FREE IDEMPOTENT GENERATED SEMIGROUPS OVER BANDS

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Abstract. We study the general structure of the free idempotent generated semigroup $\mathrm{IG}(B)$ over an arbitrary band B. We show that $\mathrm{IG}(B)$ is *always* a weakly abundant semigroup with the congruence condition, but not necessarily abundant. We then prove that if B is a normal band or a quasi-zero band for which $\mathrm{IG}(B)$ satisfies Condition (P), then $\mathrm{IG}(B)$ is an abundant semigroup. In consequence, if Y is a semilattice, then $\mathrm{IG}(Y)$ is adequate, that is, it belongs to the quasivariety of unary semigroups introduced by Fountain over 30 years ago. Further, the word problem of $\mathrm{IG}(B)$ is solvable if B is quasi-zero. We also construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

1. Introduction

Let S be a semigroup with a set of idempotents $E = E(S)$. The structure of the set E, equipped with the restriction of the quasi-orders $\leq_{\mathcal{R}}$ or $\leq_{\mathcal{L}}$ defined on S, can naturally be described as a *biordered set,* a notion arising as a generalisation of the semilattice of idempotents in an inverse semigroup. Moreover, Easdown [6] shows every biordered set E occurs as $E(S)$ for some semigroup S. Given a biordered set E, i.e. the set E of idempotents of some semigroup S, there is a free object in the category of semigroups that are generated by E, called the *free idempotent generated semigroup* over E, given by the following presentation:

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle,$$

where $\overline{E} = \{\bar{e} : e \in E\}$.[1] Note that $\{e, f\} \cap \{ef, fe\} \neq \emptyset$ implies both ef and fe are idempotents of E. Clearly, there is a natural morphism φ from $\mathrm{IG}(E)$ to $\langle E \rangle$, the subsemigroup of S generated by E. In fact, $E(\mathrm{IG}(E)) = \overline{E}$, and the restriction $\varphi|_{\overline{E}} : \overline{E} \longrightarrow E$ is an isomorphism of biordered sets [6]. We refer our readers to [14] for other properties of $\mathrm{IG}(E)$.

Given the universal nature of free idempotent generated semigroups, it is natural to enquire into their structure. A popular theme is to investigate their maximal subgroups. It has been proved that *every* group is a maximal subgroup of $\mathrm{IG}(E)$ for some biordered set E [13,14] and E may be taken to be a band [5].

Whereas much of the former work in the literature of $\mathrm{IG}(E)$ has focused on the maximal subgroups, the aim of the current paper is to investigate the general structure of $\mathrm{IG}(B)$ for a band B. Our main result is that for an arbitrary band B, $\mathrm{IG}(B)$ is a weakly abundant semigroup with the congruence condition.

We proceed as follows. In Section 2 we recall some basics of Green's relations and regular semigroups, and of generalised Green's relations and (weakly) abundant semigroups. We briefly describe how $\mathrm{IG}(B)$ naturally induces a reduction system. In Section 3, we begin our

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[1] It is more usual to identify elements of E with those of $\overline{E}$, but it helps the clarity of our later arguments to make this distinction.

investigation of IG(B) by looking at a semilattice Y. We prove that every element of IG(Y) has a unique normal form. Consequently, we show that IG(Y) is abundant, and hence adequate. We remark here that this result can be obtained as a corollary of Proposition 7.2, however, the straightforward proof makes clear the strategies we subsequently use in other contexts. In Section 4, we show that for any rectangular band B, IG(B) is regular. We then proceed to look at a general band B in Section 5. Unlike the case of semilattices and rectangular bands, here we may lose uniqueness of normal forms. To overcome this problem, the concept of *almost normal forms* is introduced. It is proved that for any band B, IG(B) is a weakly abundant semigroup with the congruence condition, but need not be abundant.

We then consider some sufficient condition for IG(B) to be abundant. In Section 6, we introduce a class of bands B, which are in general not normal, for which the word problem of IG(B) is solvable. Then in Section 7, we show that if B is a quasi-zero band or a normal band for which IG(B) satisfies a condition we label (P), then IG(B) is an abundant semigroup. We then find two classes of normal bands satisfying Condition (P). One would naturally ask here that whether IG(B) is abundant for an arbitrary normal band B. In Section 8, we construct a 10-element normal band with 4 $\mathcal{D}$-classes for which IG(B) is not abundant.

2. Preliminaries: (weakly) abundant semigroups and Reduction systems

Throughout this paper, for $n \in \mathbb{N}$ we write $[1, n]$ to denote $\{1, \cdots, n\} \subseteq \mathbb{N}$. The free semigroup on a set A is denoted by A^+.

We start by introducing an important tool for analysing ideals of a semigroup S and related notions of structure, called *Green's relations*. There are five equivalence relations that characterize the elements of S in terms of the principal ideals they generate. The two most basic of Green's relations are $\mathcal{L}$ and $\mathcal{R}$, are defined by

$$a \; \mathcal{L} \; b \Longleftrightarrow S^1 a = S^1 b, a \; \mathcal{R} \; b \Longleftrightarrow aS^1 = bS^1,$$

where S^1 denotes S with an identity element adjoined (unless S already has one.) Furthermore, we denote the intersection $\mathcal{L} \cap \mathcal{R}$ by $\mathcal{H}$, while, the join $\mathcal{L} \vee \mathcal{R}$ is denoted by $\mathcal{D}$. It is known that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, and hence $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

An element $a \in S$ is called *regular* if there exists $x \in S$ such that $a = axa$. A semigroup S is called a *regular semigroup* if contains entirely of regular elements. We say that S is *inverse* if it is regular and the set of all idempotents of S forms a semilattice. It is well known that S is regular (inverse) if and only if each $\mathcal{L}$-class ($\mathcal{R}$-class) contains a (unique) idempotent. The definition of regular semigroups is copied from von Neumann's definition of regular rings; regular semigroups are particularly amenable to analysis using Green's relations.

As a generalisation of Green's relations, the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are defined on a semigroup S by the rule that

$$a \; \mathcal{L}^* \; b \iff (\forall x, y \in S^1) \; (ax = ay \Leftrightarrow bx = by)$$

and

$$a \; \mathcal{R}^* \; b \iff (\forall x, y \in S^1) \; (xa = ya \Leftrightarrow xb = yb).$$

It is easy to see that $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$, and if S is regular, then $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$. We denote by $\mathcal{H}^*$ the intersection $\mathcal{L}^* \cap \mathcal{R}^*$, and by $\mathcal{D}^*$ the join of $\mathcal{L}^* \vee \mathcal{R}^*$. Note that unlike Green's relations, generally $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. A binary relation $\mathcal{J}^*$ may also be defined on S, which is not required here.

A semigroup S is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. An abundant semigroup is *adequate* if its idempotents form a semilattice. In view of the comment above, regular semigroups are abundant while inverse semigroups are adequate. In the theory of abundant semigroups the relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$, $\mathcal{D}^*$ and $\mathcal{J}^*$ play a role which is analogous to that of Green's relations in the theory of regular semigroups.

As an easy but useful consequence of the definition of $\mathcal{L}^*$, we have the following lemma (a dual result holds for $\mathcal{R}^*$).

Lemma 2.1. *[9] Let S be a semigroup with $a \in S$ and $e^2 = e \in E(S)$. Then the following statements are equivalent:*
(i) $a\ \mathcal{L}^*\ e$;
(ii) $ae = a$ and for any $x, y \in S^1$, $ax = ay$ implies $ex = ey$.

A third set of relations, extending the stared versions of Green's relations, and useful for semigroups that are not abundant, were introduced in [20].

Let $E(S)$ be the set of idempotents of a semigroup S. The relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ on a semigroup S are defined by the rule

$$a\ \widetilde{\mathcal{L}}\ b \iff (\forall e \in E(S))\ (ae = a \Leftrightarrow be = b)$$

and

$$a\ \widetilde{\mathcal{R}}\ b \iff (\forall e \in E(S))\ (ea = a \Leftrightarrow eb = b)$$

for any $a, b \in S$.

We remark here that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$. Moreover, if S is regular, then $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R} = \mathcal{R}^* = \widetilde{\mathcal{R}}$

A semigroup S is *weakly abundant* if each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$-class contains an idempotent. Whereas $\mathcal{L}^*$ and $\mathcal{R}^*$ are always right and left congruences on S, respectively, the same is not necessarily true for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$. We say that a weakly abundant semigroup S satisfies the *congruence condition* if $\widetilde{\mathcal{L}}$ is a right congruence and $\widetilde{\mathcal{R}}$ is a left congruence.

The following lemma is an analogue of Lemma 2.1. Of course, a dual result holds for $\widetilde{\mathcal{R}}$.

Lemma 2.2. *[20] Let S be a semigroup with $a \in S$ and $e^2 = e \in E(S)$. Then the following statements are equivalent:*
(i) $a\ \widetilde{\mathcal{L}}\ e$;
(ii) $ae = a$ and for any $f \in E(S)$, $af = a$ implies $ef = e$.

From easy observation, we have the following lemma.

Lemma 2.3. *Let S be a semigroup with $e, f \in E(S)$. Then $e\ \mathcal{L}\ f$ if and only if $e\ \widetilde{\mathcal{L}}\ f$ and $e\ \mathcal{R}\ f$ if and only if $e\ \widetilde{\mathcal{R}}\ f$.*

Lemma 2.4. *Let S be a semigroup, and let $a \in S$, $f^2 = f \in E(S)$ be such that $a\ \widetilde{\mathcal{R}}\ f$. Then a is not $\mathcal{R}^*$-related to f implies that a is not $\mathcal{R}^*$-related to any idempotent of S.*

Proof. Suppose that $a\ \mathcal{R}^*\ e$ for some idempotent $e \in E(S)$. Then $a\ \widetilde{\mathcal{R}}\ e$, as $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$, so that $e\ \widetilde{\mathcal{R}}\ f$ by assumption, and so $e\ \mathcal{R}\ f$ by Lemma 2.3. Hence $a\ \mathcal{R}^*\ f$ as $\mathcal{R} \subseteq \mathcal{R}^*$, a contradiction. □

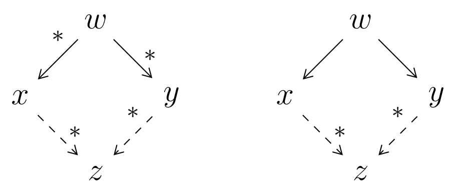
Lemma 2.5. *Let S be a weakly abundant semigroup with $a \in S$ and $e^2 = e \in E(S)$ such that $a \, \widetilde{\mathcal{R}} \, e$. Then $a \, \mathcal{R}^* \, e$ if and only if for any $x, y \in S$, $xa = ya$ implies that $xe = ye$.*

Proof. Suppose that for all $x, y \in S$, if $xa = ya$ then $xe = ye$. By Lemma 2.1, we need only show that if $x \in S$ and $xa = a$, then $xe = e$. Suppose therefore that $x \in S$ and $xa = a$. As $a \, \widetilde{\mathcal{R}} \, e$, we have $xa = a = ea$, so that by assumption, $xe = ee = e$. □

In the rest of this section we recall the definition of reduction systems and their properties. As far as possible we follow standard notation and terminology, as may be found in [1].

Let A be a set of objects and $\longrightarrow$ a binary relation on A. We call the structure $(A, \longrightarrow)$ a *reduction system* and the relation $\longrightarrow$ a *reduction relation*. The reflexive, transitive closure of $\longrightarrow$ is denoted by $\xrightarrow{*}$, while $\xleftrightarrow{*}$ denotes the smallest equivalence relation on A that contains $\longrightarrow$. We denote the equivalence class of an element $x \in A$ by $[x]$. An element $x \in A$ is said to be *irreducible* if there is no $y \in A$ such that $x \longrightarrow y$; otherwise, x is *reducible.* For any $x, y \in A$, if $x \xrightarrow{*} y$ and y is irreducible, then y is a *normal form* of x. A reduction system $(A, \longrightarrow)$ is *noetherian* if there is no infinite sequence $x_0, x_1, \cdots \in A$ such that for all $i \geq 0$, $x_i \longrightarrow x_{i+1}$.

