# Closed Submonoids of the Symmetric Inverse Monoid

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# Symmetric Inverse Monoids

## Definition (Symmetric Inverse Monoid)

Bijections between subsets of a set $X$.

$$I_X = \{\, \text{bijections } A \mapsto B \mid A, B \text{ subsets of } X \,\}$$

The semigroup operation is *“compose wherever possible”*.

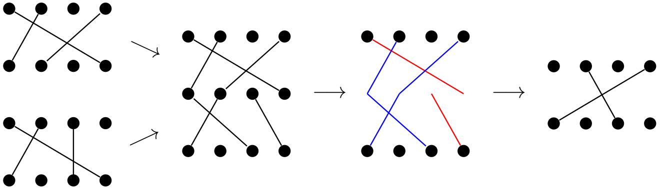
$\alpha =$ $\beta =$  $= \alpha\beta$

Figure: Composition of the charts $\alpha$ and $\beta$ in $I_4$.

# Representation Theorem

**Definition (Inverse Semigroup)**

A semigroup $S$ is *inverse* if

$$(\forall x \in S)(\exists! y \in S) \quad xyx = x \ \wedge \ yxy = y.$$

Such a $y$ is usually denoted by $x^{-1}$.

**Theorem (Wagner-Preston, 1952-1954)**

*If S is an inverse semigroup, then there is a faithful representation*

$$\varphi : S \mapsto I_S.$$

*That is, every inverse semigroup is isomorphic to a subsemigroup of some symmetric inverse monoid.*

# Topological Algebra

## Definition (Topological Semigroup)

A semigroup $S$ with a topology $\mathcal{T}$ such that the multiplication map

$$\mu : S \times S \mapsto S : (x, y) \mapsto xy$$

is continuous with respect to the product topology on $S \times S$.

## Definition (Topological Inverse Semigroup)

A topological semigroup $S$ such that the inversion map

$$\iota : S \mapsto S : x \mapsto x^{-1}$$

is continuous with respect to the semigroup topology.

## Definition (Topological Group)

A topological inverse semigroup that happens to be a group.

# Polish Topologies

A topological space $(X, \mathcal{T})$ is called *Polish* if it is:

1. Completely metrizable
   - There exists a complete metric on $X$, which induces $\mathcal{T}$.
2. Separable
   - There exists a countable dense subset of $X$.

## Examples of Polish Groups

- $(\mathbb{R}, +)$ with the standard topology.
- $\mathrm{Sym}(X)$ with the pointwise topology.

# The Pointwise Topology

The full transformation monoid $X^X$ equipped with the product topology, where $X$ has the discrete topology.

- This topology and any of its subspace topologies are referred to as the *pointwise topology*.

## Subbasis for the Pointwise Topology

$$U_{x,y} = \{\, f \in X^X \mid (x)f = y \,\}$$

## Basic Open Sets in the Pointwise Topology

$$[\sigma : \tau] = \{\, f \in X^X \mid (\sigma_i)f = \tau_i \text{ for all } i \in n \,\}$$

Where $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ and $\tau = (\tau_0, \tau_1, ..., \tau_{n-1})$ are sequences of elements of $X$, and $n \in \omega$.

# Polish Semigroup Topology

**Claim**

*If $X$ is countable, then the pointwise topology is Polish on $X^X$.*

Enumerate $X$ over the natural numbers $\omega$.

1. Let $d(f,g) = \frac{1}{m+1}$, where $m = \min(n \in \omega \mid (x_n)f \neq (x_n)g)$.
2. Let $Q = \{ f \in X^X \mid |\{ x \in X \mid (x)f \neq x \}| < \aleph_0 \}$.

Then $d$ is a complete metric on $X^X$ and $Q$ is a countable dense subset.

**Theorem (Elliott et al., 2023)**

*If $X$ is countable, then the pointwise topology is the unique Polish semigroup topology on $X^X$.*

The symmetric inverse monoid $I_X$ has two minimal Polish semigroup topologies.

