Ordered Covers

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Motivation

In $\mathbf{S}\text{-Act}$, and in $\mathbf{S}\text{-Pos}$ we have the following relations exists

$$Pr \Rightarrow SF = P + E$$

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\[ \mathcal{P}r \Rightarrow \mathcal{S} \mathcal{F} = \mathcal{P} + \mathcal{E} \]

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- Roghaieh, Majid and Mojtaba (2009), $P$-covers and $SF$-covers, are defined for $S$-acts;
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(i) $S$ left is po-perfect;
(ii) $S$ satisfies Condition $(A^0)$ and $(M_R)$;
(iii) $SF = Pr$;
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Definitions

A pomonoid is a monoid $S$ partially ordered by $\leq$, such that $\leq$ is compatible with the semigroup operation.

- A pomonoid $S$ is ordered left reversible if $sS \cap (tS) \neq \emptyset$ for all $s, t \in S$, or for any $s, t \in S$ there exists $u, v \in S$ such that $su \leq tv$.

- We say $S$ is right collapsible if for any $s, t \in S$ there exists $u \in S$ such that $su = tu$.

We note that notion of ordered right collapsible pomonoid and right collapsible pomonoid coincides.

- $(\text{FP}_0)$: if every subpomonoid generated by idempotents have a right zero; i.e. $M = \langle e : e \in E(S) \rangle$ have a right zero element in $M$.
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Let $S$ be a monoid and let $A$ be a non-empty set. We say that $A$ is a *left $S$-act* if with the following function $S \times A \rightarrow A$, it satisfies the following conditions:

(i) $1.a = a$ for all $a \in A$;

(ii) $(st)a = s(ta)$ for all $s, t \in S$ and $a \in A$. 

$S$-Acts
Let $S$ be a monoid and let $A$ be a non-empty set. We say that $A$ is a \textit{left $S$-act} if with the following function $S \times A \to A$, it satisfies the following conditions:

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A map $\alpha : A \to B$ from a left $S$-act $A$ to a left $S$-act $B$ called an \textit{$S$-morphism} if it preserves the action of $S$, that is $(sa)\psi = s(a\psi)$ for all $a \in A$ and $s \in S$. 

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We will denote the category of left $S$-acts and $S$-morphism by $S$-Act.
Let $S$ be a pomonoid and let $A$ be a partially order set. We say that $A$ is *left $S$-poset* if it is an $S$-act and in addition if $s \leq t$ then $sa \leq ta$ and if $a \leq b$ then $sa \leq sb$ for all $s, t \in S$ and $a, b \in A$. 
\textbf{S-Posets}

- Let $S$ be a pomonoid and let $A$ be a partially order set. We say that $A$ is \textit{left S-poset} if it is an $S$-act and in addition if $s \leq t$ then $sa \leq ta$ and if $a \leq b$ then $sa \leq sb$ for all $s, t \in S$ and $a, b \in A$.

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S-Posets

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An $S$-act congruence $\rho$ on $A$ is called an $S$-poset congruence on $A$ if $A/\rho$ can be partially ordered such that it becomes an $S$-poset and the natural map $\nu : A \to A/\rho$ is a $S$-pomorphism.
Definitions

- **Projective S-Posets**: have the standard categorical definition and will be denoted by $\mathcal{Pr}$;
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- **Condition (P)**: A left \( S \)-poset satisfies Condition (P) if, for some \( s, t \in S \) and \( a, b \in A \), if \( sa \leq tb \) then there exists \( c \in A, u, v \in S \) such that \( a = uc, b = vc \) with \( su \leq tv \). We will denote the class of left \( S \)-posets satisfy Condition (P) by \( \mathcal{P} \).
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- **Condition (E)**: A left $S$-poset satisfies Condition (E) if, for some $s, t \in S$ and $a \in A$, if $sa \leq ta$ then there exists $c \in A, u \in S$ such that $a = uc$ with $su \leq tu$. We will denote the class of left $S$-posets satisfy Condition (E) by $\mathcal{E}$.