We say that a reduction system $(A, \longrightarrow)$ is *confluent* if whenever $w, x, y \in A$, are such that $w \xrightarrow{*} x$ and $w \xrightarrow{*} y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure below on the left, and $(A, \longrightarrow)$ is *locally confluent* if whenever $w, x, y \in A$, are such that $w \longrightarrow x$ and $w \longrightarrow y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure below on the right.



Lemma 2.6. *[1] Let $(A, \longrightarrow)$ be a reduction system. Then the following statements hold:*
(i) If $(A, \longrightarrow)$ is noetherian and confluent, then for each $x \in A$, $[x]$ contains a unique normal form.
(ii) If $(A, \longrightarrow)$ is noetherian, then it is confluent if and only if it is locally confluent.

Let E be a biordered set. We use $\overline{E}^+$ to denote the free semigroup on $\overline{E} = \{\overline{e} : e \in E\}$.

Lemma 2.7. *Let $\mathrm{IG}(E)$ be the free idempotent generated semigroup over a biordered set E, and let R be the relation on $\overline{E}^+$ defined by*

$$R = \{(\bar{e}\bar{f}, \overline{ef}) : \ (e, f) \textit{ is a basic pair}\}.$$

Then $(\overline{E}^+, \longrightarrow)$ forms a noetherian reduction system, where $\longrightarrow$ is defined by

$$u \longrightarrow v \Longleftrightarrow (\exists \ (l, r) \in R) \ (\exists \ x, y \in \overline{E}^+) \ u = xly \textit{ and } v = xry.$$

Proof. The proof follows directly from the definitions of the reduction system and the binary relation $\longrightarrow$. □

We remark here that in the reduction system $(\overline{E}^+, \longrightarrow)$ induced by $\mathrm{IG}(E)$, the smallest equivalence relation $\xleftrightarrow{*}$ on $\overline{E}^+$ is exactly the congruence generated by R.

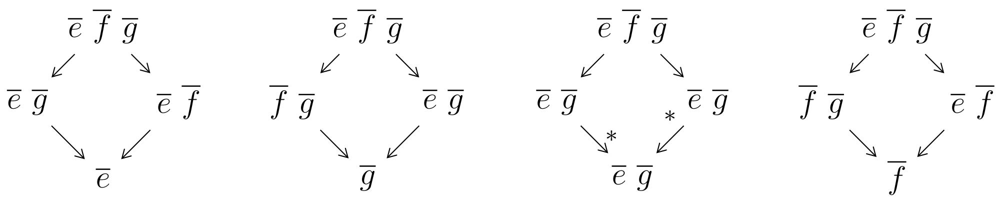
3. Free idempotent generated semigroups over semilattices

We start our investigation of free idempotent generated semigroups IG(B) over bands B, by looking at the special case of semilattices. Throughout this section we will use the letter Y to denote a semilattice. It is proved that IG(Y) is an adequate semigroup; however, it need not be regular.

It follows from Lemma 2.7 that IG(Y) induces naturally a noetherian reduction system $(\overline{Y}^{+}, \longrightarrow)$.

Lemma 3.1. *Let Y be a semilattice. Then every element in* IG(Y) *has a unique normal form.*

Proof. By Lemma 2.6, to show the required result we only need to argue that $(\overline{Y}^{+}, \longrightarrow)$ is locally confluent. For this purpose, it is sufficient to consider an arbitrary word of length 3, say $\overline{e}\ \overline{f}\ \overline{g} \in \overline{Y}^{+}$, where e, f and f, g are comparable. There are four cases, namely, $e \leq f \leq g$, $e \geq f \geq g$, $e \leq f, f \geq g$ and $e \geq f, f \leq g$, for which we have the following 4 diagrams:



Thus $(\overline{Y}^{+}, \longrightarrow)$ is locally confluent, so that every element in IG(Y) has a unique normal form. $\square$

Note that an element $\overline{x_1}\ \cdots\ \overline{x_n} \in$ IG(Y) is in normal form if and only if x_i and x_{i+1} are incomparable, for all $i \in [1, n-1]$. By uniqueness of normal forms in IG(Y), we can easily deduce that two words of IG(B) are equal if and only the corresponding normal forms of them are identical word in $\overline{E}^{+}$, and hence the word problem of IG(Y) is solvable.

Proposition 3.2. *The free idempotent generated semigroup* IG(Y) *over a semilattice Y is an abundant semigroup.*

Proof. Let $\overline{x_1} \cdots\ \overline{x_n}, \overline{y_1} \cdots\ \overline{y_m} \in$ IG(B) be in normal form. We begin with considering the product $(\overline{x_1} \cdots\ \overline{x_n})(\overline{y_1} \cdots\ \overline{y_m})$. Either x_n, y_1 are incomparable, $x_n \geq y_1$ or $x_n \leq y_1$. In the first case it is clear that $\overline{x_1} \cdots\ \overline{x_n}\ \overline{y_1} \cdots\ \overline{y_m}$ is a normal form. If $x_n \geq y_1$, then either $\overline{x_1} \cdots\ \overline{x_{n-1}}\ \overline{y_1} \cdots\ \overline{y_m}$ is in normal form, or y_1 and x_{n-1} are comparable. If y_1 and x_{n-1} are comparable, then $y_1 < x_{n-1}$, for we cannot have $x_{n-1} \leq y_1$ else $x_{n-1} \leq x_n$, a contradiction. Continuing in this manner we obtain $(\overline{x_1} \cdots\ \overline{x_n})(\overline{y_1} \cdots\ \overline{y_m})$ has normal form $\overline{x_1} \cdots\ \overline{x_{t-1}}\ \overline{y_1} \cdots\ \overline{y_m}$, where $1 \leq t \leq n$, $x_n, \cdots, x_t \geq y_1$, and either $t = 1$ (in which case $\overline{x_1}\ \cdots\ \overline{x_{t-1}}$ is the empty product) or x_{t-1}, y_1 are incomparable. Similarly, if $x_n \leq y_1$, then $(\overline{x_1} \cdots\ \overline{x_n})(\overline{y_1} \cdots\ \overline{y_m})$ has normal form $\overline{x_1} \cdots\ \overline{x_n}\ \overline{y_{t+1}} \cdots\ \overline{y_m}$, where $1 \leq t \leq m$, $x_n \leq y_1, \cdots y_t$, and $t = m$ or x_n, y_{t+1} are incomparable.

Suppose now that $\overline{x_1} \cdots\ \overline{x_n}, \overline{z_1} \cdots\ \overline{z_k}$ and $\overline{y_1} \cdots\ \overline{y_m} \in$ IG(Y) are in normal form such that

$$\overline{x_1} \cdots\ \overline{x_n}\ \overline{y_1} \cdots\ \overline{y_m} = \overline{z_1} \cdots\ \overline{z_k}\ \overline{y_1} \cdots\ \overline{y_m}$$

in IG(Y). Here we assume $n, k \geq 0$ and $m \geq 1$. We proceed to prove that

$$\overline{x_1} \cdots\ \overline{x_n}\ \overline{y_1} = \overline{z_1} \cdots\ \overline{z_k}\ \overline{y_1}$$

in IG(Y). If $n = k = 0$ there is nothing to show. Note that the result is clearly true if $m = 1$, so in what follows we assume $m \geq 2$.

First we assume that $n \geq 1$ and $k = 0$ (i.e. $\overline{z_1} \cdots \overline{z_k}$ is empty), so that

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} \cdots \overline{y_m} = \overline{y_1} \cdots \overline{y_m}.$$

In view of Lemma 3.1, x_n and y_1 must be comparable. If $x_n \geq y_1$, then it follows from the above observation that $y_1 \leq x_1, \cdots, x_n$, so that $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{y_1}$. On the other hand, if $x_n \leq y_1$, then

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_{t+1}} \cdots \overline{y_m} = \overline{y_1} \cdots \overline{y_m}$$

for $1 \leq t \leq m$ such that $x_n \leq y_1, \cdots, y_t$ and $t = m$ or x_n, y_{t+1} are incomparable. Then $x_n = y_t$, so that to avoid the contradiction $y_t \leq y_{t-1}$ we must have $t = 1$. Clearly then $n = 1$ and $x_1 = x_n = y_1$ so that $\overline{x_1}\, \overline{y_1} = \overline{y_1}$. Hence certainly holds for $n + k + m \leq 3$.

Suppose that $n + k + m \geq 4$ and the result is true for all $n' + k' + m' < n + k + m$. Recall we are assuming that $m \geq 2$ and in view of the above we may take $n, k \geq 1$.

If x_n, y_1 and z_k, y_1 are incomparable pairs, then it follows from uniqueness of normal form that $k = n$ and $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$.

Suppose now that $y_1 \leq x_n$. Then

$$\overline{x_1} \cdots \overline{x_{n-1}}\, \overline{y_1} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1} \cdots \overline{y_m}$$

so that our induction gives us

$$\overline{x_1} \cdots \overline{x_{n-1}}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$$

and hence $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$. A similar result holds for the case $y_1 \leq z_k$.

Suppose now that $y_1 \nleq x_n$ and $y_1 \nleq z_k$ and at least one of x_n, y_1 or z_k, y_1 are comparable. Without loss of generality assume that $x_n < y_1$. As above $x_n \leq y_1, \cdots, y_t$ for some $1 \leq t \leq m$ with $t = m$ or x_n, y_{t+1} incomparable. Further, there is an r with $0 \leq r \leq m$ such that $z_k \leq y_1, \cdots, y_r$ and $r = m$ or z_k, y_{r+1} incomparable. Thus both sides of

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_{t+1}} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_{r+1}} \cdots \overline{y_m}$$

are in normal form and so $n - t = k - r$. If $n > k$, then $r < t$, so $x_n = y_t$. To avoid the contradiction $y_t \leq y_{t-1}$, we must have $t = 1$, but then $x_n = y_1$ a contradiction. Similarly, we can not have $k > n$. Hence $n = k$, and hence $\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_k}$, so that certainly $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$ as required. □

We remark here that Proposition 3.2 can also be obtained as a corollary of Proposition 7.2, but for the sake of our readers, we have proved this special case to outline our strategy in a simple case.

Corollary 3.3. *The free idempotent generated semigroup* IG(Y) *over a semilattice* Y *is an adequate semigroup.*

Proof. We have already remarked in the beginning of Introduction that the biordered set of idempotents of IG(Y) is isomorphic to Y, which is a semilattice, so that IG(Y) is an adequate semigroup. □

Example 3.4. Consider a semilattice $Y = \{e, f, g\}$ with $e, f \geq g$ and e, f incomparable.

First, we observe that

$$\mathrm{IG}(Y) = \{\overline{e}, \overline{f}, \overline{g}, (\overline{e}\ \overline{f})^n, (\overline{f}\ \overline{e})^n, (\overline{e}\ \overline{f})^n\ \overline{e}, (\overline{f}\ \overline{e})^n\ \overline{f} :\ n \in \mathbb{N}\}.$$

It is easy to check that for any $n \in \mathbb{N}$, $(\overline{e}\ \overline{f})^n \in \mathrm{IG}(Y)$ is not regular, as for any $w \in \mathrm{IG}(Y)$, $(\overline{e}\ \overline{f})^n w (\overline{e}\ \overline{f})^n = \overline{g}$ if w contains $\overline{g}$ as a letter; otherwise $(\overline{e}\ \overline{f})^n w (\overline{e}\ \overline{f})^n = (\overline{e}\ \overline{f})^m$ for some $m \geq 2n \in \mathbb{N}$. Therefore, $\mathrm{IG}(Y)$ is not a regular semigroup.

