Endomorphisms of the random graph

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All graphs considered are **countable simple** graphs: No multiple edges and no loops.
The random graph $R$

[Arises in model theory]

Start with vertices: $v_1, v_2, \ldots$.

For each pair of vertices, toss a coin:

If $H$ the vertices are joined; if $T$ the vertices are not joined by an edge.

With probability 1, the resulting graph, the **random graph** $R$, is **existentially closed**:

If $A$ and $B$ are disjoint finite sets of vertices, there exists some vertex $v$ that is joined to all the vertices in $A$ and to none of the vertices in $B$.

This property uniquely characterises $R$.

A back-and-forth argument shows that any two countable graphs satisfying the condition are isomorphic.
More properties of the random graph

\( R \) is homogeneous:

Every isomorphism \( \phi : \Gamma_1 \to \Gamma_2 \) between finite subgraphs of \( R \) can be extended to an automorphism \( \hat{\phi} \) of \( R \).

\( R \) is the Fraïssé limit of the finite graphs

The class \( C \) of finite graphs satisfy the hereditary property, joint embedding property and amalgamation property. Fraïssé’s Theorem says \( C \) has a Fraïssé limit. This is the random graph \( R : \ \text{age}(R) = C \).

Theorem (Truss, 1985)

*The automorphism group of \( R \) is simple.*
Construction of the random graph

If \( \Gamma = (V, E) \) is any countable graph, enumerate the finite subsets of \( V \) as \((A_i)_{i \in \mathbb{N}}\). Define \( G(\Gamma) \) to be the graph with vertices

\[
V \cup \{ v_i \mid i \in \mathbb{N} \},
\]

edges \( E \) plus new edges joining each \( v_i \) to each vertex in \( A_i \) for all \( i \in \mathbb{N} \). Then

- \( \Gamma \) is a subgraph of \( G(\Gamma) \),
- given two disjoint finite subsets \( A \) and \( B \) of \( V \), there exists some \( v \) joined to every vertex of \( A \) and to none of the vertices in \( B \) (namely \( v_i \) when \( A = A_i \)).

Now define \( \Gamma_0 = \Gamma \) and \( \Gamma_{n+1} = G(\Gamma_n) \) for each \( n \).

**Observation**

\( \Gamma_\infty = G^\infty(\Gamma) = \lim_{n \to \infty} \Gamma_n = \bigcup_{n=0}^\infty \Gamma_n \) is isomorphic to the random graph \( R \).
Green’s relations on $M = \text{End } R$

\[
f \mathcal{L} g \quad \text{when } Mf = Mg \quad \text{(R sim.)}
\]

\[
\mathcal{H} = \mathcal{L} \cap \mathcal{R}
\]

\[
\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}
\]

- **Maximal subgroups** of $\text{End } R$ are the $\mathcal{H}$-classes of idempotents $(f^2 = f)$.
- **Regular** $\mathcal{D}$-classes are those that contain group $\mathcal{H}$-classes.
- If $f$ is an idempotent, then $f|_{\text{im } f} = \text{id}$ and

\[
H_f \cong \text{Aut(\text{im } f)}.
\]

Indeed, if $g \in \text{Aut(\text{im } f)}$, then $fg \in \text{End } R$ satisfies

\[
(fg^{-1})(fg) = f, \quad (fg)f = fg, \quad (fg)(fg^{-1}) = f, \quad f(fg) = fg
\]

so $fg \mathcal{H} f$. The isomorphism is $fg \leftrightarrow g$. 
Idempotents in $\text{End } R$

Upshot: Need to understand the idempotent endomorphisms $f$ of $R$.

Note that since $R$ is existentially closed, it is also algebraically closed:

**a.c.:** If $A$ is any finite set of vertices, there exists some vertex $v$ joined to all vertices in $A$.

This is inherited by images: $\text{im } f$ is algebraically closed. Conversely, if $\Gamma$ is a.c., we can extend the identity map to a homomorphism $G^\infty(\Gamma) \to \Gamma$.

