Ordered Covers

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Motivation

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- Roghaieh, Majid and Mojtaba (2009), $\mathcal{P}$-covers and $\mathcal{SF}$-covers, are defined for S-acts;
- Gould and Shaheen (2009), investigate pomonoids for which $\mathcal{P}r = \mathcal{SF}$ in **S-Pos**.

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Theorem: For a pomonoid S the following are equivalent;

(i) S left is po-perfect;

(ii) S satisfies Condition $(\mathrm{A}^{\mathrm{o}})$ and (M_R);

(iii) $\mathcal{SF} = \mathcal{P}r$;

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($\mathbf{A^o}$): Ascending chain condition on cyclic S-subposets.

Definitions

A pomonoid is a monoid S partially ordered by $\leq$, such that $\leq$ is compatible with the semigroup operation.

- A pomonoid S is *ordered left reversible* if $sS \cap (tS] \neq \emptyset$ for all $s, t \in S$, or for any $s, t \in S$ there exists $u, v \in S$ such that $su \leq tv$;

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- (FP_0): if every subpomonoid generated by idempotents have a right zero; i.e
  $$M =< e : e \in E(S) >$$
  have a right zero element in M.

S-Acts

- Let S be a monoid and let A be a non-empty set. We say that A is a *left S-act* if with the following function $S \times A \to A$, it satisfies the following conditions;

  (i) $1.a = a$ for all $a \in A$;

  (ii) $(st)a = s(ta)$ for all $s, t \in S$ and $a \in A$.

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- A map $\alpha : A \to B$ from a left S-act A to a left S-act B called an *S-morphism* if it preserves the action of S, that is $(sa)\psi = s(a\psi)$ for all $a \in A$ and $s \in S$.

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- We will denote the category of left S-acts and S-morphism by **S-Act**.

S-Posets

- Let S be a pomonoid and let A be a partially order set. We say that A is *left S-poset* if it is an S-act and in addition if $s \leq t$ then $sa \leq ta$ and if $a \leq b$ then $sa \leq sb$ for all $s, t \in S$ and $a, b \in A$.

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- An order preserving map $\psi : A \to B$ from a left S-poset A to a left S-poset B called an *S-pomorphism* if it preserves the action of S, that is $(sa)\psi = s(a\psi)$ for all $a \in A$ and $s \in S$. We will denote the category of left S-posets and S-pomorphisms by **S-Pos**.

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- An S-act congruence ρ on A is called an *S-poset congruence* on A if A/ρ can be partially ordered such that it becomes an S-poset and the natural map $\nu : A \to A/\rho$ is a S-pomorphism.

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- **Condition (P)**: A left S-poset satisfies Condition (P) if, for some $s, t \in S$ and $a, b \in A$, if $sa \leq tb$ then there exists $c \in A, u, v \in S$ such that $a = uc, b = vc$ with $su \leq tv$. We will denote the class of left S-posets satisfy Condition (P) by $\mathcal{P}$.

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- **Condition (E)**: A left S-poset satisfies Condition (E) if, for some $s, t \in S$ and $a \in A$, if $sa \leq ta$ then there exists $c \in A, u \in S$ such that $a = uc$ with $su \leq tu$. We will denote the class of left S-posets satisfy Condition (E) by $\mathcal{E}$.

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- **Condition (E)**: A left S-poset satisfies Condition (E) if, for some $s, t \in S$ and $a \in A$, if $sa \leq ta$ then there exists $c \in A, u \in S$ such that $a = uc$ with $su \leq tu$. We will denote the class of left S-posets satisfy Condition (E) by $\mathcal{E}$.

- **Strongly flat S-posets**: are those S-posets which satisfy Conditions (P) and (E), and will be denoted by $\mathcal{SF}$.

Cyclic S-posets satisfying Condition (P)

- Let S be a pomonoid and let B be a ordered left reversible subpomonoid of S and let ρ be the relation on S defined by $s\rho t$ if and only if $s\,\sigma\,t\,\sigma\,s$ where $s\sigma t$ if there exists $n \in \mathbb{N}$ and $p_1, q_1, \cdots p_n, q_n \in B$ and $u_1, \cdots u_n \in S$ such that

$$s \leq u_1 p_1,\ u_1 q_1 \leq u_2 p_2, \cdots, u_n q_n \leq t$$

then
(i) ρ is a left congruence ;
(ii) $B \subseteq [1]_\rho$;
(iii) S/ρ satisfies Condition (P).