- **Strongly flat S-posets**: are those $S$-posets which satisfy Conditions (P) and (E), and will be denoted by $SF$. 

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Cyclic $S$-posets satisfying Condition (P)

Let $S$ be a pomonoid and let $B$ be a ordered left reversible subpomonoid of $S$ and let $\rho$ be the relation on $S$ defined by $s \rho t$ if and only if $s \sigma t \sigma s$ where $s \sigma t$ if there exists $n \in \mathbb{N}$ and $p_1, q_1, \cdots p_n, q_n \in B$ and $u_1, \cdots u_n \in S$ such that

$$s \leq u_1 p_1, \quad u_1 q_1 \leq u_2 p_2, \cdots, \quad u_n q_n \leq t$$

then

(i) $\rho$ is a left congruence ;
(ii) $B \subseteq [1]_{\rho}$;
(iii) $S/\rho$ satisfies Condition (P).

We note that left congruence relation $\rho$ defined above is the congruence generated by the relation $B \times B$. 
Cyclic $S$-posets satisfying Condition (P)

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Let $\rho$ be a congruence on $S$ such that $S/\rho$ satisfies Condition (P) and $R = [1]$. Then $R$ is a ordered left reversible subpomonoid of $S$. 
Cyclic $S$-posets which are Strongly Flat

Let $P \subseteq S$ be a right collapsible subpomonoid and let $\rho$ be the relation on $S$ defined by $s \rho t$ if and only if $s \sigma t \sigma s$, where $s \sigma t$ if there exists $n \in \mathbb{N}$ and $p_1, q_1, \ldots, p_n, q_n \in P$ and $u_1, \ldots, u_n \in S$ such that

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Then

(i) $\rho$ is a left congruence;
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(iii) $S/\rho$ is strongly flat.

Let $\rho$ be a left congruence on $S$ such that $S/\rho$ is strongly flat and let $P = [1]$. Then $P$ is a right collapsible subpomonoid.
Covers of cyclic $S$-posets

A left $S$-poset $A$ over a pomonoid $S$ is called a cover for a left $S$-poset $B$, if there exists an $S$-poset epimorphism $\beta : A \rightarrow B$, such that any restriction of $\beta$ to a proper $S$-subposet of $A$ is not an $S$-poepimorphism. Such a map $\beta$ is called coessential (minimal) $S$-poepimorphism.

▶ **Theorem**: Let $S$ be a pomonoid and $\sigma$, $\sigma'$ are left congruences on $S$. Then $S/\sigma'$ is isomorphic to a cyclic $S$-subposet of $S/\sigma$ if and only if there exists $u \in S$ such that $\sigma' = \{(s, t) \in S \times S : (su, tu) \in \sigma\}$. 
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▶ **Theorem:** Consider the $S$-pomonomorphism $h : S/\sigma' \to S/\sigma$ for some $u \in S$ as defined above, then $h$ is onto if and only if $Su \cap [1]_\sigma \neq \emptyset$. 
Covers of cyclic $S$-posets

**Theorem:** Let $S$ be a pomonoid and $\rho$ a left pocongruence on $S$. If $\sigma$ is a left pocongruence on $S$ such that $f : S/\sigma \to S/\rho$ is a coessential $S$-poset epimorphism then there exists $u \in S$ such that $S/\sigma_u \cong S/\sigma$ and $f' : S/\sigma_u \to S/\rho$ given by $(s \sigma_u)f' = [s]_\rho$ is a $S$-poset coessential epimorphism. In particular $[1]_{\sigma_u} \subseteq [1]_{\rho}$.
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- **Theorem**: Let $S$ be a pomonoid, then the $S$-pomorphism $f : S/\sigma \to S/\rho$ given by $[s]_\sigma \mapsto [s]_\rho$ is coessential if and only if $\sigma \subseteq \rho$ and for all $u \in [1]_\rho$, $Su \cap [1]_\sigma \neq \emptyset$. 
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- **Theorem:** Let $S$ be a pomonoid, and $S/\rho$ a cyclic $S$-sposet. If $R$ is a subpomonoid of $[1]_{\rho}$ such that $Su \cap R \neq \emptyset$ then there exists a left pocongruence $\sigma$ such that $R \subseteq [1]_{\sigma}$ and $S/\sigma$ is a cover of $S/\rho$. 
Covers of cyclic $S$-posets