**Theorem (Bonato–Delić, 2000)**

There is an idempotent endomorphism $f$ of $R$ with $\text{im } f \cong \Gamma$ if and only if $\Gamma$ is a.c.
Uncountably many idempotent endomorphisms with given image

Suppose $\Gamma_0 = \Gamma$ is a.c. \hspace{1cm} $\Gamma_{n+1} = \mathcal{G}(\Gamma_n)$.
Assume we’ve constructed $f_n : \Gamma_n \to \Gamma$ with $f_n|_{\Gamma} = \text{id}$.

In $\Gamma_{n+1}$ have vertices $v_i$ corresponding to finite $A_i \subseteq V(\Gamma_n)$.

Extend $f_n$ as follows:

- Assume images of $v_1, v_2, \ldots, v_k$ have already been specified; i.e.,
  have defined $f_{n+1}$ on the subgraph induced by $V \cup \{v_1, v_2, \ldots, v_k\}$.
- $\Gamma$ is a.c. $\Rightarrow \exists w$ adjacent to every vertex of
  $(A_{k+1} \cup \{v_1, \ldots, v_k\})f_{n+1}$.
- Extend: Define $v_{k+1} \mapsto w$.

There are infinitely many choices for $w$. (Need only more than one!)

\textbf{Conclusion:} $2^{\aleph_0}$ extensions to $\Gamma_{\infty} \cong \mathbb{R}$.
Let $\Gamma$ be any countable graph and $S \subseteq \{2, 3, 4, \ldots\}$.

Construct $L_S$:

Write $\dagger$ to denote the complement. Then

$$(\Gamma \cup L_S)^\dagger$$

is a.c.

and, provided $L_S \not\cong \Gamma$,

$$\text{Aut}(\Gamma \cup L_S)^\dagger \cong \text{Aut}(\Gamma \cup L_S) \cong \text{Aut} \Gamma \times \text{Aut} L_S \cong \text{Aut} \Gamma.$$

Conclusion: $2^{\aleph_0}$ a.c. graphs with specified automorphism group.
The maximal subgroups of $\text{End } R$

Theorem (DGMMQ)

(i) Let $\Gamma$ be a countable graph. There are $2^{\aleph_0}$ regular $\mathcal{D}$-classes of $\text{End } R$ whose group $\mathcal{H}$-classes are isomorphic to $\text{Aut } \Gamma$.

(ii) Every regular $\mathcal{D}$-class of $\text{End } R$ contains $2^{\aleph_0}$ group $\mathcal{H}$-classes.

Every group that could appear as a maximal subgroup of $\text{End } R$ occurs and does so as many times as it possibly could.
The maximal subgroups of $\text{End } R$

**Theorem (DGMMQ)**

(i) *Let $\Gamma$ be a countable graph.*

There are $2^{\aleph_0}$ regular $\mathcal{D}$-classes of $\text{End } R$ whose group $\mathcal{H}$-classes are isomorphic to $\text{Aut } \Gamma$.

(ii) *Every regular $\mathcal{D}$-class of $\text{End } R$ contains $2^{\aleph_0}$ group $\mathcal{H}$-classes.*

**Proof.**

(i) Take $S \subseteq \{2, 3, \ldots \}$ with $L_S \not\cong \Gamma$. There is an idempotent $f_S$ with image $\cong (\Gamma \cup L_S)^\dagger$. Then

$$H_{f_S} \cong \text{Aut}(\text{im } f_S) \cong \text{Aut } \Gamma.$$  

For $S \neq T$, these lie in different $\mathcal{D}$-classes because $L_S \not\cong L_T$, so $\text{im } f_S \not\cong \text{im } f_T$.

(ii) For each a.c. graph $\Gamma$, there are $2^{\aleph_0}$ idempotents with image $\cong \Gamma$. □
Theorem (DGMMQ)

Every regular $D$-class in $\text{End} \, R$ contains $2^{\aleph_0}$ many $L$- and $R$-classes.