We note that left congruence relation ρ defined above is the congruence generated by the relation $B \times B$.

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- Let ρ be a congruence on S such that S/ρ satisfies Condition (P) and $R = [1]$. Then R is a ordered left reversible subpomonoid of S.

Cyclic S-posets which are Strongly Flat

- Let $P \subseteq S$ be a right collapsible subpomonoid and let ρ be the relation on S defined by $s\,\rho\,t$ if and only if $s\,\sigma\,t\,\sigma\,s$, where $s\,\sigma\,t$ if there exists $n \in \mathbb{N}$ and $p_1, q_1, \ldots, p_n, q_n \in P$ and $u_1, \ldots, u_n \in S$ such that

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  Then
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- Let ρ be a left congruence on S such that S/ρ is strongly flat and let $P = [1]$. Then P is a right collapsible subpomonoid.

Covers of cyclic S-posets

A left S-poset A over a pomonoid S is called a *cover* for a left S-poset B, if there exists an S-poset epimorphism $\beta : A \to B$, such that any restriction of β to a proper S-subposet of A is not an S-poepimorphism. Such a map β is called *coessential (minimal)* S-poepimorphism.

- **Theorem**: Let S be a pomonoid and σ, $\sigma^{'}$ are left congruences on S. Then $S/\sigma^{'}$ is isomorphic to a cyclic S-subposet of S/σ if and only if there exists $u \in S$ such that $\sigma^{'} = \{(s,t) \in S \times S : (su, tu) \in \sigma\}$.

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- **Theorem**: Consider the S-pomonomorphism $h : S/\sigma^{'} \to S/\sigma$ for some $u \in S$ as defined above, then h is onto if and only if $Su \cap [1]_\sigma \neq \emptyset$.

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- **Theorem**: Let S be a pomonoid and ρ a left pocongruence on S. If σ is a left pocongruence on S such that $f : S/\sigma \to S/\rho$ is a coessential S-poset epimorphism then there exists $u \in S$ such that $S/\sigma_u \cong S/\sigma$ and $f' : S/\sigma_u \to S/\rho$ given by $(s\,\sigma_u)f' = [s]_\rho$ is a S-poset coessential epimorphism. In particular $[1]_{\sigma_u} \subseteq [1]_\rho$.

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- **Theorem**: Let S be a pomonoid, then the S-pomorphism $f : S/\sigma \to S/\rho$ given by $[s]_\sigma \mapsto [s]_\rho$ is coessential if and only if $\sigma \subseteq \rho$ and for all $u \in [1]_\rho,\ \ Su \cap [1]_\sigma \neq \emptyset$.

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- **Theorem**: Let S be a pomonoid, then the S-pomorphism $f : S/\sigma \to S/\rho$ given by $[s]_\sigma \mapsto [s]_\rho$ is coessential if and only if $\sigma \subseteq \rho$ and for all $u \in [1]_\rho,\ \ Su \cap [1]_\sigma \neq \emptyset$.
- **Theorem**: Let S be a pomonoid, and S/ρ a cyclic S-sposet. If R is a subpomonoid of $[1]_\rho$ such that $Su \cap R \neq \emptyset$ then there exists a left pocongruence σ such that $R \subseteq [1]_\sigma$ and S/σ is a cover of S/ρ.

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- **Theorem**: Let S be a pomonoid, and S/ρ a cyclic S-sposet. If R is a subpomonoid of $[1]_\rho$ such that $Su \cap R \neq \emptyset$ then there exists a left pocongruence σ such that $R \subseteq [1]_\sigma$ and S/σ is a cover of S/ρ.
- **Theorem**: Let S be a pomonoid, and S/ρ a cyclic S-sposet, then the natural map $S \to S/\rho$ is coessential if and only if $[1]_\rho$ is a subgroup of S.