### Subbasis for $\mathcal{I}_2$

$$U_{x,y} = \{ f \in I_X \mid (x)f = y \}$$

$$\text{and} \quad W_x = \{ f \in I_X \mid x \notin \operatorname{dom}(f) \}.$$

### Subbasis for $\mathcal{I}_3$

$$U_{x,y} = \{ f \in I_X \mid (x)f = y \}$$

$$\text{and} \quad W_x^{-1} = \{ f \in I_X \mid x \notin \operatorname{im}(f) \}.$$

# The Topologies $\mathcal{I}_2$ and $\mathcal{I}_3$

**Theorem (Elliott et al., 2023)**

*Let $X$ be an infinite set. Then the following hold:*

1. *$\mathcal{I}_2$ and $\mathcal{I}_3$ are distinct;*
2. *$I_X$ with $\mathcal{I}_2$ and $I_X$ with $\mathcal{I}_3$ are homeomorphic Hausdorff topological semigroups;*
3. *every $T_1$ semigroup topology for $I_X$ contains $\mathcal{I}_2$ or $\mathcal{I}_3$;*
4. *if $X$ is countable, then $\mathcal{I}_2$ and $\mathcal{I}_3$ are Polish.*

However, neither $\mathcal{I}_2$ nor $\mathcal{I}_3$ are inverse semigroup topologies!

# The Topology $\mathcal{I}_4$

The topology $\mathcal{I}_4$, which is generated by $\mathcal{I}_2 \cup \mathcal{I}_3$, is the unique Polish inverse semigroup topology on $I_X$.

### Subbasis for $\mathcal{I}_4$

$$U_{x,y} = \{\, f \in I_X \mid (x)f = y \,\}\,, \quad W_x = \{\, f \in I_X \mid x \notin \mathrm{dom}(f) \,\}\,,$$

$$\text{and} \quad W_x^{-1} = \{\, f \in I_X \mid x \notin \mathrm{im}(f) \,\}\,.$$

# The Topology $\mathcal{I}_4$

**Theorem (Elliott et al., 2023)**

*Let $X$ be an infinite set. Then the following hold:*

1. *the topology $\mathcal{I}_4$ is a Hausdorff inverse semigroup topology for $I_X$;*
2. *if $\mathcal{T}$ is a semigroup topology for $I_X$ and $\mathcal{T}$ induces the pointwise topology on $\mathrm{Sym}(X)$, then $\mathcal{T}$ is contained in $\mathcal{I}_4$;*
3. *$\mathcal{I}_4$ is the unique $T_1$ inverse semigroup topology on $I_X$ inducing the pointwise topology on $\mathrm{Sym}(X)$;*
4. *if $X$ is countable, then $\mathcal{I}_4$ is the unique Polish inverse semigroup topology on $I_X$.*

**Definition (Left Semitopological Semigroup)**

Semigroup $S$ with topology $\mathcal{T}$ such that for all $s \in S$ the left multiplication map $\lambda_s$ is continuous.

$$\lambda_s : S \mapsto sS : x \mapsto sx$$

**Definition (Right Semitopological Semigroup)**

Semigroup $S$ with topology $\mathcal{T}$ such that for all $s \in S$ the right multiplication map $\rho_s$ is continuous.

$$\rho_s : S \mapsto Ss : x \mapsto xs$$

**Definition (Semitopological semigroup)**

Semigroup $S$ with topology $\mathcal{T}$ such that $\mathcal{T}$ is left and right semitopological for $S$.

# The Topology $\mathcal{I}_1$

$I_X$ has a minimal $T_1$ semitopological semigroup topology.

## Subbasis for $\mathcal{I}_1$

$$U_{x,y} = \{\, f \in I_X \mid (x)f = y \,\} \quad \text{and} \quad V_{x,y} = \{\, f \in I_X \mid (x)f \neq y \,\}$$

## Theorem (Elliott et al. 2023)

*Let $X$ be an infinite set. Then the following hold:*

1. *the topology $\mathcal{I}_1$ is compact, Hausdorff, and semitopological for $I_X$ and inversion is continuous;*
2. *$\mathcal{I}_1$ is the least $T_1$ topology that is semitopological for $I_X$;*
3. *if $X$ is countable, then $\mathcal{I}_1$ is Polish.*

# Relational Structures

**Definition (Relation)**

An *$n$-ary relation* on a set $X$ is a subset $R \subseteq X^n$.

**Definition (Relational structure)**

A *relational structure* $\mathcal{R}$ is a set of relations on a common set $X$.

**Theorem (P. J. Cameron, 1990)**

*A subgroup $G$ of $\mathrm{Sym}(X)$ is closed in the pointwise topology if and only if $G$ is the group $\mathrm{Aut}(\mathcal{R})$ of automorphisms of a relational structure $\mathcal{R}$ on $X$.*

**Theorem (Cameron & Nešetřil, 2006)**

*A submonoid $S$ of $X^X$ is closed in the pointwise topology if and only if $S$ is the monoid $\mathrm{End}(\mathcal{R})$ of endomorphisms of a relational structure $\mathcal{R}$ on $X$.*

# Partial Automorphisms

## Definition (Partial Automorphism)

A chart $f \in I_X$ is a *partial automorphism* of a relational structure $\mathcal{R}$ if it is an isomorphism between induced substructures.