For $f, g$ regular:

- $f \ L \ g$ iff $Vf = Vg$
- $f \ R \ g$ iff $\ker f = \ker g$
- $f \ D \ g$ iff $\text{im} \, f \cong \text{im} \, g$

[$\Rightarrow$ holds without the regularity assumption.]

$2^{\aleph_0}$ $R$-classes: Given an a.c. graph $\Gamma$, there are $2^{\aleph_0}$ idempotents with image $\cong \Gamma$ (extend the identity map on $\Gamma$).

All such $f$ are $L$-related, but not $R$-related.
Uncountably many regular $\mathcal{L}$-classes

Start with an a.c. graph $\Gamma$ (having vertices $v_i$).

**Construct** $\Gamma^\#$ with vertices

$$V^\# = \{ v_{i,0}, v_{i,1} \mid i \in \mathbb{N} \}$$

and edges

$$\{ (v_{i,0}, v_{j,0}), (v_{i,0}, v_{j,1}), (v_{i,1}, v_{j,0}), (v_{i,1}, v_{j,1}) \}$$

whenever $(v_i, v_j)$ is an edge in $\Gamma$.

Note

- $\Gamma^\#$ is also algebraically closed.
- For any sequence $b = (b_i)$ with $b_i \in \{0, 1\}$, the subgraph $\Lambda_b$ induced by $\{ v_{i,b_i} \mid i \in \mathbb{N} \}$ is isomorphic to $\Gamma$.

Build a copy of $R$ (as $\mathcal{G}^\infty(\Gamma^\#)$) around $\Gamma^\#$.

Hence construct idempotent $f$ in $\text{End} R$ with $\text{im} f = \Gamma^\#$. Given $b$, apply the map $\phi_b$ that maps $v_{i,0}, v_{i,1} \mapsto v_{i,b_i}$.

Note the $f\phi_b$ are $\mathcal{D}$-related but not $\mathcal{L}$-related.
What about non-regular $\mathcal{D}$-classes?

Our conclusions are less complete.

Write $R = (V, E)$.

If $f \in \text{End } R$, the key is understanding the difference between

$$\text{im } f = (Vf, Ef) \quad \text{vs.} \quad \langle Vf \rangle = (Vf, E \cap (Vf \times Vf)).$$

$f \in \text{End } R$ is regular if $\exists g$ with $fgf = f$.

$$f \text{ regular } \Rightarrow \text{ im } f = (Vf, Ef) = \langle Vf \rangle$$

**Proposition (Cameron–Nešetřil, 2006)**

Let $\Gamma = (V', E')$ be a countable graph. Then $\Gamma$ is algebraically closed if and only if $(V', F) \cong R$ for some $F \subseteq E'$.

We use this to construct a injective homomorphism $f : R \to \Gamma$ such that $\text{im } f = (V', F) \neq \langle Vf \rangle = (V', E')$. 
Let $\Gamma$ be an a.c. graph with $\Gamma \ncong R$.
Create $\Gamma^\#$ with vertices $\{v_{i,0}, v_{i,1} \mid i \in \mathbb{N}\}$. Set $\Lambda_0 = \langle v_{i,0} \mid i \in \mathbb{N} \rangle \cong \Gamma$.
Build $R = \mathcal{G}^\infty(\Gamma^\#) = (V, E)$.

Use Cameron–Nešetřil: there is an injective endomorphism $f : R \rightarrow R$ with $Vf = \{v_{i,0} \mid i \in \mathbb{N}\}$. So $\text{im } f \cong R$ and $\langle Vf \rangle = \Lambda_0 \cong \Gamma$.
In particular, $f$ is not regular.

If $b = (b_i) \in \{0, 1\}^\mathbb{N}$, the map $v_{i,j} \mapsto v_{i,j+b_i}$ is an automorphism of $\Gamma^\#$. It extends to an automorphism $\psi_b$ of $R$.
Then $f \psi_b$ is $R$-related to $f$.
No pair of these are $L$-related.