On the other hand, by Proposition 3.2 we have that $\mathrm{IG}(Y)$ is an abundant semigroup. Furthermore,

$$\mathcal{R}^* = \{\{\overline{e}, (\overline{e}\ \overline{f})^n, (\overline{e}\ \overline{f})^n\ \overline{e} :\ n \in \mathbb{N}\}, \{\overline{f}, (\overline{f}\ \overline{e})^n, (\overline{f}\ \overline{e})^n\ \overline{f} :\ n \in \mathbb{N}\}, \{\overline{g}\}\}$$

and

$$\mathcal{L}^* = \{\{\overline{e}, (\overline{f}\ \overline{e})^n, (\overline{e}\ \overline{f})^n\ \overline{e} :\ n \in \mathbb{N}\}, \{\overline{f}, (\overline{e}\ \overline{f})^n, (\overline{f}\ \overline{e})^n\ \overline{f} :\ n \in \mathbb{N}\}, \{\overline{g}\}\}$$

Note that we have

$$\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$$

in $\mathrm{IG}(Y)$, *and there are two* $\mathcal{D}^*$*-classes of* $\mathrm{IG}(Y)$, *namely,*

$$\{\overline{g}\}\ \textit{and}\ \{\overline{e}, (\overline{e}\ \overline{f})^n, (\overline{e}\ \overline{f})^n\ \overline{e}, \overline{f}, (\overline{f}\ \overline{e})^n, (\overline{f}\ \overline{e})^n\ \overline{f} : n \in \mathbb{N}\},$$

the latter of which can be depicted by the following so called egg-box picture:

$\overline{e}, (\overline{e}\ \overline{f})^n\overline{e}$	$(\overline{e}\ \overline{f})^n$
$(\overline{f}\ \overline{e})^n$	$\overline{f}, (\overline{f}\ \overline{e})^n\overline{f}$

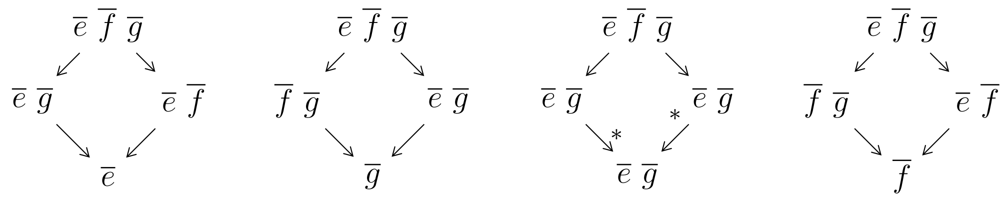
4. Free idempotent generated semigroups over rectangular bands

In this section we are concerned with the free idempotent generated semigroup $\mathrm{IG}(B)$ over a rectangular band B. Recall from [16] that a band B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, and the B_α's are the $\mathcal{D} = \mathcal{J}$-classes of B. Thus $B = \bigcup_{\alpha \in Y} B_\alpha$ where each B_α is a rectangular band and $B_\alpha B_\beta \subseteq B_{\alpha\beta}$, $\forall \alpha, \beta \in Y$. At times we will use this notation without specific comments. We show that $\mathrm{IG}(B)$ is a regular semigroup. It follows that if B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, then any word in $\overline{B_\alpha}^+$ is regular in $\mathrm{IG}(B)$.

Lemma 4.1. *Let B be a rectangular band. Then every element in* $\mathrm{IG}(B)$ *has a unique normal form.*

Proof. We have already remarked that the reduction system $(\overline{B}^+, \longrightarrow)$ induced by $\mathrm{IG}(B)$ is noetherian, so that according to Lemma 2.6, to show the uniqueness of normal form of elements in $\mathrm{IG}(B)$, we only need to prove that $(\overline{B}^+, \longrightarrow)$ is locally confluent.

For this purpose, it is sufficient to consider an arbitrary word of length 3, say $\overline{e}\ \overline{f}\ \overline{g} \in \overline{Y}^+$, where e, f and f, g are comparable. Clearly, there are four cases, namely, $e\ \mathcal{L}\ f\ \mathcal{L}\ g$, $e\ \mathcal{R}\ f\ \mathcal{R}\ g$, $e\ \mathcal{L}\ f\ \mathcal{R}\ g$ and $e\ \mathcal{R}\ f\ \mathcal{L}\ g$. Then we have the following 4 diagrams:



Hence (B^*, R) is locally confluent. $\square$

Lemma 4.2. *Suppose that B is a rectangular band and $\overline{u_1}\ \cdots\ \overline{u_n} \in \mathrm{IG}(B)$. Then we have $\overline{u_n}\ \mathcal{L}\ \overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}$, and hence $\mathrm{IG}(B)$ is a regular semigroup.*

Proof. Let $w = \overline{u_1}\ \cdots\ \overline{u_n} \in \mathrm{IG}(B)$. First we claim that

$$\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}\ \cdots\ \overline{u_{n-1}}.$$

Observe that (u_n, u_nu_{n-1}) and (u_{n-1}, u_nu_{n-1}) are both basic pairs. Hence we have

$$\begin{aligned}\overline{u_1}\ \cdots\ \overline{u_{n-1}}\ \overline{u_n}\ \overline{u_nu_{n-1}} &= \overline{u_1}\ \cdots\ \overline{u_{n-1}}\ \overline{u_nu_nu_{n-1}}\\ &= \overline{u_1}\ \cdots\ \overline{u_{n-1}}\ \overline{u_nu_{n-1}}\\ &= \overline{u_1}\ \cdots\ \overline{u_{n-1}u_nu_{n-1}}\\ &= \overline{u_1}\ \cdots\ \overline{u_{n-1}},\end{aligned}$$

so that $\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}\ \cdots\ \overline{u_{n-1}}$. By finite induction we obtain that $\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}$.

Similarly, we can show that $\overline{u_1}\ \overline{u_2}\ \cdots\ \overline{u_n}\ \mathcal{L}\ \overline{u_n}$. Certainly then $\mathrm{IG}(B)$ is regular. $\square$

Corollary 4.3. *Let B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then for any $x_1, \cdots, x_n \in B_\alpha$, $\overline{x_1}\ \cdots\ \overline{x_n}$ is a regular element of $\mathrm{IG}(B)$.*

Proof. It is clear from the presentations of $\mathrm{IG}(B_\alpha)$ and $\mathrm{IG}(B)$ that there is a well defined morphism

$$\overline{\psi} : \mathrm{IG}(B_\alpha) \longrightarrow \mathrm{IG}(B), \text{ such that } \overline{e}\ \overline{\psi} = \overline{e}$$

for each $e \in B_\alpha$. It follows from Lemma 4.2 that for any $x_1, \cdots, x_n \in B_\alpha$, $\overline{x_1}\ \cdots\ \overline{x_n}$ is regular in $\mathrm{IG}(B_\alpha)$. Since $\overline{\psi}$ preserves the regularity, we have that $(\overline{x_1}\ \cdots\ \overline{x_n})\ \overline{\psi} = \overline{x_1}\ \cdots\ \overline{x_n}$ is regular in $\mathrm{IG}(B)$. $\square$

5. Free idempotent generated semigroups over bands

Our aim here is to investigate the general structure of $\mathrm{IG}(B)$ for an arbitrary band B. We prove that for any band B, $\mathrm{IG}(B)$ is a weakly abundant semigroup with the congruence condition. However, we demonstrate a band B for which $\mathrm{IG}(B)$ is not abundant.

Lemma 5.1. *Let S and T be semigroups with biordered sets of idempotents $U = E(S)$ and $V = E(T)$, respectively, and let $\theta : S \longrightarrow T$ be a morphism. Then the map from $\overline{U}$ to $\overline{V}$ defined by $\overline{e} \mapsto \overline{e\theta}$, for all $e \in U$, lifts to a well defined morphism $\overline{\theta} : \mathrm{IG}(U) \longrightarrow \mathrm{IG}(V)$.*

Proof. Since θ is a morphism by assumption, we have that (e, f) is a basic pair in U implies $(e\theta, f\theta)$ is a basic pair in V, so that there exists a morphism $\overline{\theta} : \mathrm{IG}(U) \longrightarrow \mathrm{IG}(V)$ defined by $\overline{e}\ \overline{\theta} = \overline{e\theta}$, for all $e \in U$. $\square$

Let B be a band. Write B as a semilattice Y of rectangular bands B_α, $\alpha \in Y$. The mapping θ defined by

$$\theta : B \longrightarrow Y, x \mapsto \alpha$$

where $x \in B_\alpha$, is a morphism with kernel $\mathcal{D}$. Hence, by applying Lemma 5.1 to our band B and the associated semilattice Y, we have the following corollary.

Corollary 5.2. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then a map $\overline{\theta} : \mathrm{IG}(B) \longrightarrow \mathrm{IG}(Y)$ defined by*

$$(\overline{x_1} \ \cdots \ \overline{x_n})\, \overline{\theta} = \overline{\alpha_1} \ \cdots \ \overline{\alpha_n}$$

is a morphism, where $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$.

To proceed further we need the following definition of *left to right significant indices* of elements in $\mathrm{IG}(B)$, where B is a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$.

Let $\overline{x_1} \cdots \ \overline{x_n} \in \overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Then a set of numbers

$$\{i_1, \cdots, i_r\} \subseteq [1, n] \text{ with } i_1 < \cdots < i_r$$

is called the *left to right significant indices* of $\overline{x_1} \cdots \ \overline{x_n}$, if these numbers are picked out in the following manner:
i_1 : the largest number such that $\alpha_1, \cdots, \alpha_{i_1} \geq \alpha_{i_1}$;
k_1 : the largest number such that $\alpha_{i_1} \leq \alpha_{i_1}, \alpha_{i_1+1}, \cdots, \alpha_{k_1}$.

We pause here to remark that $\alpha_{i_1}, \alpha_{k_1+1}$ are incomparable. Because, if $\alpha_{i_1} \leq \alpha_{k_1+1}$, we add 1 to k_1, contradicting the choice of k_1; and if $\alpha_{i_1} > \alpha_{k_1+1}$, then $\alpha_1, \cdots, \alpha_{i_1}, \cdots, \alpha_{k_1} \geq \alpha_{k_1+1}$, contradicting the choice of i_1. Now we continue our process:

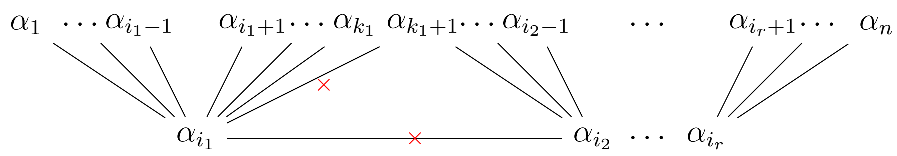
i_2 : the largest number such that $\alpha_{k_1+1}, \cdots, \alpha_{i_2} \geq \alpha_{i_2}$;
k_2 : the largest number such that $\alpha_{i_2} \leq \alpha_{i_2}, \alpha_{i_2+1}, \cdots, \alpha_{k_2}$.
$\vdots$
i_r : the largest number such that $\alpha_{k_{r-1}+1}, \cdots, \alpha_{i_r} \geq \alpha_{i_r}$;
$k_r = n$: here we have $\alpha_{i_r} \leq \alpha_{i_r}, \alpha_{i_r+1}, \cdots, \alpha_n$. Of course, we may have $i_r = k_r = n$.