- **Theorem:** Let $S$ be a pomonoid and $\rho$ a left pocongruence on $S$. If $\sigma$ is a left pocongruence on $S$ such that $f : S/\sigma \twoheadrightarrow S/\rho$ is a coessential $S$-poset epimorphism then there exists $u \in S$ such that $S/\sigma_u \cong S/\sigma$ and $f' : S/\sigma_u \twoheadrightarrow S/\rho$ given by $(s \sigma_u)f' = [s]_\rho$ is a $S$-poset coessential epimorphism. In particular $[1]_{\sigma_u} \subseteq [1]_\rho$.

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- **Theorem:** Let $S$ be a pomonoid, and $S/\rho$ a cyclic $S$-sposet. If $R$ is a subpomonoid of $[1]_\rho$ such that $Su \cap R \neq \emptyset$ then there exists a left pocongruence $\sigma$ such that $R \subseteq [1]_\sigma$ and $S/\sigma$ is a cover of $S/\rho$.

- **Theorem:** Let $S$ be a pomonoid, and $S/\rho$ a cyclic $S$-sposet, then the natural map $S \twoheadrightarrow S/\rho$ is coessential if and only if $[1]_\rho$ is a subgroup of $S$. 

Strongly flat Covers

Let $S$ be a pomonoid and $A$ be an $S$-poset, we say that $A$ has a *strongly flat cover* if there exists an coessential $S$-poepimorphism $\beta : C \to A$ where $C$ is a strongly flat $S$-poset.

**Condition (L₀):** every left pounitary subpomonoid $B$ of $S$ contains a right collapsible subpomonoid $R$ such that for all $u \in B$, $Su \cap R \neq \emptyset$.

▶ **Theorem:** Let $S$ be a pomonoid then the cyclic $S$-poset $S/\rho$ has a strongly flat cover if and only if every left pounitary subpomonoid $B$ contains a right collapsible subpomonoid $R$ such that for all $u \in B$, $Su \cap R \neq \emptyset$. 
Let $S$ be a pomonoid and $A$ be an $S$-poset, we say that $A$ has a **strongly flat cover** if there exists an coessential $S$-poepimorphism $\beta : C \to A$ where $C$ is a strongly flat $S$-poset.

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- **Corollary:** A pomonoid $S$ satisfies Condition (L₀) if and only if every cyclic $S$-poset has a strongly flat cover.
Condition (P) Covers

Let $S$ be a pomonoid and $A$ be an $S$-poset, we say that $A$ has a \((P)\) cover if there exists an coessential poepimorphism $\beta : C \rightarrow A$ where $C$ is an $S$-poset satisfying Condition (P).

**Condition (K\(O\)):** every left pounitary subpomonoid $B$ of $S$ contains a left reversible subpomonoid $R$ such that for all $u \in B$, $Su \cap R \neq \emptyset$.

- **Theorem:** Let $S$ be a pomonoid, then every cyclic $S$-poset has a (P)-cover if and only if every left pounitary subpomonoid $B$ of $S$ contains a left reversible subpomonoid $R$ such that for all $u \in B$, $Su \cap R \neq \emptyset$. 


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**Condition (K⁰):** every left pounitary subpomonoid $B$ of $S$ contains a left reversible subpomonoid $R$ such that for all $u \in B$, $Su \cap R \neq \emptyset$.