## Proposition

*The set* $\mathrm{pAut}(\mathcal{R}) =$
$\{ f \in I_X \mid (\forall R \in \mathcal{R})(\forall a \in \mathrm{dom}(f)^{\dim(R)}) \quad (a)f \in R \Leftrightarrow a \in R \}$
*of all partial automorphisms of a relational structure* $\mathcal{R}$ *is a full inverse submonoid of* $I_X$*.*

## Definition (Full Subsemigroup)

Given a semigroup $S$, a subsemigroup $T \subseteq S$ is called *full* if $T$ contains all the idempotents of $S$.

# Partial Automorphisms Theorem

Restricting to the full submonoids of $I_X$ yields a result analogous to those of Cameron and Nešetřil.

**Theorem**

*Let $X$ be an infinite set and $M$ a full inverse submonoid of $I_X$. Then the following are equivalent:*

1. *$M$ is closed in the topology $\mathcal{I}_1$;*
2. *$M$ is closed in the topology $\mathcal{I}_4$;*
3. *There exists a relational structure $\mathcal{R}$ on $X$ such that $M = \mathrm{pAut}(\mathcal{R})$.*

The topologies $\mathcal{I}_1$ and $\mathcal{I}_4$ coincide on these submonoids!

# Partial Automorphisms Proof

*Proof of $M \subseteq \mathrm{pAut}(\mathcal{R})$:* For each $n \in \omega$ and each $a \in X^n$ we take an $n$-ary relation $R_a$ defined by

$$R_a = \{ (a)f \in X^n \mid f \in M \wedge a \in \mathrm{dom}(f)^n \}.$$

Let $\mathcal{R} = \{ R_a \mid a \in \bigcup_{n\in\omega} X^n \}$. Take $f \in M$ and $b \in \mathrm{dom}(f)^n$. Then $(b)f \in R_a$ if and only if $b \in R_a$, since $b \in R_a$ implies there exists $g \in M$ such that $(a)g = b$ (then $(b)f = (a)gf$) and $M$ is inverse (so if $b \notin R_a$, then $a \notin R_b$).

$\square$

# Closed submonoids in $\mathcal{I}_1$

This makes a strong case for studying specifically the full inverse submonoids of $I_X$. The following result further enforces this idea.

**Proposition**

*Let $X$ be a countable set, $E_X \subseteq I_X$ the semilattice of idempotents, and $S \subseteq I_X$ a subsemigroup closed in the topology $\mathcal{I}_1$. Then $S' = \langle S, E_X \rangle$ is also closed in $\mathcal{I}_1$.*

The closed inverse subsemigroups if $I_X$ in the topology $\mathcal{I}_1$ are equivalent to the monoids of partial automorphisms modulo the semilattice of idempotents.

**Definition (Relative Rank)**

Given a semigroup $S$ and $A \subseteq S$, then the *relative rank* of $A$ in $S$, denoted $\operatorname{rank}(S : A)$, is the minimum cardinality of a set $B \subseteq S$ such that $\langle A, B \rangle = S$.

**Theorem (Galvin, 1995)**

*If* $G \subseteq \operatorname{Sym}(X)$*, then either* $\operatorname{rank}(\operatorname{Sym}(X) : G) \leq 2$ *or* $\operatorname{rank}(\operatorname{Sym}(X) : G) > |X|$*.*

**Theorem (Sierpiński, 1935)**

*If* $S \subseteq X^X$*, then either* $\operatorname{rank}(X^X : S) \leq 2$ *or* $\operatorname{rank}(X^X : S) > |X|$*.*

**Theorem (Hyde & Péresse, 2012)**

*If* $M \subseteq I_X$*, then either* $\operatorname{rank}(I_X : M) \leq 2$ *or* $\operatorname{rank}(I_X : M) > |X|$*.*

# Relative Rank Theorems

The following two results on relative rank hold when $X$ is a countably infinite set.

**Theorem (Bergman & Shelah, 2006)**

*Let $G$ be a closed subgroup of $\mathrm{Sym}(X)$ in the pointwise topology. Then $G$ has finite relative rank in $\mathrm{Sym}(X)$ if and only if for all finite $\Gamma \subseteq X$ the pointwise stabilizer subgroup $G_{(\Gamma)}$ has at least one infinite orbit in $X$.*

**Theorem**

*Let $\mathcal{R}$ be a relational structure on $X$ such that $|\mathcal{R}|$ is finite. Then $\mathrm{pAut}(\mathcal{R})$ has finite relative rank in $I_X$.*

# Rooted Trees

- Almost disjoint families.
- Semigroups of path bijections on rooted trees.
- Trivial $\mathcal{H}$-classes.
- Used for counter-examples.
- Easy to visualise.