Corresponding to the so called left to right significant indices $i_1, \cdots, i_r$, we have

$$\alpha_{i_1}, \cdots, \alpha_{i_r} \in Y.$$

We claim that for all $1 \leq s \leq r-1$, α_{i_s} and $\alpha_{i_{s+1}}$ are incomparable. If not, suppose that there exists some $1 \leq s \leq r-1$ such that $\alpha_{i_s} \leq \alpha_{i_{s+1}}$. Then $\alpha_{i_s} \leq \alpha_{k_s+1}$ as $\alpha_{i_{s+1}} \leq \alpha_{k_s+1}$, a contradiction; if $\alpha_{i_s} \geq \alpha_{i_{s+1}}$, then $\alpha_{i_{s+1}} \leq \alpha_{i_{s+1}}, \alpha_{i_{s+1}-1}, \cdots, \alpha_{k_{s-1}+1}$ with $k_0 = 0$, contradicting our choice of i_s. Therefore, we deduce that $\overline{\alpha_{i_1}} \ \cdots \ \overline{\alpha_{i_r}}$ is the unique normal form of $\overline{\alpha_1} \ \cdots \ \overline{\alpha_n}$ in $\mathrm{IG}(Y)$.

We can use the following *Hasse diagram* to depict the relationship among $\alpha_1, \cdots, \alpha_{i_r}$:



Dually, we can define the *right to left significant indices* $\{l_1, \cdots, l_s\} \subseteq [1,n]$ of the element $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$, where $l_1 < \cdots < l_s$. Note that as $\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}}$ must equal to $\overline{\alpha_{l_1}} \cdots \overline{\alpha_{l_s}}$ in $\overline{B}^+$, we have $r = s$.

Lemma 5.3. *Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with $x_i \in \alpha_i$, for all $i \in [1,n]$, and left to right significant indices $i_1, \cdots, i_r$. Suppose also that $\overline{y_1} \cdots \overline{y_m} \in \overline{B}^+$ with $y_i \in \beta_i$, for all $i \in [1,m]$, and left to right significant indices $l_1, \cdots, l_s$. Then*

$$\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$$

in IG(B) *implies $s = r$ and $\alpha_{i_1} = \beta_{l_1}, \cdots, \alpha_{i_r} = \beta_{l_r}$.*

Proof. It follows from Lemma 5.2 and the discussion above that

$$\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}} = \overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\beta_1} \cdots \overline{\beta_m} = \overline{\beta_{l_1}} \cdots \overline{\beta_{l_s}}$$

in IG(Y). By uniqueness of normal form, we have that $s = r$ and $\alpha_{i_1} = \beta_{l_1}, \cdots, \alpha_{i_r} = \beta_{l_r}$. □

In view of the above observations, we introduce the following notions.

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$, and let $w = \overline{x_1} \cdots \overline{x_n}$ be a word in $\overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $i \in [1,n]$. Suppose that w has left to right significant indices $i_1, \cdots, i_r$. Then we call the natural number r the *Y-length*, and $\alpha_{i_1}, \cdots, \alpha_{i_r}$ the *ordered Y-components* of the equivalence class of w in IG(B).

In all what follows whenever we write $w \sim w'$ for $w, w' \in \overline{B}^+$, we mean that the word w' can be obtained from the word w from a single splitting step or a single squashing step.

Lemma 5.4. *Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with left to right significant indices $i_1, \cdots, i_r$, where $x_i \in B_{\alpha_i}$, for all $i \in [1,n]$. Let $\overline{y_1} \cdots \overline{y_m} \in \overline{B}^+$ be an element obtained from $\overline{x_1} \cdots \overline{x_n}$ from a single step, and suppose that the left to right significant indices of $\overline{y_1} \cdots \overline{y_m}$ are $j_1, \cdots, j_r$. Then for all $l \in [1,r]$, we have*

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l}}\ \overline{u}$$

and $y_{j_l} = u' x_{i_l} u$, where $u' = \varepsilon$ or $u' \in B_\sigma$ with $\sigma \geq \alpha_{i_l}$, and either $u = \varepsilon$, or $u \in B_\delta$ for some $\delta > \alpha_{i_l}$, or $u \in B_{\alpha_{i_l}}$ and there exists $v \in B_\theta$ with $\theta > \alpha_{i_l}$, $vu = u$ and $uv = x_{i_l}$.

Proof. Suppose that we split $x_k = uv$ for some $k \in [1,n]$, where uv is a basic product with $u \in B_\mu$ and $v \in B_\tau$, so that $\alpha_k = \mu\tau$. Then

$$\overline{x_1} \cdots \overline{x_n} \sim \overline{x_1} \cdots \overline{x_{k-1}}\ \overline{u}\ \overline{v}\ \overline{x_{k+1}} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}.$$

If $k < i_l$, then clearly $y_{j_l} = x_{i_l}$ and

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{k-1}}\ \overline{u}\ \overline{v}\ \overline{x_{k+1}} \cdots \overline{x_{i_l}} = \overline{x_1} \cdots \overline{x_{i_l}},$$

so we may take $u = u' = \varepsilon$.

If $k = i_l$, then $\mu\tau = \alpha_{i_l}$. If $\mu \geq \tau$, then $y_{j_l} = v$ and again

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l-1}}\ \overline{u}\ \overline{v} = \overline{x_1} \cdots \overline{x_{i_l}}.$$

As $x_{i_l} = uv\ \mathcal{L}\ v$, we have $y_{j_l} = v = v x_{i_l}$. Also, $x_{i_l} = uv = u y_{j_l}$.

On the other hand, if $\mu < \tau$, then $y_{j_l} = u$. As uv is a basic product, $uv = u = x_{i_l}$ or $vu = u$. If $uv = u = x_{i_l}$, then

$$\overline{y_1} \cdots \overline{y_{j_1}} = \overline{x_1} \cdots \overline{x_{i_l-1}}\ \overline{u} = \overline{x_1} \cdots \overline{x_{i_l}},$$

and $y_{j_l} = u = uv = x_{i_l}$. If $vu = u$, then as $x_k = uv \,\mathcal{R}\, u$ and $u = uvu$,

$$\overline{y_1} \cdots \overline{y_{j_1}} = \overline{x_1} \cdots \overline{x_{i_l-1}}\ \overline{u} = \overline{x_1} \cdots \overline{x_{k-1}}\ \overline{uv}\ \overline{u} = \overline{x_1} \cdots \overline{x_{i_l}}\ \overline{u}$$

and $y_{j_l} = x_{i_l} u$ where $vu = u$. Also,

$$\overline{x_1} \cdots \overline{x_{i_l}} = \overline{x_1} \cdots \overline{x_{k-1}}\ \overline{uv} = \overline{x_1} \ \cdots\ \overline{x_{i_l}}\ \overline{u}\ \overline{v} = \overline{y_1} \ \cdots\ \overline{y_{j_l}}\ \overline{v}$$

and $x_{i_l} = y_{j_l} v$.

Finally, suppose that $k > i_l$. Then it is obviously that $j_l = i_l$, $x_{i_l} = y_{j_l}$ and

$$\overline{y_1} \ \cdots\ \overline{y_{j_l}} = \overline{x_1} \ \cdots\ \overline{x_{i_l}}.$$

□

It follows immediately from Lemma 5.4 that

Corollary 5.5. *Suppose that $\overline{y_1} \cdots \overline{y_m} = \overline{x_1} \ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ with left to right significant indices $j_1, \cdots, j_r$ and $i_1, \cdots, i_r$, respectively, and suppose $x_i \in B_{\alpha_i}$ for all $i \in [1, n]$. Then for all $l \in [1, r]$, we have*

$$\overline{y_1} \ \cdots\ \overline{y_{i_l}} = \overline{x_1} \ \cdots\ \overline{x_{i_l}}\ \overline{u_1}\ \overline{u_2} \cdots \overline{u_s}$$

and $y_{j_l} = u'_s \cdots u'_1 x_{i_l} u_1 \cdots u_s$, where for all $t \in [1, s]$, $u'_t = \varepsilon$ or $u'_t \in B_{\sigma_t}$ for some $\sigma_t \geq \alpha_{i_l}$, and either $u_t = \varepsilon$ or $u_t \in B_{\delta_t}$ for some $\delta_t > \alpha_{i_l}$, or $u_t \in B_{\alpha_{i_l}}$ and there exists $v_t \in B_{\theta_t}$ with $\theta_t > \alpha_{i_l}$ and $v_t u_t = u_t$. Consequently, $\overline{y_1} \ \cdots\ \overline{y_{j_l}} \,\mathcal{R}\, \overline{x_1} \ \cdots\ \overline{x_{i_l}}$, and hence $y_1 \ \cdots\ y_{j_l} \,\mathcal{R}\, x_1 \ \cdots\ x_{i_l}$.

Proof. The proof follows from Lemma 5.4 by finite induction. □

Note that the duals of Lemma 5.4 and Corollary 5.5 hold for right to left significant indices.

From Lemmas 3.1 and 4.1, we know that every element in $\mathrm{IG}(B)$ has a unique normal form, if B is a semilattice or a rectangular band. However, it may not true for an arbitrary band B, even if B is normal. Recall that a normal band

$$B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$$

is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, such that for all $\alpha \geq \beta$ in Y there exists a morphism $\phi_{\alpha,\beta} : B_\alpha \longrightarrow B_\beta$ such that

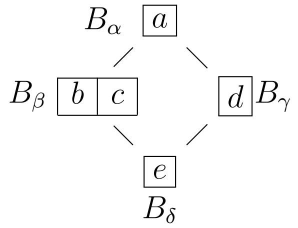
(B1) for all $\alpha \in Y$, $\phi_{\alpha,\alpha} = 1_{B_\alpha}$;

(B2) for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$,

and for all $\alpha, \beta \in Y$ and $x \in B_\alpha, y \in B_\beta$,

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}).$$

Example 5.6. Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice $Y = \{\alpha, \beta, \gamma, \delta\}$ of rectangular bands B_α, $\alpha \in Y$ (see the figure below), such that $\phi_{\alpha,\beta}$ is defined by $a\phi_{\alpha,\beta} = b$, the remaining morphisms being defined in the obvious unique manners.



By an easy calculation, we have

$$\overline{c}\ \overline{d} = \overline{c}\ \overline{ad} = \overline{c}\ \overline{a}\ \overline{d} = \overline{ca}\ \overline{d} = \overline{b}\ \overline{d}$$

in IG(B), so that not every element in IG(B) has a unique normal form.

Lemma 5.7. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. Let $\overline{x_1}\ \cdots\ \overline{x_n} \in$ IG(B) with $x_i \in B_{\alpha_i}$, for all $i \in [1,n]$, and let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $i \in [1,n]$. Then in* IG(B) *we have*

$$\overline{x_1}\ \cdots\ \overline{x_n}\ \overline{y} = \overline{x_1 \cdots x_n y x_n \cdots x_1}\ \cdots\ \overline{x_{n-1} x_n y x_n x_{n-1}}\ \overline{x_n y x_n}\ \overline{y}$$

and

$$\overline{y}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y}\ \overline{x_1 y x_1}\ \overline{x_2 x_1 y x_1 x_2}\ \cdots\ \overline{x_n \cdots x_1 y x_1 \cdots x_n}.$$

Proof. First, we notice that for any $x \in B_\alpha, y \in B_\beta$ such that $\alpha \geq \beta$, we have $yx\ \mathcal{R}\ y$, so that (y, yx) is a basic pair and $(yx)y = y$. On the other hand, as $(yx)x = yx$, we have that (x, yx) is a basic pair, so that

$$\overline{x}\ \overline{y} = \overline{x}\ \overline{(yx)y} = \overline{x}\ \overline{yx}\ \overline{y} = \overline{xyx}\ \overline{y}.$$

Thus, the first required equality follows from the above observation by finite induction. Dually, we can show the second one. □

Corollary 5.8. *Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $\overline{x_1}\ \cdots\ \overline{x_n} \in$ IG(B) be such that $x_i \in B_{\alpha_i}$, for all $i \in [1,n]$. Let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $i \in [1,n]$. Then*

$$\overline{x_1}\ \cdots\ \overline{x_n}\ \overline{y} = \overline{x_1 \phi_{\alpha_1,\beta}}\ \cdots\ \overline{x_n \phi_{\alpha_n,\beta}}\ \overline{y}$$

and

$$\overline{y}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y}\ \overline{x_1 \phi_{\alpha_1,\beta}}\ \cdots\ \overline{x_n \phi_{\alpha_n,\beta}}.$$

Corollary 5.9. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a chain Y of rectangular bands B_α, $\alpha \in Y$. Then* IG(B) *is a regular semigroup.*

Proof. Let $\overline{u_1}\ \cdots\ \overline{u_n}$ be an element in IG(B). From Lemma 5.7 it follows that $\overline{u_1}\ \cdots\ \overline{u_n}$ can be written as an element of IG(B) in which all letters come from B_γ, where γ is the minimum of $\{\alpha_1, \cdots, \alpha_n\}$, so that $\overline{u_1}\ \cdots\ \overline{u_n}$ is regular by Lemma 4.3. □

Given the above observations, we now introduce the idea of *almost normal form* for elements in IG(B).

