS.$$

Full generating set

We say that A is a **full generating set** if $A \subseteq A^2$.

Theorem: A semigroup S has a full generating set A if and only if $S = SS$. If S is finitely generated, then A can be chosen to be finite.

Obtaining a full generating set $S = SS$

Let $A = \{a_1, \ldots, a_n\}$ be a finite generating set for S and let $a \in A$. Then,

$$a = b_1 b_2 b_3 \ldots b_{m-1} b_m$$

$$a = b_1 \underbrace{b_2 b_3 \ldots b_{m-1} b_m}_{w_1}$$

$$a = b_1 b_2 \underbrace{b_3 \ldots b_{m-1} b_m}_{w_2}$$

$$\vdots$$

$$a = b_1 b_2 b_3 \ldots \underbrace{b_{m-1} b_m}_{w_{m-2}}$$

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$$\vdots$$

$$a = b_1 b_2 b_3 \dots \underbrace{b_{m-1} b_m}_{w_{m-2} = \alpha_{m-2}}$$

$A \cup \overline{A}$ is a full generating set of S.

Further properties

$$a = b_1\alpha_1 = b_1 b_2\alpha_2 = b_1 b_2 b_3\alpha_3 = \ldots = b_1 b_2 b_3 \ldots b_m\alpha_m.$$

Theorem (ED): Let $S = \langle\ A\ \rangle$ be a semigroup satisfying $S = SS$ and assume that (A, M) is an automatic structure for S. Assume that the finite set B also generates S. Then there exists a regular language N over B such that (B, N) forms an automatic structure for S.

Changing generators in $S = SS$

Step 1:

- (A, M) is an automatic structure for S;
- B is also a generating set for S;
- (C, K) is an automatic structure for S, where $B \subset C$;
- C is a subset of a full generating set containing B.

Key idea in the proof:

$$a_i\eta = (u_i p_i)\varphi = (u_i v_i d_i)\varphi$$

$$\xi : A^+ \to C^+; a_i \to w_i$$

$$M\xi = K \quad \text{regular language}$$

Changing generators in $S = SS$

$$a_i\eta = (u_i p_i)\varphi = (u_i v_i d_i)\varphi$$

$$u_i \in B^*, \quad p_i \in B, \quad d_i \in C \setminus B$$

$$(a_1 a_2 \dots a_j)\xi = u_1 v_1 d_1 u_2 v_2 d_2 \dots u_j v_j d_j$$

Changing generators in $S = SS$

$$a_i\eta = (u_i p_i)\varphi = (u_i v_i d_i)\varphi$$

$$u_i \in B^*, \quad p_i \in B, \quad d_i \in C \setminus B$$

$$(a_1 a_2 \ldots a_j)\xi = u_1 v_1 \underline{d_1} u_2 v_2 \underline{d_2} \ldots u_j v_j \underline{d_j}$$

Changing generators in $S = SS$

Step 2:

- (C, K) is an automatic structure for S, where $B \subset C$;
- C is a subset of a full generating set containing B;
- Construct an automatic structure (B, N) from the automatic structure (C, K) obtained in the first step by removing generators in $C \setminus B$

Changing generators in $S = SS$

Step 2:

- Removing generators in $C \setminus B$ involves recursively constructing languages

$$K_1, \ldots, K_{(|C \setminus B|)}$$

  where $K_j \subseteq C^+$
- $K_{(|C \setminus B|)}$ is a language over B.
- The language K_j is constructed from K_{j-1} by substituting in every $w \in K_{j-1}$ certain occurrences of $v_j d_j$ by $v_j \gamma_j$ and certain occurrences of $v_j d_j$ by p_j , while keeping track of the length of the modified word.

Step 2 details

Let $v \equiv x_1x_2x_3$ and $\gamma \equiv y_1y_2y_3y_4y_5$. Then $|vd| = 4$ and $|v\gamma| = 8$. Hence, if w is the word

$$\underbrace{x_1x_2x_3d}\,\underbrace{x_1x_2x_3d}\,\underbrace{x_1x_2x_3d}\,\underbrace{x_1x_2x_3d}\,\underbrace{x_1x_2x_3d}\,\underbrace{x_1x_2x_3d}\,\underbrace{x_1x_2x_3d}$$

$$\underbrace{x_1x_2x_3y_1y_2y_3y_4y_5}\,\underbrace{x_1x_2x_3y_1y_2y_3y_4y_5}\,\underbrace{x_1x_2x_3y_1y_2y_3y_4y_5}\,\underbrace{\;p\;}\,\underbrace{\;p\;}\,\underbrace{\;p\;}\,\underbrace{\;p\;}$$

Corollary

Corollary: Let S be a regular semigroup. If S is automatic with respect to a finite generating set then it is automatic with respect to any other finite generating set.

Examples: Inverse semigroups, locally inverse semigroups, orthodox semigroups, completely regular semigroups, in particular completely simple semigroups and Clifford semigroups.

Thank you for your attention!