# Free idempotent generated Semigroups

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# Outline

- Idempotent generated semigroups
- Biordered sets
- Free idempotent generated semigroups over biordered sets
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# idempotent generated semigroups

Let $S$ be a semigroup with set of idempotents $E(S)$, and let $\langle E(S) \rangle$ denote the subsemigroup of $S$ generated by $E(S)$. We say $S$ is an idempotent generated semigroup if $S = \langle E(S) \rangle$.

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- J.M. Howie [2] proved the subsemigroup of all non-invertible transformations of the full transformation monoid $T_x$ on a finite set $X$ is idempotent generated. Furthermore, every semigroup can be embedded into an idempotent generated semigroup.

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Groups arise as the Maximal subgroups of semigroups have received considerable attentions.

Given an idempotent $e$ of any semigroup $S$, the maximal subgroups $H_e$ of $S$ with identity $e$ is the group of units of the submonoid $eSe$ of $S$. For example, if $e$ is an idempotent of rank $r$ in $M_n(Q)$ over a division ring $Q$, then $H_e \cong GL_r(Q)$, the general linear group of size $r$ over $Q$.

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which groups can arise as the maximal subgroups of a free idempotent generated semigroup over some biordered set $E$?

Let $E$ be a partial algebra, by which we mean a set $E$ together with a partial binary operation on $E$. We will use $D_E$ to denote the domain of $E$. On $E$ we define:

$$\omega^r = \{(e, f) : fe = e\}, \ \omega^l = \{(e, f) : ef = e\}$$

and

$$\mathcal{R} = \omega^r \cap (\omega^r)^{-1}, \ \mathcal{L} = \omega^l \cap (\omega^l)^{-1}, \text{ and } \omega = \omega^l \cap \omega^r.$$

# Biordered sets

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Let $E$ be a partial algebra. Then $E$ is a biordered set if the following axioms and their duals hold;

$(B1)$ $\omega^r$ and $\omega^l$ are quasiorders on $E$ and

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}.$$

(B21) $f \in \omega^r(e) \Rightarrow f \mathcal{R} fe \omega e$.

(B22) $g \omega^l f,\ f, g \in \omega^r(e) \Rightarrow ge \omega^l fe$.

(B31) $g \omega^r f \omega^r e \Rightarrow gf = (ge)f$.

(B32) $g \omega^l f,\ f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge)$.

Let $M(e, f)$ denote the quasiordered set $(\omega^l(e) \cap \omega^r(f), <)$, where $<$ is defined by

$$g < h \Leftrightarrow eg \omega^r eh, gf \omega^l hf$$

.

# Biordered sets

Then the set

$$S(e,f) = \{h \in M(e,f) : g < h, (\forall g \in M(e,f))\}$$

is called the sandwich set of $e$ and $f$.

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(Regular) Biordered Sets $\longleftrightarrow$ (Regular) Semigroups

It was shown by Nambooripad and Easdown that if $S$ is a (regular) semigroup, then $E(S)$ is a (regular) biordered set. Conversely, if $E$ is a (regular) biordered set, then there exists a (regular) semigroup $S$ with $E \simeq E(S)$ a biordered set.

Suppose $E$ is a biordered set. We denote $IG(E)$ the semigroup with presentation

$$IG(E) = \langle E : e.f = ef, (e, f) \text{ is a basic pair} \rangle$$

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and if $E$ is regular biordered set, then we define

$$RIG(E) = \langle E : e.f = ef, \text{ if } (e,f) \text{ is a basic pair and } e.f = e.h.f \text{ for all } e,f \in E, h \in S(e,f) \rangle.$$

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$IG(E)$ and $RIG(E)$ can be very different when $E$ is regular biordered set. Also, the regular elements of $IG(E)$ do not form a subsemigroup in general, even if $E$ is a regular biordered set. (example see [5]).

The biordered set of idempotent of $IG(E)$ is $E$. In particular, every biordered set is the biordered set of some semigroup $S$. If $S$ is any idempotent generated semigroup with biordered set of idempotents isomorphic to $E$, then the natural map $E \to S$ extends uniquely to a homomorphism $IG(E) \to S$. (See [6])

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If $E$ is a regular biordered set then $RIG(E)$ is a regular semigroup with biordered set of idempotents $E$. If $S$ is any regular idempotent generated semigroup with biordered set of idempotents isomorphic to $E$, then the natural map $E \to S$ extends uniquely to a homomorphism $RIG(E) \to S$. (See [7] and [8])

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It has been conjectured that the maximal subgroups of $IG(E)$ are free, when $E$ is regular (See [9]). Indeed, there are several papers in the literature prove that the maximal subgroups are free for certain class of biordered set (See [8],[9] and [10]).

Brittenham, Margolis and Meakin [5] provided the first counterexample to this conjecture by showing that the free Abelian group $Z \times Z$ can arise as a maximal subgroup of $IG(E)$ for some biordred set $E$.

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Later, Brittenham, Margolis and Meakin [11] showed that if $Q$ is a division ring, then the maximal subgroups of $IG(E(M_n(Q)))$ containing an idempotent of rank 1 is $Q^*$, the multiplicative group of units of $Q$, where $n \geqslant 3$.

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The maximal subgroups of $IG(E(M_n(Q)))$ containing an idempotent of rank $n-1$ is a free group.

They also conjectured that the maximal subgroups of an idempotent of rank $r$ with $r < n-1$ is isomorphic to the $r$-dimensional general linear group $GL_r(Q)$ over $Q$, at least for $r < n/2$ and $n \geqslant 3$.

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I. Dolinka and R. Gray [15] solved the conjecture in paper [11] by showing that if $e$ is an idempotent with rank $r < n/3$, $n \geqslant 4$, then the maximal subgroup of $IG(E(M_n(Q)))$ containing $e$ is isomorphic to the $r$-dimensional general linear group $GL_r(Q)$ over $Q$.

I. Dolinka [16] investigated the free idempotent generated semigroups over bands and it is shown that there is a regular band $B$ such that $IG(B)$ has a maximal subgroup isomorphic to the free Abelian group of rank 2.

# Free idempotent generated semigroups over biordered sets

R. Gray and N. Ruskuc [13] gave a complete description of maximal subgroups of the free idempotent generated semigroups arising from finite full transformation semigroups.

It was shown that the maximal subgroup of $IG(E(T_n))$ containing an idempotent $e$ with rank $r$ $(1 \leqslant r \leqslant n-2)$ is isomorphic to the symmetric group $S_r$.

If $e$ is the identity mapping then the maximal subgroup containing $e$ is trivial. If $|Im(e)| = n-1$, then the maximal subgroup containing $e$ is free.

The maximal subgroups containing $e$ in $T_n$ and in $IG(E(T_n))$ are identical!

# Question

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The question is: if $A$ is an independence algebra of rank $n$, and $E$ is the biordered set of idempotents of $EndA$, for which $1 \leqslant r \leqslant n-1$, is the maximal subgroup $H_e$ of $IG(E)$ isomorphic to the automorphism monoid of a rank $r$ subalgebra of $A$?

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Our work may provide a route to proving the corresponding results for $T_n$ and $M_n(Q)$.

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