- **Theorem:** Let $S$ be a pomonoid, then every cyclic $S$-poset has a (P)-cover if and only if every left pounitary subpomonoid $B$ of $S$ contains a left reversible subpomonoid $R$ such that for all $u \in B$, $Su \cap R \neq \emptyset$.

- **Corollary:** A pomonoid $S$ satisfies Condition (K⁰) if and only if every cyclic $S$-poset has a (P)-cover.
Theorem: Let $A$ be a left $S$-poset that satisfies condition $(P)$ and which also satisfies the ascending chain condition for cyclic subposets. If $A$ is indecomposable then $A$ is cyclic.
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Theorem: If $S$ satisfies condition ($A^0$), then every left $S$-poset satisfies condition ($P$) is a disjoint union of cyclic left $S$-posets satisfies condition ($P$).
**$\mathcal{SF}$-perfect, $\mathcal{P}$-perfect**

**Definition:** A pomonoid $S$ is $\mathcal{SF}$-perfect ($\mathcal{P}$-perfect) if every $S$-poset has a strongly flat cover ($\mathcal{P}$-cover).

**Theorem:** Let $S$ be a pomonoid. The following conditions are equivalent;

- (i) $S$ is $\mathcal{SF}$-perfect ($\mathcal{P}$-perfect);
\textbf{Definition}: A pomonoid $S$ is $SF$-perfect ($P$-perfect) if every $S$-poset has a strongly flat cover ($P$-cover).

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- \hspace{1cm} (i) $S$ is $SF$-perfect ($P$-perfect);

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$SF$-perfect, $P$-perfect

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- (ii) $S$ satisfies condition $(A^0)$, and every cyclic left $S$-poset has a strongly flat cover ($P$-cover);
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Pomonoids for which condition \((P)\) implies projective

- **Theorem**: Let \(S\) be a pomonoid. All cyclic left \(S\)-posets that satisfy condition \((P)\) are projective if and only if \(S\) satisfies the condition \((K^0)\): if \(P \subseteq S\) is a ordered left reversible and right po-unitary subpomonoid then \(P\) contains a right zero.
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- **Theorem**: Let \(S\) be a pomonoid. All cyclic left \(S\)-posets that satisfy condition \((P)\) are projective if and only if \(S\) satisfies the condition \((K'\,^0)\): if \(P \subseteq S\) is an ordered left reversible and right po-unitary subpomonoid then \(P\) contains a right zero.

- **Theorem**: Let \(S\) be a pomonoid, \(S\) satisfies condition \((A^0)\) and \((K'\,^0)\) if and only if all \(S\)-posets that satisfy condition \((P)\) are projective.
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- **Theorem**: Let \(S\) be a pomonoid such that every left \(S\)-poset satisfies Condition \((P)\) is projective. Then \(S\) satisfies \(M_R\).

- **Corollary**: Let \(S\) be a pomonoid such that every left \(S\)-poset satisfies Condition \((P)\) is projective then \(S\) is left po-perfect.
Pomonoids for which condition \( (P) \) implies \( SF \)

A pomonoid \( S \) is called *aperiodic* if for every element \( x \in S \) there exists \( n \in \mathbb{N} \) such that \( x^n = x^{n+1} \).

▶ **Theorem**: \( S \) be a aperiodic pomonoid if and only if every non-trivial left reversible (ordered left reversible) monogenic subpomonoid of \( S \) contains a right zero.
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Pomonoids for which condition \((P)\) implies \(SF\)

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  \item \textbf{Theorem}: For any pomonoid \(S\) if all cyclic left \(S\)-posets that satisfy condition \((P)\) are strongly flat then \(S\) is aperiodic pomonoid.
\end{itemize}
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Theorem: Let $S$ be a pomonoid such that every cyclic left $S$-poset which satisfies condition ($P$) is projective then $S$ is aperiodic and satisfies ($FP_0$).
Theorem: Let $S$ be a pomonoid which satisfies condition $(FP_0)$, if $S$ satisfies condition $(A)$ and is aperiodic then all left $S$-poset that satisfy condition $(P)$ are strongly flat.
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Theorem: Let $A$ be a strongly flat left $S$-poset then $A$ is a strongly flat as a left $S$-act.
Locally Cyclic $S$-posets

We say that a left $S$-poset $A$ over a pomonoid $S$ is locally cyclic if every finitely generated $S$-subposet of $A$ is contained in a cyclic $S$-poset.