Definition 5.10. *An element $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ is said to be in almost normal form if there exists a sequence*

$$1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n$$

with

$$\{x_1, \cdots, x_{i_1}\} \subseteq B_{\alpha_1}, \{x_{i_1+1}, \cdots, x_{i_2}\} \in B_{\alpha_2}, \cdots, \{x_{i_{r-1}+1}, \cdots x_n\} \subseteq B_{\alpha_r}$$

where α_i, α_{i+1} are incomparable for all $i \in [1, r-1]$.

It is obvious that the element $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ defined as above has left to right significant indices $i_1, i_2, \cdots, i_{r-1}, i_r = n$ (right to left significant indices $1, i_1+1, \cdots, i_{r-2}+1, i_{r-1}+1$), Y-length r and ordered Y-components $\alpha_1, \cdots, \alpha_r$. Note that, in general, the almost normal forms of elements of IG(B) are not unique. Further, if $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ are in almost normal form, then they have the same Y-length and ordered Y-components, but the left to right significant indices of them can be quite different.

The next result is immediate from the definition of significant indices and Lemma 5.7.

Lemma 5.11. *Every element of* $\mathrm{IG}(B)$ *can be written in almost normal form.*

We have the following lemma regarding the almost normal form of the product of two almost normal forms.

Lemma 5.12. *Let* $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ *be in almost normal form with* Y*-length* r*, left to right significant indices* $i_1, \cdots, i_r = n$ *and ordered* Y*-components* $\alpha_1, \cdots, \alpha_r$*, let* $\overline{y_1}\ \cdots\ \overline{y_m} \in \mathrm{IG}(B)$ *be in almost normal form with* Y*-length* s*, left to right significant indices* $l_1, \cdots, l_s = m$ *and ordered* Y*-components* $\beta_1, \cdots, \beta_s$*. Then (with* $i_0 = 0$*)*

(i) α_r *and* β_1 *incomparable implies that* $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$ *is in almost normal form;*

(ii) $\alpha_r \geq \beta_1$ *implies*

$$\overline{x_1} \cdots \overline{x_{i_t}}\ \overline{x_{i_t+1} \cdots x_{i_r} y_1 x_{i_r} \cdots x_{i_t+1}}\ \cdots\ \overline{x_{i_r} y_1 x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$*, for some* $t \in [0, r-1]$ *such that* $\alpha_r, \cdots, \alpha_{t+1} \geq \beta_1$ *and* $t = 0$ *or* α_t, β_1 *are incomparable;*

(iii) $\alpha_r \leq \beta_1$ *implies*

$$\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1 x_{i_r} y_1}\ \cdots\ \overline{y_{l_v} \cdots y_1 x_{i_r} y_1 \cdots y_{l_v}}\ \overline{y_{l_v+1}}\ \cdots\ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$ *for some* $v \in [1, s]$ *such that* $\alpha_r \leq \beta_1, \cdots, \beta_v$ *and* $v = s$ *or* β_{v+1}, α_r *are incomparable;*

Proof. Clearly, the statement (i) is true. We now aim to show (ii). Since $\alpha_r \geq \beta_1$, we have

$$\overline{x_{i_{r-1}+1}}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1} = \overline{x_{i_{r-1}+1} \cdots x_{i_r} y_1 x_{i_r} \cdots x_{i_{r-1}+1}}\ \cdots\ \overline{x_{i_r} y_1 x_{i_r}}\ \overline{y_1}$$

by Corollary 5.7. Consider α_{r-1} and β_1, then we either have $\alpha_{r-1} \geq \beta_1$ or they are incomparable, as $\alpha_{r-1} < \beta_1$ would imply $\alpha_r > \alpha_{r-1}$, which contradicts the almost normal form of $\overline{x_1}\ \cdots\ \overline{x_{i_r}}$. By finite induction we have that

$$\overline{x_1} \cdots \overline{x_{i_t}}\ \overline{x_{i_t+1} \cdots x_{i_r} y_1 x_{i_r} \cdots x_{i_t+1}}\ \cdots\ \overline{x_{i_r} y_1 x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \overline{y_1}\ \cdots\ \overline{y_{l_s}}$, for some $t \in [0, r-1]$, such that $\alpha_r, \cdots, \alpha_{t+1} \geq \beta_1$ and $t = 0$ or α_t, β_1 are incomparable. Similarly, we can show (iii). □

Theorem 5.13. *Let* $B = \bigcup_{\alpha \in Y} B_\alpha$ *be a semilattice* Y *of rectangular bands* $B_\alpha, \alpha \in Y$*. Then* $\mathrm{IG}(B)$ *is a weakly abundant semigroup with the congruence condition.*

Proof. Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, and Y-components $\alpha_1, \cdots, \alpha_r$. Clearly $\overline{x_1}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{x_1}\ \cdots\ \overline{x_n}$. Let $e \in B_\delta$ be such that $\overline{e}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{x_1}\ \cdots\ \overline{x_n}$. Then by Corollary 5.2, that applying $\overline{\theta}$ we have $\overline{\delta}\ \overline{\alpha_1}\ \cdots\ \overline{\alpha_r} = \overline{\alpha_1}\ \cdots\ \overline{\alpha_r}$. It follows from Lemma 3.1 that $\delta \geq \alpha_1$, so that by Corollary 5.5 we have

$$e x_1 \cdots x_{i_1}\ \mathcal{R}\ x_1 \cdots x_{i_1}.$$

On the other hand, $x_1 \cdots x_{i_1}\ \mathcal{R}\ x_1$ so that $e x_1\ \mathcal{R}\ x_1$, thus we have $x_1 \leq_{\mathcal{R}} e$. Thus $\overline{e}\ \overline{x_1} = \overline{e x_1} = \overline{x_1}$. Therefore $\overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{x_1}$. Dually, $\overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{L}}\ \overline{x_n}$, so that $\mathrm{IG}(B)$ is a weakly abundant semigroup as required.

Next we show that IG(B) satisfies the congruence condition.

Let $\overline{x_1}\,\cdots\,\overline{x_n}\in$ IG(B) be defined as above and let $\overline{y_1}\,\cdots\,\overline{y_m}\in$ IG(B) be in almost normal form with Y-length u, left to right significant indices $l_1,\cdots,l_u=m$ and ordered Y-components $\beta_1,\cdots,\beta_u$. From the above and a comment in Section 1, we have $\overline{x_1}\,\cdots\,\overline{x_n}\,\widetilde{\mathcal{R}}\,\overline{y_1}\,\cdots\,\overline{y_m}$ if and only if $x_1\;\mathcal{R}\;y_1$. Suppose now that $x_1\;\mathcal{R}\;y_1$, so that $\alpha_1=\beta_1$. Let $\overline{z_1}\,\cdots\,\overline{z_s}\in$ IG(B), where, without loss of generality, we can assume it is in almost normal form with Y-length t, left to right significant indices $j_1,\cdots,j_t=s$, and Y-components $\gamma_1,\cdots,\gamma_t$. We aim to show that

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}\,\widetilde{\mathcal{R}}\,\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}.$$

We consider the following three cases.

(i) If $\alpha_1=\beta_1,\gamma_t$ are incomparable, then it is clear that

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}\text{ and }\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}$$

are in almost normal form, so clearly we have

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}\,\widetilde{\mathcal{R}}\,\overline{z_1}\;\mathcal{R}\;\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}.$$

(ii) If $\beta_1=\alpha_1\leq\gamma_1$, then by Lemma 5.12

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}=\overline{z_1}\,\cdots\,\overline{z_{j_v}}\,\overline{z_{j_v+1}\cdots z_sx_1z_s\cdots z_{j_v+1}}\,\cdots\,\overline{z_sx_1z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}$$

and

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}=\overline{z_1}\,\cdots\,\overline{z_{j_v}}\,\overline{z_{j_v+1}\cdots z_sy_1z_s\cdots z_{j_v+1}}\,\cdots\,\overline{z_sy_1z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}$$

where $v\in[0,t-1]$, $\gamma_{v+1},\cdots,\gamma_t\geq\alpha_1=\beta_1$ and γ_v,β_1 are incomparable or $v=0$. Clearly, the right hand sides are in almost normal form.

If $v\geq 1$, then clearly the required result is true, as the above two almost normal forms begin with the same idempotent. If $v=0$, then we need to show that

$$z_1\cdots z_sx_1z_s\cdots z_1\;\mathcal{R}\;z_1\cdots z_sy_1z_s\cdots z_1$$

Since $x_1\;\mathcal{R}\;y_1$, it follows from the structure of B that

$$z_1\cdots z_sx_1z_s\cdots z_1\;\mathcal{R}\;z_1\cdots z_sx_1\;\mathcal{R}\;z_1\cdots z_sy_1\;\mathcal{R}\;z_1\cdots z_sy_1z_s\cdots z_1$$

as required.

(iii) If $\beta=\alpha_1\geq\gamma_1$, then by Lemma 5.12

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}=\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1z_sx_1}\,\cdots\,\overline{x_{i_k}\cdots x_1z_sx_1\cdots x_{i_k}}\,\overline{x_{i_k+1}}\,\cdots\,\overline{x_n}$$

and

$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}=\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1z_sy_1}\,\cdots\,\overline{y_{l_p}\cdots y_1z_sy_1\cdots y_{l_p}}\,\overline{y_{l_p+1}}\,\cdots\,\overline{y_m},$$

where $k\in[1,r]$, $\alpha_1,\cdots,\alpha_k\geq\gamma_1$, and α_{k+1},γ_1 are incomparable or $k=r$, and $p\in[1,u]$, $\beta_1,\cdots,\beta_p\geq\gamma_1$, and β_{p+1},γ_1 are incomparable or $p=u$. Clearly, the right hand sides are in almost normal form, so that

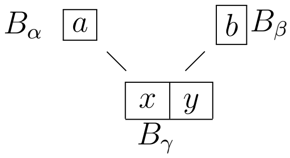
$$\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{x_1}\,\cdots\,\overline{x_n}\,\widetilde{\mathcal{R}}\,\overline{z_1}\,\widetilde{\mathcal{R}}\,\overline{z_1}\,\cdots\,\overline{z_s}\,\overline{y_1}\,\cdots\,\overline{y_m}.$$

Similarly, we can show that $\widetilde{\mathcal{L}}$ is a right congruence, so that IG(B) is a weakly abundant semigroup satisfying the congruence condition. This completes the proof. □

We finish this section by constructing a band B for which IG(B) is not an abundant semigroup.