Strongly flat Covers

Let S be a pomonoid and A be an S-poset, we say that A has a *strongly flat cover* if there exists an coessential S-poepimorphism $\beta : C \to A$ where C is a strongly flat S-poset.

Condition ($\mathbf{L^o}$): every left pounitary subpomonoid B of S contains a right collapsible subpomonoid R such that for all $u \in B$, $Su \cap R \neq \emptyset$.

- **Theorem**: Let S be a pomonoid then the cyclic S-poset S/ρ has a strongly flat cover if and only if every left pounitary subpomonoid B contains a right collapsible subpomonoid R such that for all $u \in B$, $Su \cap R \neq \emptyset$.

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- **Corollary**: A pomonoid S satisfies Condition (L^O) if and only if every cyclic S-poset has a strongly flat cover.

Condition (P) Covers

Let S be a pomonoid and A be an S-poset, we say that A has a *(P) cover* if there exists an coessential poepimorphism $\beta : C \to A$ where C is an S-poset satisfying Condition (P).
Condition ($\mathbf{K^o}$): every left pounitary subpomonoid B of S contains a left reversible subpomonoid R such that for all $u \in B$, $Su \cap R \neq \emptyset$.

- **Theorem**: Let S be a pomonoid, then every cyclic S-poset has a (P)-cover if and only if every left pounitary subpomonoid B of S contains a left reversible subpomonoid R such that for all $u \in B$, $Su \cap R \neq \emptyset$.

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- **Corollary**: A pomonoid S satisfies Condition (K^o) if and only if every cyclic S-poset has a (P)-cover.

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- **Theorem**: If S satisfies condition $(\mathrm{A}^{\mathrm{o}})$, then every left S-poset satisfies condition (P) is a disjoint union of cyclic left S-posets satisfies condition (P).

$\mathcal{SF}$-perfect, $\mathcal{P}$-perfect

Definition: A pomonoid S is $\mathcal{SF}$-perfect ($\mathcal{P}$-perfect) if every S-poset has a strongly flat cover (P-cover).

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- (ii) S satisfies condition (A^o), and every cyclic left S-poset has a strongly flat cover (P-cover);
- (iii) S satisfies condition (A^o), and (L^o)((K^o))

Pomonoids for which condition (P) implies projective

- **Theorem**: Let S be a pomonoid. All cyclic left S-posets that satisfy condition (P) are projective if and only if S satisfies the condition
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- **Corollary**: Let S be a pomonoid such that every left S-poset satisfies Condition (P) is projective then S is left po-perfect.

Pomonoids for which condition (P) implies $\mathcal{SF}$

A pomonoid S is called *aperiodic* if for every element $x \in S$ there exists $n \in \mathbb{N}$ such that $x^n = x^{n+1}$.

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Locally Cyclic S-posets

We say that a left S-poset A over a pomonoid S is locally cyclic if every finitely generated S-subposet of A is contained in a cyclic S-poset.

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- **Corollaries**; For a pomonoid S, the following are true;
  (i) every projective cover of a locally cyclic left S-poset is cyclic;
  (ii) every $\mathcal{P}$-cover of a locally cyclic left S-act is locally cyclic.

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Theorem: Let S be a pomonoid, then the following are equivalent;

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- (iv) every locally cyclic strongly flat left S-poset has a projective cover.

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- **Theorem**: Let S be a pomonoid, and let A be a locally cyclic left S-poset. Then A is strongly flat if and only if A satisfies condition (E).

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- **Corollary**: every $\mathcal{SF}$-cover of a locally cyclic left S-poset is locally cyclic.

Examples

- Let S be a ordered right cancellative (right cancellative) pomonoid. The cyclic S-poset S/ρ has a strongly flat cover if and only if $[1]_\rho$ is a subgroup of S.

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- Let S be a simple pomonoid, then S satisfies condition (A) and each cyclic S-poset have a cover which satisfies condition (P). Thus S is $\mathcal{P}$-perfect.

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We have following facts relating to covers

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- To characterise those conditions on S such that $\mathcal{SF}$-cover and $\mathcal{P}$-cover are unique;