▶ **Theorem**: Let $S$ be a pomonoid, and let $A$ be a left $S$-poset that satisfy condition $(P)$. If $A$ is indecomposable then $A$ is locally cyclic.
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- **Theorem**: Any cover of a locally cyclic left $S$-poset is indecomposable.

- **Corollaries**: For a pomonoid $S$, the following are true;
  (i) every projective cover of a locally cyclic left $S$-poset is cyclic;
  (ii) every $\mathcal{P}$-cover of a locally cyclic left $S$-act is locally cyclic.
Po-Perfect Pomonoids

**Theorem:** Let $S$ be a pomonoid, then the following are equivalent;

- (i) $S$ is left po-perfect;
- (ii) every strongly flat left $S$-poset has a projective cover;
- (iii) $S$ satisfies condition (A) and every cyclic strongly flat left $S$-poset has a projective cover;
- (iv) every locally cyclic strongly flat left $S$-poset has a projective cover.
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**Theorem:** Let \(S\) be a pomonoid, then the following are equivalent;

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**Theorem:** Let $S$ be a pomonoid, then the following are equivalent;

- (i) $S$ is left $SF$-perfect ($P$-perfect);
- (ii) $S$ satisfies condition ($A^0$) and every locally cyclic left $S$-poset has a strongly flat cover (condition (P) cover);
Theorem: Let $S$ be a pomonoid, and let $A$ be a locally cyclic left $S$-poset. Then $A$ is strongly flat if and only if $A$ satisfies condition $(E)$.
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Corollary: every $SF$-cover of a locally cyclic left $S$-poset is locally cyclic.
Examples

- Let $S$ be a ordered right cancellative (right cancellative) pomonoid. The cyclic $S$-poset $S/\rho$ has a strongly flat cover if and only if $[1]_\rho$ is a subgroup of $S$. 

- Let $S$ be a pomonoid, and $S/\rho$ a cyclic $S$-poset. If $S$ contains a ordered right zero (right zero) say $z$ such that $z \in [1]_\rho$ then $S/\rho$ has a strongly flat cover.

- Not all cyclic $S$-poset need have a strongly flat cover. For example $(\mathbb{N},.)$ under usual ordering having only subgroup $\{1\}$.

- Then all cyclic $S$-poset having $[1]_\rho \neq \{1\}$ do not have a strongly flat cover.

- If $S$ is a group then every cyclic $S$-poset has $S$ as a cover. Also each cyclic $S$-poset have a strongly flat cover.

- Let $S$ be a simple pomonoid, then $S$ satisfies condition (A) and each cyclic $S$-poset have a cover which satisfies condition (P). Thus $S$ is P-perfect.
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- Let $S$ be an ordered right cancellative (right cancellative) pomonoid. The cyclic $S$-poset $S/\rho$ has a strongly flat cover if and only if $[1]_\rho$ is a subgroup of $S$.

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Summary

We have following facts relating to covers

- Projective Cover $\Rightarrow$ Strongly flat Cover $\Rightarrow$ (P)- Cover;
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- Projective Cover $\Rightarrow$ Strongly flat Cover $\Rightarrow$ (P)- Cover;
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Open Problems

- We would like to know either \((K^0)\) implies \((K'^0)\) or not.
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Open Problems

▶ We would like to know either \( (K^0) \) implies \( (K'{}^0) \) or not.

▶ To characterised those pomonoids for which each \( S \)-poset has flat cover, po-flat cover, weakly flat cover and principally weakly flat cover.

▶ To characterise those conditions on \( S \) such that \( SF \)-cover and \( P \)-cover are unique;