Example 5.14. Let $B = B_\alpha \cup B_\beta \cup B_\gamma$ be a band with semilattice decomposition structure and multiplication table defined by

	a	b	x	y
a	a	y	x	y
b	y	b	x	y
x	x	y	x	y
y	y	y	x	y



First, it is easy to check that B is indeed a semigroup. We now show that $\mathrm{IG}(B)$ is not abundant by arguing that the element $\overline{a}\,\overline{b} \in \mathrm{IG}(B)$ is not $\mathcal{R}^*$-related to any idempotent of $\overline{B}$. It follows from Theorem 5.13 that $\overline{a}\,\overline{b}\,\widetilde{\mathcal{R}}\,\overline{a}$. However, $\overline{a}\,\overline{b}$ is not $\mathcal{R}^*$-related to $\overline{a}$, because

$$\overline{x}\,\overline{a}\,\overline{b} = \overline{y} = \overline{y}\,\overline{a}\,\overline{b} \text{ but } \overline{x}\,\overline{a} = \overline{x} \neq \overline{y} = \overline{y}\,\overline{a},$$

so that from Lemma 2.4, $\overline{a}\,\overline{b}$ is not $\mathcal{R}^*$-related to any idempotent of $\overline{B}$, and hence $\mathrm{IG}(B)$ is not an abundant semigroup.

6. Free idempotent generated generated semigroups over quasi-zero bands

In this section we will introduce a class of bands B for which the word problem of $\mathrm{IG}(B)$ is solvable. Further, in Section 7, we will show that for any quasi-zero band B, the semigroup $\mathrm{IG}(B)$ is abundant.

Definition 6.1. *Let B be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. We say that B is a quasi zero band if for all $\alpha, \beta \in Y$ with $\beta > \alpha$, $u \in B_\alpha$ and $v \in B_\beta$, we have $uv = vu = u$.*

It is easy to deduce that if B is quasi-zero, then for any $\alpha, \beta \in Y$ with $\alpha < \beta$, $u \in B_\alpha$ and $v \in B_\beta$, the products uv and vu are basic.

Lemma 6.2. *Let B be a quasi-zero band, and let $\overline{x_1} \cdots \overline{x_n}$, $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ have left to right significant indices $i_1, \cdots, i_r$; $j_1, \cdots, j_r$, respectively. If $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$, then for any $l \in [1, r]$, $\overline{x_1} \cdots \overline{x_{i_l}} = \overline{y_1} \cdots \overline{y_{j_l}}$.*

Proof. Suppose that $x_i \in B_{\alpha_i}$ for all $i \in [1, r]$. It is enough to consider a single step, say,

$$\overline{x_1} \cdots \overline{x_n} \sim \overline{w_1} \cdots \overline{w_s}.$$

Suppose that the significant indices of $\overline{w_1} \cdots \overline{w_s}$ are $k_1, \cdots, k_r$. By Lemma 5.4, for any $l \in [1, r]$, we have

$$\overline{w_1} \cdots \overline{w_{k_l}} = \overline{x_1} \cdots \overline{x_{i_l}}\,\overline{u}$$

and $w_{k_l} = u' x_{i_l} u$, where $u' = \varepsilon$ or $u' \in B_\sigma$ with $\sigma \geq \alpha_{i_l}$, and either $u = \varepsilon$, or $u \in B_\delta$ for some $\delta > \alpha_{i_l}$, or $u \in B_{\alpha_{i_l}}$ and there exists $v \in B_\theta$ with $\theta > \alpha_{i_l}$, $vu = u$ and $uv = x_{i_l}$. By the comment proceeding Lemma 6.2 we see that in each case, $\overline{x_{i_l}}\,\overline{u} = \overline{x_{i_l}}$, so that clearly, $\overline{w_1} \cdots \overline{w_{k_l}} = \overline{x_1} \cdots \overline{x_{i_l}}$. ∎

Lemma 6.3. *Let B be a quasi-zero band, let $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant $i_1, \cdots, i_r = n$ and ordered Y-components $\alpha_1, \cdots, \alpha_r$, and let $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length s, left to right significant indices $j_1, \cdots, j_s = m$ and ordered Y-components $\beta_1, \cdots, \beta_s$. Then $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$ in $\mathrm{IG}(B)$ if and only if $r = s$, $\alpha_l = \beta_l$ and $\overline{x_{i_{l-1}+1}} \cdots \overline{x_{i_l}} = \overline{y_{j_{l-1}+1}} \cdots \overline{x_{j_l}}$ in $\mathrm{IG}(B)$, for each $l \in [1, r]$, where $i_0 = j_0 = 0$.*

Proof. The sufficiency is obvious. Suppose now that $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$ in $\mathrm{IG}(B)$. Then it follows from Lemma 5.3 that $r = s$ and $\alpha_i = \beta_i$ for all $i \in [1, r]$. From Lemma 6.2, we have that $\overline{x_1} \cdots \overline{x_{i_l}} = \overline{y_1} \cdots \overline{y_{j_l}}$ in $\mathrm{IG}(B)$, for all $l \in [1, r]$. Then by the dual of Lemma 6.2, $\overline{x_{i_{l-1}+1}} \cdots \overline{x_{i_l}} = \overline{y_{j_{l-1}+1}} \cdots \overline{x_{j_l}}$ in $\mathrm{IG}(B)$. □

Lemma 6.4. *Let B be a quasi-zero band and $w = \overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with $x_i \in B_{\alpha_i}$ for each $i \in [1, n]$. Suppose that there exists an $\alpha \in Y$ such that for all $i \in [1, n]$, $\alpha_i \geq \alpha$ and there is at least one $j \in [1, n]$ such that $\alpha = \alpha_j$. Suppose also that p is a word in $\overline{B}^+$ obtained by single step on w. Then we have that $w' = p'$ in $\mathrm{IG}(B_\alpha)$, where w' and p' are words obtained by deleting all letters in w and p which do not lie in B_α.*

Proof. Suppose that we split $x_k = uv$ for some $k \in [1, n]$, where $u \in B_\nu$ and $v \in B_\tau$. Then we have

$$w = \overline{x_1} \cdots \overline{x_{k-1}}\, \overline{x_k}\, \overline{x_{k+1}} \cdots \overline{x_n} \sim \overline{x_1} \cdots \overline{x_{k-1}}\, \overline{u}\, \overline{v}\, \overline{x_{k+1}} \cdots \overline{x_n} = p.$$

If $\alpha_k > \alpha$, then $\nu, \tau > \alpha$. Hence $w' = p'$ in $\overline{B_\alpha}^+$; of course, they are also equal in $\mathrm{IG}(B_\alpha)$.

If $\alpha_k = \alpha$ and $\mu = \tau = \alpha$, then $u \,\mathcal{L}\, v$ or $u \,\mathcal{R}\, v$, so that uv is basic in B_α. In this case, $\overline{x_k} = \overline{uv} = \overline{u}\, \overline{v}$ in $\mathrm{IG}(B_\alpha)$, so that certainly,

$$p' = (\overline{x_1} \cdots \overline{x_{k-1}})'\, \overline{u}\, \overline{v}\, (\overline{x_{k+1}} \cdots \overline{x_n})' = (\overline{x_1} \cdots \overline{x_{k-1}})'\, \overline{x_k}\, (\overline{x_{k+1}} \cdots \overline{x_n})' = w'$$

in $\mathrm{IG}(B_\alpha)$.

If $\alpha_k = \alpha$ and $\nu > \tau = \alpha$, then we have $x_k = uv = v$ as B is a quasi-zero band, so that

$$\begin{aligned} p' &= (\overline{x_1} \cdots \overline{x_{k-1}})'\, (\overline{u}\, \overline{v})'\, (\overline{x_{k+1}} \cdots \overline{x_n})' \\ &= (\overline{x_1} \cdots \overline{x_{k-1}})'\, \overline{v}\, (\overline{x_{k+1}} \cdots \overline{x_n})' \\ &= (\overline{x_1} \cdots \overline{x_{k-1}})'\, \overline{x_k}\, (\overline{x_{k+1}} \cdots \overline{x_n})' \\ &= w' \end{aligned}$$

in $\overline{B_\alpha}^+$, so that certainly $p' = w'$ in $\mathrm{IG}(B_\alpha)$.

A similar argument holds if $\alpha_k = \alpha$ and $\alpha = \nu < \tau$. □

Lemma 6.5. *Let B be a quasi-zero band and let $x_1, \cdots, x_n, y_1, \cdots, y_m \in B_\alpha$ for some $\alpha \in Y$. Then with $w = \overline{x_1} \cdots \overline{x_n}$ and $p = \overline{y_1} \cdots \overline{y_m}$ we have $w = p$ in $\mathrm{IG}(B_\alpha)$ if and only if $w = p$ in $\mathrm{IG}(B)$.*

Proof. The sufficiency is clear, as any basic pair in B_α is basic in B. Conversely, if $w = p$ in $\mathrm{IG}(B)$, there exists a finite sequence

$$w = w_0 \sim w_1 \sim w_2 \cdots \sim w_{s-1} \sim w_s = p.$$

Let $w'_0, w'_1, w'_2, \cdots, w'_{s-1}, w'_s$ be the words obtained by deleting letters x within the word such that $x \in B_\beta$ with $\beta \neq \alpha$. From Lemma 6.4, we have that $w'_0 = w'_1 = w'_2 = \cdots = w'_{s-1} = w'_s$ in $\mathrm{IG}(B_\alpha)$. Note that $w'_0 = w_0 = w \in \overline{B_\alpha}^+$ and $w'_s = w_s = p \in \overline{B_\alpha}^+$, so that $w = p$ in $\mathrm{IG}(B_\alpha)$. □

Lemma 6.6. *Let B be a quasi-zero band. Then the word problem of $\mathrm{IG}(B)$ is solvable.*

Proof. The result is immediate from Lemmas 4.1, 6.3 and 6.5. □

7. Free idempotent generated semigroups with condition (P)

From the above discussion, we know that for any band B, the semigroup $\mathrm{IG}(B)$ is always weakly abundant with the congruence condition, but not necessarily abundant. The aim of this section is devoted to finding some special kinds of bands B for which $\mathrm{IG}(B)$ is abundant.

Definition 7.1. *We say that the semigroup* $\mathrm{IG}(B)$ *satisfies Condition* (P) *if for any two almost normal forms* $\overline{u_1}\ \cdots\ \overline{u_n} = \overline{v_1}\ \cdots\ \overline{v_m} \in \mathrm{IG}(B)$ *with* Y*-length* r*, left to right significant indices* $i_1, \cdots, i_r = n$ *and* $l_1, \cdots, l_r = m$*, respectively, and ordered* Y*-components* $\alpha_1, \cdots, \alpha_r$*, the following statements (with* $i_0 = l_0 = 0$*) hold:*
(i) $u_{i_s}\ \mathcal{L}\ v_{l_s}$ *implies* $\overline{u_1}\ \cdots \overline{u_{i_s}} = \overline{v_1}\ \cdots\ \overline{v_{l_s}}$*, for all* $s \in [1, r]$*.*
(ii) $u_{i_t+1}\ \mathcal{R}\ v_{l_t+1}$ *implies* $\overline{u_{i_t+1}}\ \cdots \overline{u_n} = \overline{v_{l_t+1}}\ \cdots\ \overline{v_m}$*, for all* $t \in [0, r-1]$*.*

Proposition 7.2. *Let* B *be a band for which* $\mathrm{IG}(B)$ *satisfies Condition* (P)*. In addition, suppose that* B *is normal (so that* $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$*) or quasi-zero. Then* $\mathrm{IG}(B)$ *is an abundant semigroup.*

Proof. Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. By Theorem 5.13, $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \widetilde{\mathcal{R}}\ \overline{x_1}$. We aim to show that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{R}^*\ \overline{x_1}$. From Lemma 2.5, we only need to show that for any two almost normal forms $\overline{y_1}\ \cdots\ \overline{y_m} \in \mathrm{IG}(B)$ with Y-length m, left to right significant indices $l_1, \cdots, l_s = m$, and ordered Y-components $\beta_1, \cdots, \beta_s$, and $\overline{z_1}\ \cdots\ \overline{z_h} \in \mathrm{IG}(B)$ with Y-length t, left to right significant indices $j_1, \cdots, j_t = h$, and ordered Y-components $\gamma_1, \cdots, \gamma_t$, we have that

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$$

implies that $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

Suppose now that

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}.$$

We consider the following cases:

(i) If γ_t, α_1 and β_s, α_1 are incomparable, then both sides of the above equality are in almost normal form, so that by Condition (P)