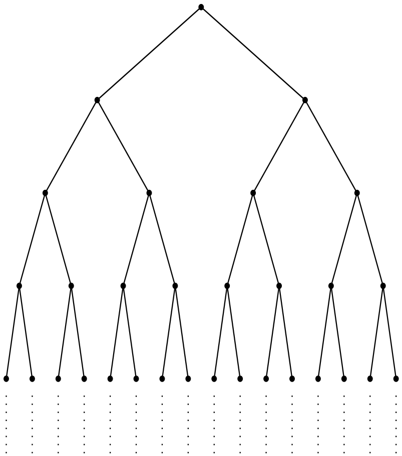
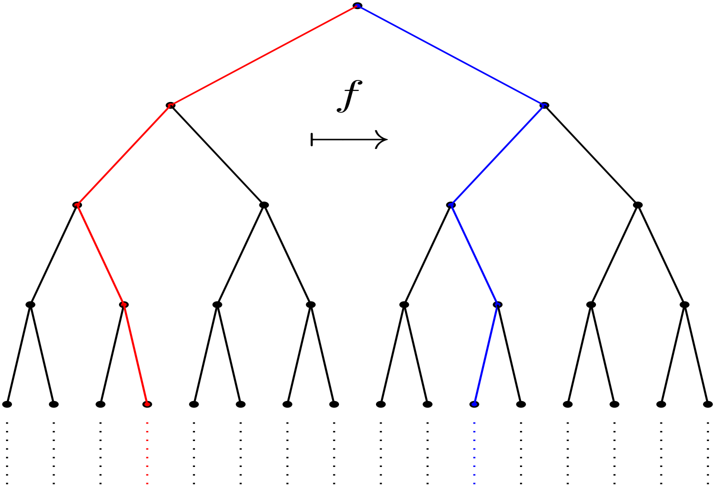


Figure: Binary rooted tree $T^{(2)}$

# Semigroups on Trees

$$\mathcal{T} = \{\, f \in \mathrm{pAut}(T) \mid \mathrm{dom}(f), \mathrm{im}(f) \in P \text{ and } (r)f = r \,\}$$



# Infinitary Rooted Trees

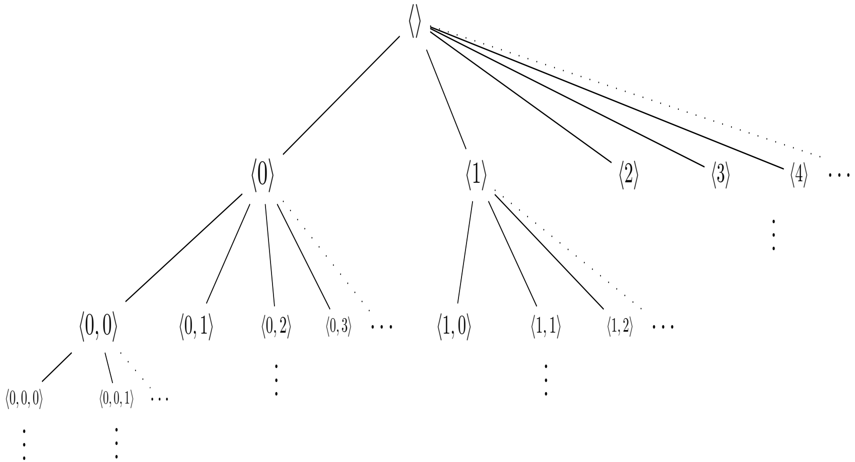



Figure: Infinite rooted tree $T^{(\aleph_0)}$.

# Classification of Trees

All rooted trees can be sorted into 3 distinct classes.

**Theorem**

*Let $T$ be a rooted tree on a countable set $X$ with root $r \in X$ and $P$ the set of all paths in $T$ starting at $r$. Then one of the following holds:*

1. *$T$ contains a subtree for which every vertex has a descendent of infinite degree and $\mathcal{T} \approx I_X$.*
2. *$T$ does not satisfy* (i) *but contains a subtree for which every vertex has a descendent of degree at least 3 and $\mathcal{T} \approx \mathcal{T}^{(2)}$.*
3. *$P$ and $\mathcal{T}$ are countable.*