Can also create $2^{\aleph_0}$ many $R$-classes in $D_f$.
Varying $\Gamma$ yields $2^{\aleph_0}$ many $\mathcal{D}$-classes.
Summary for non-regular $\mathcal{D}$-classes

Theorem (DGMMQ)

(i) There exists a non-regular injective endomorphism $f$ of $R$ such that the $\mathcal{D}$-class of $f$ contains $2^{\aleph_0}$ many $L$- and $R$-classes.

(ii) There are $2^{\aleph_0}$ many non-regular $\mathcal{D}$-classes in $\text{End } R$.

Questions

1. Can the injectivity condition in (i) be removed?
2. Does (i) hold for all non-regular $\mathcal{D}$-classes?
Schützenberger Groups

If the $\mathcal{H}$-class of $f \in \text{End} R$ is not a group, can create the Schützenberger group $S_H$.
This highlights the distinction between $\text{im } f = (Vf, Ef)$ and $\langle Vf \rangle$ for certain $f$ arising via Cameron–Nešetřil:
Let $\Gamma_0 = (V_0, E_0)$ be a.c. and construct $R$ as $R = G^\infty(\Gamma_0)$. There is an injective endomorphism $f$ with $Vf = V_0$.

Proposition

Let $H = H_f$ for such $f$. Then

$$S_H \cong \text{Aut}(\text{im } f) \cap \text{Aut}\langle Vf \rangle.$$ 

By a suitable construction of $\Gamma_0$ around a particular graph $\Gamma$ obtain:

Theorem (DGMMQ)

Let $\Gamma$ be a countable graph. There are $2^{\aleph_0}$ many non-regular $\mathcal{D}$-classes in $\text{End} R$ that have Schützenberger groups isomorphic to $\text{Aut}\Gamma$. 

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Directed graphs & bipartite graphs

Also have analogous results for the endomorphism of the countable universal homogeneous directed graph $D$ and the countable universal homogeneous bipartite graph $B$.

Definition of bipartite graphs?
The partition is preserved by a homomorphism, but the parts may be interchanged.

Some unusual observations for bipartite graphs: e.g., the finite complete bipartite graphs are a.c.
Maximal subgroups / group $\mathcal{H}$-classes:

**Theorem (DGMMQ)**

1. Let $\Gamma$ be a countable graph. There are $2^{\aleph_0}$ regular $\mathcal{D}$-classes of $\text{End } B$ whose group $\mathcal{H}$-classes are isomorphic to $\text{Aut } \Gamma$.

2. Let $f$ be an idempotent.
   - If $\text{im } f \not\cong K_{1,1}$, then $D_f$ contains $2^{\aleph_0}$ many group $\mathcal{H}$-classes.
   - If $\text{im } f \cong K_{1,1}$, then $D_f$ contains $\aleph_0$ many group $\mathcal{H}$-classes (each $\cong C_2$).
Some results for bipartite graphs, II

\( \mathcal{L} \)- and \( \mathcal{R} \)-classes in regular \( \mathcal{D} \)-classes:

**Theorem (DGMMQ)**

Let \( f \) be a regular endomorphism of \( B \).

1. If \( \text{im} \ f \) is infinite, \( D_f \) contains \( 2^{\aleph_0} \) many \( \mathcal{L} \)- and \( \mathcal{R} \)-classes.
2. If \( \text{im} \ f \) is finite but not \( K_{1,1} \), then \( D_f \) contains \( \aleph_0 \) many \( \mathcal{L} \)-classes and \( 2^{\aleph_0} \) many \( \mathcal{R} \)-classes.
3. If \( \text{im} \ f \cong K_{1,1} \), then \( D_f \) contains \( \aleph_0 \) many \( \mathcal{L} \)-classes and one \( \mathcal{R} \)-class.

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Thank you for your attention!