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}.$$

Since $\overline{x_1} \cdots\ \overline{x_{i_1}}\ \mathcal{R}\ \overline{x_{i_1}}$ by Lemma 4.2, we have $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

(ii) If $\gamma_t \leq \alpha_1$ and β_s, α_1 are incomparable, then by Lemma 5.12, $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$ has an almost normal form

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}},$$

for some $v \in [1, r]$, where $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable. Hence we have

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}.$$

Note that both sides of the above equality are in almost normal form. It follows from Corollary 5.2 that

$$(\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}})\ \overline{\theta} = (\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}})\ \overline{\theta}$$

and so

$$\overline{\gamma_1}\ \cdots\ \overline{\gamma_t}\ \overline{\alpha_{v+1}}\ \cdots\ \overline{\alpha_r} = \overline{\beta_1}\ \cdots\ \overline{\beta_s}\ \overline{\alpha_1}\ \cdots\ \overline{\alpha_r}.$$

Since $v \geq 1$, we have $\gamma_t = \alpha_v$. To avoid contradiction, $v = 1$, so $x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1} = x_{i_1}$, and hence by Condition (P)

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}.$$

and so

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

so that $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_1}}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

(iii) If $\gamma_t \leq \alpha_1$ and $\beta_s \leq \alpha_1$, then by Lemma 5.12 we have the following two almost normal forms for $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$ and $\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$, namely,

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}}$$

where $v \in [1, r]$ such that $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable, and

$$\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1 y_{l_s} x_1}\ \cdots\ \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}\ \overline{x_{i_u+1}}\ \cdots\ \overline{x_{i_r}}$$

where $u \in [1, r]$ with $\beta_s \leq \alpha_1, \cdots, \alpha_u$ and $u = r$ or β_s, α_{u+1} are incomparable. Hence by Corollary 5.2,

$$\overline{\gamma_1}\ \cdots\ \overline{\gamma_t}\ \overline{\alpha_{v+1}}\ \cdots\ \overline{\alpha_r} = \overline{\beta_1}\ \cdots\ \overline{\beta_s}\ \overline{\alpha_{u+1}}\ \cdots\ \overline{\alpha_r}$$

If $v > u$, then $\gamma_t = \alpha_v$, to avoid contradiction $v = 1$, so $u = 0$, contradiction. Similarly, $v < u$ is impossible. If $v = u$, then $t = s$ and $\beta_s = \gamma_t$. If B is a normal band satisfying Condition (P),

$$x_1 z_{j_t} x_1 = x_1 \phi_{\alpha_1, \gamma_t} = x_1 \phi_{\alpha_1, \beta_s} = x_1 y_{l_s} x_1$$
$$\vdots$$
$$x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v} = x_{i_v} \phi_{\alpha_v, \gamma_t} = x_{i_u} \phi_{\alpha_u, \beta_s} = x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}$$

so that by Condition (P), we have

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1 y_{l_s} x_1}\ \cdots\ \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}.$$

On the other hand, we have

$$\overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} = \overline{x_1 y_{l_s} x_1}\ \cdots\ \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}$$

which is $\mathcal{R}$-related to $x_1 z_{j_t} x_1$, and so

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1 y_{l_s} x_1},$$

and hence

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}.$$

Suppose now that B is a quasi-zero band. First suppose that $v = u = 1$. Then by Lemma 6.2 we have

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1 z_{j_t} x_1}\ \cdots\ \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1 y_{l_s} x_1}\ \cdots\ \overline{x_{i_1} \cdots x_1 y_{l_s} x_1 \cdots x_{i_1}}$$

and so

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

so that

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}.$$

Suppose now that $v=u>1$. By assumption $\beta_s=\gamma_t\le\alpha_1,\cdots,\alpha_v$. We claim that there exists no $j\in[1,v]$ such that $\gamma_t=\alpha_j$; otherwise we will have α_j,α_{j+1} are comparable if $v>j$ or α_v,α_{v-1} are comparable if $v=j$. Hence $\gamma_t=\beta_s<\alpha_1,\cdots,\alpha_v$. Since B is a quasi-zero band, we have

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1z_{j_t}x_1}\ \cdots\ \overline{x_{i_v}\cdots x_1z_{j_t}x_1\cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}}=\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}}$$

and

$$\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1y_{l_s}x_1}\ \cdots\ \overline{x_{i_u}\cdots x_1y_{l_s}x_1\cdots x_{i_u}}\ \overline{x_{i_u+1}}\ \cdots\ \overline{x_{i_r}}=\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}}$$

so that it follows from Lemma 6.2 that

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}=\overline{y_1}\ \cdots\ \overline{y_{l_s}}$$

and so certainly

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}=\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}.$$

(iv) If $\gamma_t\le\alpha_1$ and $\beta_s\ge\alpha_1$, then by Lemma 5.12 we have the following two almost normal forms for $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$ and $\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$, namely,

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1z_{j_t}x_1}\ \cdots\ \overline{x_{i_v}\cdots x_1z_{j_t}x_1\cdots x_{i_v}}\ \overline{x_{i_v+1}}\ \cdots\ \overline{x_{i_r}}$$

for some $v\in[1,r]$ with $\gamma_t\le\alpha_1,\cdots,\alpha_v$ and $v=r$ or γ_t,α_{v+1} are incomparable, and

$$\overline{y_1}\cdots\overline{y_{l_u}}\ \overline{y_{l_u+1}\cdots y_{l_s}x_1y_{l_s}\cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s}x_1y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$$

for some $u\in[0,s-1]$ with $\beta_{u+1},\cdots,\beta_s\ge\alpha_1$ and β_u,α_1 are incomparable or $u=0$. It follows from Corollary 5.2 that

$$\overline{\gamma_1}\ \cdots\ \overline{\gamma_t}\ \overline{\alpha_{v+1}}\ \cdots\ \overline{\alpha_r}=\overline{\beta_1}\ \cdots\ \overline{\beta_u}\ \overline{\alpha_1}\ \cdots\ \overline{\alpha_r}.$$

Note that both sides of the above equality are normal forms of IG(Y). As $v\ge1$, we have $\gamma_t=\alpha_v$, so that to avoid contradiction we have $v=1$ and so $x_{i_1}\cdots x_1z_{j_t}x_1\cdots x_{i_1}=x_{i_1}$, and hence by Condition (P)

$$\begin{aligned}&\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1z_{j_t}x_1}\ \cdots\ \overline{x_{i_1}\cdots x_1z_{j_t}x_1\cdots x_{i_1}}\\ =\ &\overline{y_1}\cdots\overline{y_{l_u}}\ \overline{y_{l_u+1}\cdots y_{l_s}x_1y_{l_s}\cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s}x_1y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\end{aligned}$$

and so

$$\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}=\overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}},$$

which implies $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}=\overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}$.

(v) If $\gamma_t\ge\alpha_1$ and $\beta_s\ge\alpha_1$, then by Lemma 5.12 we have the following two almost normal forms for $\overline{z_1}\ \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$ and $\overline{y_1}\ \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_r}}$, namely,

$$\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $v\in[0,t-1]$ such that $\gamma_{v+1},\cdots,\gamma_t\ge\alpha_1$ and γ_v,α_1 are incomparable or $v=0$, and

$$\overline{y_1}\cdots\overline{y_{l_u}}\ \overline{y_{l_u+1}\cdots y_{l_s}x_1y_{l_s}\cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s}x_1y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $u\in[0,s-1]$ such that $\beta_{u+1},\cdots,\beta_s\ge\alpha_1$ and β_u,α_1 are incomparable or $u=0$. Hence by Condition (P),

$$\begin{aligned}&\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\\ =\ &\overline{y_1}\cdots\overline{y_{l_u}}\ \overline{y_{l_u+1}\cdots y_{l_s}x_1y_{l_s}\cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s}x_1y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}},\end{aligned}$$

so that

$$\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

and hence $\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

(vi) If $\gamma_t \geq \alpha_1$ and β_s, α_1 are incomparable, then by Lemma 5.12

$$\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}} = \overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $v \in [0, t-1]$ with $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1$ and γ_v, α_1 are incomparable or $v = 0$. Note that both sides of the above equality are in almost normal form. Again by Condition (P)

$$\overline{z_1}\cdots\overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_{j_t}x_1z_{j_t}\cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t}x_1z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

so that

$$\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

and hence $\overline{z_1}\cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1}\cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

From the above discussion, we can deduce that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{R}^*\ \overline{x_1}$, and similarly we can show that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{L}^*\ \overline{x_{i_r}}$, so that IG$(B)$ is an abundant semigroup. □

We now aim to find examples of normal bands B for which IG(B) satisfies Condition (P), so that by Proposition 7.2, IG(B) is abundant.

A band $B = \bigcup_{\alpha\in Y} B_\alpha$ is called a *simple band* if it is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, where B_α is either a left zero band or a right zero band.

Lemma 7.3. *Let $B = \bigcup_{\alpha\in Y} B_\alpha$ be a simple band and let $e \in B_\alpha$ and $f \in B_\beta$. Then (e, f) is a basic pair in B if and only (α, β) is a basic pair in Y, i.e. if and only if α and β are comparable in Y.*

Proof. Since the necessity is clear, we are left with showing the sufficiency. Without loss of generality, suppose that $\alpha \leq \beta$. Then $ef, fe \in B_\alpha$. As B is a simple band, we have B_α is either a left zero band or a right zero band. If B_α is a left zero band, then $e(ef) = e$, i.e. $ef = e$, so (e, f) is a basic pair. If B_α is a right zero band, then $(fe)e = e$, i.e. $fe = e$, which again implies that (e, f) is a basic pair. □

It follows from Lemma 7.3 that for a simple band B, every element $\overline{x_1}\ \cdots\ \overline{x_n}$ of IG(B) has a special normal form (of course, which may not unique), say, $\overline{y_1}\ \cdots\ \overline{y_m} \in$ IG(B) with y_i and y_{i+1} incomparable, for all $i \in [1, m-1]$.

Lemma 7.4. *Let B be a simple band. Then* IG(B) *satisfies Condition* (P).

Proof. Let $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m} \in$ IG(B) be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, $j_1, \cdots, j_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. It then follows from Corollary 5.5 that for all $s \in [1, r]$,

$$\overline{y_1}\ \cdots\overline{y_{j_s}} = \overline{x_1}\ \cdots\ \overline{x_{i_s}}\ \overline{e_1}\ \cdots\ \overline{e_m} \text{ (in which we remove the empty word)}$$

where for all $k \in [1, m]$, $e_k \in B_{\delta_k}$ with $\delta_k \geq \alpha_{i_s}$. By Lemma 7.3, we have

$$\overline{x_{i_s}}\ \overline{e_1}\ \cdots\ \overline{e_m} = \overline{x_{i_s}e_1\cdots e_m},$$

so that if we assume $x_{i_s}\ \mathcal{L}\ y_{j_s}$, then

$$\overline{y_1}\ \cdots\overline{y_{j_s}} = \overline{y_1}\ \cdots\overline{y_{j_s}}\ \overline{x_{i_s}} = \overline{x_1}\ \cdots\ \overline{x_{i_s}e_1\cdots e_m}\ \overline{x_{i_s}} = \overline{x_1}\ \cdots\ \overline{x_{i_s}e_1\cdots e_mx_{i_s}} = \overline{x_1}\ \cdots\ \overline{x_{i_s}}.$$

Together with the dual, we have shown that IG(B) satisfies Condition (P). □

Corollary 7.5. *Let B be a simple normal band. Then* $\mathrm{IG}(B)$ *is abundant.*

Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band. We say that B is a *trivial normal band* if for every $\alpha \in Y$, there exists a $a_\alpha \in B_\alpha$ such that for all $\beta > \alpha$, $x\phi_{\beta,\alpha} = a_\alpha$.

Lemma 7.6. *Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a trivial normal band. Then* $\mathrm{IG}(B)$ *satisfies Condition (P).*

Proof. First note that since B is a trivial normal band, there exists $a_\alpha \in B_\alpha$ be such that for any $\beta > \alpha$ and $u \in B_\beta$, $u\phi_{\beta,\alpha} = a_\alpha$.

Let $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, $j_1, \cdots, j_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. It follows from Corollary 5.5 that

$$\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l}}\, \overline{u_1} \cdots \overline{u_s} \text{ (in which we remove the empty word)}$$

such that for all $k \in [1, s]$ we have $u_k \in B_{\delta_k}$ with $\delta_k > \alpha_{i_l}$, so that $u_k\phi_{\delta_k,\alpha_{i_l}} = a_{\alpha_{i_l}}$; or $u_k \in B_{\alpha_{i_l}}$ with $v_k u_k = u_k$ for some $v_k \in B_{\eta_k}$ such that $\eta_k > \alpha_{i_l}$, and in this case we have $a_{\alpha_{i_l}} u_k = u_k$, so that $a_{\alpha_{i_l}} \,\mathcal{R}\, u_k$. Thus the idempotents $u_1\phi_{\delta_1,\alpha_{i_l}}, \cdots, u_s\phi_{\delta_s,\alpha_{i_l}}$ are all $\mathcal{R}$-related, and so

$$\overline{x_{i_l}}\, \overline{u_1} \cdots \overline{u_s} = \overline{x_{i_l}}\, \overline{u_1\phi_{\delta_1,\alpha_{i_l}}} \cdots \overline{u_s\phi_{\delta_s,\alpha_{i_l}}} = \overline{x_{i_l}}\, \overline{u_1\phi_{\delta_1,\alpha_{i_l}} \cdots u_s\phi_{\delta_s,\alpha_{i_l}}}.$$

On the other hand, we have $y_{j_l} = u'_s \cdots u'_1 x_{i_l} u_1 \cdots u_s$, where $u'_k \in B_{\sigma_k}$ with $\sigma_k \geq \alpha_{i_l}$. Hence if we assume that $x_{i_l} \,\mathcal{L}\, y_{j_l}$, then $x_{i_l} = x_{i_l} u_1 \cdots u_s$, and so $x_{i_l} = x_{i_l}(u_1\phi_{\delta_1,\alpha_{i_l}}) \cdots (u_s\phi_{\delta_s,\alpha_{i_l}})$, so that

$$\overline{x_{i_l}}\, \overline{u_1\phi_{\delta_1,\alpha_{i_l}} \cdots u_s\phi_{\delta_s,\alpha_{i_l}}} = \overline{x_{i_l}(u_1\phi_{\delta_1,\alpha_{i_l}}) \cdots (u_s\phi_{\delta_s,\alpha_{i_l}})} = \overline{x_{i_l}}.$$

Hence $\overline{y_1} \cdots \overline{y_{j_l}} = \overline{x_1} \cdots \overline{x_{i_l}}$ as required. □

Corollary 7.7. *Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a trivial normal band. Then* $\mathrm{IG}(B)$ *is an abundant semigroup.*

8. A normal band B for which $\mathrm{IG}(B)$ is not abundant

From Section 7, we know that the free idempotent idempotent generated semigroup $\mathrm{IG}(B)$ over a normal band B satisfying Condition (P) is an abundant semigroup. Therefore, one would like to ask whether for any normal band B, $\mathrm{IG}(B)$ is abundant. In this section we will construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

Throughout this section, B denotes a normal band $\mathcal{B}(Y; B_\alpha, y\phi_{\alpha,\beta})$.

Lemma 8.1. *Let B be a normal band, and let $x \in B_\beta, y \in B_\gamma$ with $\beta, \gamma \geq \alpha$. Then (x, y) is a basic pair implies $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair and*

$$(x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}) = (xy)\phi_{\delta,\alpha},$$

where δ is minimum of β and γ.

Proof. Let (x, y) be a basic pair with $x \in B_\beta, y \in B_\gamma$. Then β, γ are comparable. If $\beta \geq \gamma$, then we either have $xy = y$ or $yx = y$. If $xy = y$, then $(x\phi_{\beta,\gamma})y = y$, so

$$y\phi_{\gamma,\alpha} = ((x\phi_{\beta,\gamma})y)\phi_{\gamma,\alpha} = (x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}),$$

so $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair. If $yx = y$, then $y(x\phi_{\beta,\gamma})$, so

$$y\phi_{\gamma,\alpha} = (y(x\phi_{\beta,\gamma}))\phi_{\gamma,\alpha} = (y\phi_{\gamma,\alpha})(x\phi_{\beta,\alpha}),$$

so that $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair.

A similar argument holds if $\gamma \geq \beta$. The final part of the lemma is clear. □

Lemma 8.2. *Let* $\overline{u_1}\ \cdots\ \overline{u_n} \in \mathrm{IG}(B)$ *with* $u_i \in B_{\alpha_i}$ *and* $\alpha_i \geq \alpha$ *for all* $i \in [1,n]$. *Suppose that* $\overline{v_1}\ \cdots\ \overline{v_m} \in \mathrm{IG}(B)$ *with* $v_i \in B_{\beta_i}$ *for all* $i \in [1,m]$ *is an element obtained by single step on* $\overline{u_1}\ \cdots\ \overline{u_n}$ *(note that* $\beta_i \geq \alpha$, *for all* $i \in [1,m]$*). Then in* $\mathrm{IG}(B_\alpha)$ *we have*

$$\overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}} = \overline{v_1\phi_{\beta_1,\alpha}}\ \cdots\ \overline{v_m\phi_{\beta_m,\alpha}}.$$

Proof. Suppose that $u_i = xy$ is a basic product with $x \in B_\delta, y \in B_\eta$, for some $i \in [1,n]$. Note that the minimum of δ and η is α_i. Then

$$\overline{u_1}\ \cdots\ \overline{u_n} \sim \overline{u_1}\ \cdots\ \overline{u_{i-1}}\ \overline{x}\ \overline{y}\ \overline{u_{i+1}}\ \cdots\ \overline{u_n}.$$

If follows from Lemma 8.1 that in $\mathrm{IG}(B_\alpha)$

$$\begin{aligned}\overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}} &= \overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{u_i\phi_{\alpha_i,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}}\\ &= \overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{x\phi_{\delta,\alpha}y\phi_{\eta,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}}\\ &= \overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{x\phi_{\delta,\alpha}}\ \overline{y\phi_{\eta,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}}\end{aligned}$$

as required. □

Corollary 8.3. *Let* $x_1, \cdots, x_n, y_1, \cdots, y_m \in B_\alpha$. *Then* $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ *in* $\mathrm{IG}(B_\alpha)$ *if and only if the equality holds in* $\mathrm{IG}(B)$.

Proof. The necessity is obvious, as any basic pair in B_α must also be basic in B. Suppose now that we have

$$\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$$

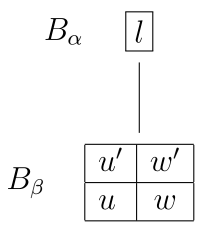
in $\mathrm{IG}(B)$. Then there exists a sequence of transitions

$$\overline{x_1}\ \cdots\ \overline{x_n} \sim \overline{u_1}\ \cdots\ \overline{u_s} \sim \overline{v_1}\ \cdots\ \overline{v_t} \sim\ \cdots\ \sim \overline{w_1}\ \cdots\ \overline{w_l} \sim \overline{y_1}\ \cdots\ \overline{y_m},$$

using basic pairs in B. Note that all idempotents involved in the above sequence lie in B_β for some $\beta \geq \alpha$, so that successive applications of Lemma 8.2 give $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ in $\mathrm{IG}(B_\alpha)$. □

We remark here that for an arbitrary band B, Corollary 8.3 need not be true.

Example 8.4. Let $B = B_\alpha \cup B_\beta$ be a band with semilattice structure and multiplication table defined by



	l	u	w	u'	w'
l	l	u'	w'	u'	w'
u	u	u	w	u	w
w	w	u	w	u	w
u'	u'	u'	w'	u'	w'
w'	w'	u'	w'	u'	w'

It is easy to check that B forms a band. By the uniqueness of normal forms in $\mathrm{IG}(B_\beta)$, we have $\overline{u'}\,\overline{w} \neq \overline{w'}$ in $\mathrm{IG}(B_\beta)$. However in $\mathrm{IG}(B)$ we have

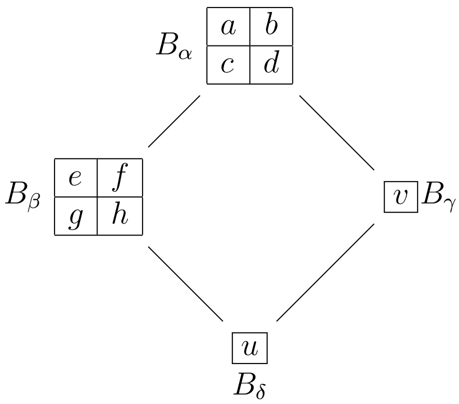
$$\begin{aligned}
\overline{u'}\,\overline{w} &= \overline{u'l}\,\overline{w} \\
&= \overline{u'}\,\overline{l}\,\overline{w} \quad \text{(as } (u',l) \text{ is a basic pair)} \\
&= \overline{u'}\,\overline{lw} \quad \text{(as } (l,w) \text{ is a basic pair)} \\
&= \overline{u'}\,\overline{w'} \\
&= \overline{w'}
\end{aligned}$$

With the above preparations, we now construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

Example 8.5. Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice $Y = \{\alpha, \beta, \gamma, \delta\}$ of rectangular bands (see the figure below), where $\phi_{\alpha,\beta} : B_\alpha \longrightarrow B_\beta$ is defined by

$$a\phi_{\alpha,\beta} = e, b\phi_{\alpha,\beta} = f, c\phi_{\alpha,\beta} = g, d\phi_{\alpha,\beta} = h$$

the remaining morphisms being defined in the obvious unique manner.



Now we consider an element $\overline{e}\,\overline{v} \in \mathrm{IG}(B)$, then we have

$$\begin{aligned}
\overline{e}\,\overline{v} &= \overline{e}\,\overline{dv} \\
&= \overline{e}\,\overline{d}\,\overline{v} \quad \text{(as } (d,v) \text{ is a basic pair)} \\
&= \overline{e}\,\overline{h}\,\overline{v} \quad \text{(as } \overline{e}\,\overline{d} = \overline{e}\,\overline{d\phi_{\alpha,\beta}} = \overline{e}\,\overline{h} \text{ by Corollary 5.8)} \\
&= \overline{e}\,\overline{h}\,\overline{av} \\
&= \overline{e}\,\overline{h}\,\overline{a}\,\overline{v} \quad \text{(as } (a,v) \text{ is a basic pair)} \\
&= \overline{e}\,\overline{h}\,\overline{e}\,\overline{v} \quad \text{(as } \overline{h}\,\overline{a} = \overline{h}\,\overline{a\phi_{\alpha,\beta}} = \overline{h}\,\overline{e} \text{ by Corollary 5.8)}
\end{aligned}$$

However, $\overline{e}\,\overline{h}\,\overline{e} \neq \overline{e}$ in $\mathrm{IG}(B_\beta)$ by the uniqueness of normal forms, so by Corollary 8.3, we have $\overline{e}\,\overline{h}\,\overline{e} \neq e$ in $\mathrm{IG}(B)$, which implies $\overline{e}\,\overline{v}$ is not $\mathcal{R}^*$-related to $\overline{e}$. On the other hand, we have known from Theorem 5.13 that $\overline{e}\,\overline{v}\ \widetilde{\mathcal{R}}\ \overline{e}$, so that by Lemma 2.4 that $\overline{e}\,\overline{v}$ is not $\mathcal{R}^*$-related any idempotent of B, so that $\mathrm{IG}(B)$ is not an abundant semigroup.

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