Reconsidering MacLane: the foundations of categorical coherence

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Talks is based on:

- Talk at AbramskyFest (Oxford, June 2013)

**Topic of talk:**

Foundations of category theory & “MacLane’s Theorem”
Category Theory is simply a calculus of mathematical\(^1\) structures.

It studies:

- Mathematical structures.
- Structure-preserving mappings.
- Transformations between structures.

\(^1\) or logical, or computational, or linguistic, or . . .
It arose from work by:

- Samuel Eilenberg,
- Saunders MacLane,

in *Algebraic Topology*.

Later applied (despite protests) in other subjects:

- Theoretical Computing
- Linguistics
- Logic
- Quantum Mechanics
- Foundations of Mathematics
History & prehistory

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- Foundations of Mathematics
Precursors to category theory

John von Neumann (1925): Axiomatic theory of classes.

A formalism for working with **proper classes**: 

\[ \text{All sets, all monoids, all lattices, &c.} \]

Later became the **von Neumann, Gödel, Bernay** formalism

- **von Neumann** originated the theory. *(proto-cat. theory)*
- **Gödel** made it logically consistent.
- **Bernay** rewrote it to look like ZFC set theory ....
Applications of category theory in various fields

... a large range of texts.

The underlying theory of categories:

“Categories for the Working Mathematician”
— S. MacLane (1971)

... examples & applications taken from algebraic topology.
Applications of category theory in various fields

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The underlying theory of categories:

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... examples & applications taken from algebraic topology.
A category $\mathcal{C}$ consists of

- A class of objects, $\text{Ob}(\mathcal{C})$.
- For all objects $A, B \in \text{Ob}(\mathcal{C})$, a set of arrows $\mathcal{C}(A, B)$.

We will work diagrammatically:

An arrow $f \in \mathcal{C}(A, B)$ is drawn as

\[ A \xrightarrow{f} B \]
Matching arrows can be composed

Composition is associative

There is an identity $1_A$ at each object $A$
Examples of categories

- **Monoid**
  - (Objects:) *all monoids*.
  - (Arrows:) *homomorphisms*.

- **Set**
  - (Objects:) *all sets*.
  - (Arrows:) *functions*.

- **Poset**
  - (Objects:) *all partially ordered sets*.
  - (Arrows:) *order-preserving functions*.
Identities and equations are usually expressed graphically.

A diagram in the category \textbf{Set}

\[
\begin{array}{c}
\mathbb{Z} \\
\downarrow \quad \quad \downarrow \\
\mathbb{N} \\
\end{array}
\]

\[
\begin{array}{c}
x \mapsto x^2 \\
\uparrow \\
x \mapsto \text{abs}(x) \\
\end{array}
\quad \quad \quad \quad 
\begin{array}{c}
n \mapsto n^2 \\
\end{array}
\]

A diagram \textbf{commutes} when all paths with the same source / target describe the same arrow.
Commuting diagrams can be pasted along a common edge.

Both the above diagrams commute . . .
Commuting diagrams can be pasted along a common edge.

\[ \mathbb{Z} \xrightarrow{x \mapsto x^2} \mathbb{N} \]

\[ x \mapsto \text{abs}(x) \]

\[ \mathbb{N} \xrightarrow{n \mapsto n^2} \mathbb{N} \]

\[ n \mapsto n \pmod{2} \]

\[ n \mapsto n \pmod{2} \]

\[ \mathbb{N} \xrightarrow{n \mapsto \{0, 1\}} \]

... this diagram also commutes!
Edges can be deleted in commuting diagrams.

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{x \mapsto x^2} & \mathbb{N} \\
\downarrow & & \downarrow \\
\mathbb{N} & \xrightarrow{n \mapsto n^2} & \mathbb{N} \\
\downarrow & & \downarrow \\
\mathbb{N} & \xrightarrow{n \mapsto \{0, 1\}} & \{0, 1\} \\
\end{array}
\]
Edges can be **deleted** in commuting diagrams.

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{x \mapsto x^2} & \mathbb{N} \\
\downarrow & & \downarrow \\
\mathbb{N} & \xrightarrow{n \mapsto n \ (\text{mod} \ 2)} & \{0, 1\}
\end{array}
\]

\[x \mapsto \text{abs}(x)\]

\[n \mapsto n \ (\text{mod} \ 2)\]

\ldots this is still a commuting diagram.
A mapping between categories $C$ and $D$ is a functor $\Gamma : C \to D$.

- **Objects** of $C$ are mapped to **objects** of $D$.
- **Arrows** of $C$ are mapped to **arrows** of $D$. 

\[ \begin{array}{c}
\text{Category } C \\
\Gamma(A) \quad \Gamma(f) \quad \Gamma(B) \\
\text{Category } D
\end{array} \]
Functors must preserve **composition** and **identities**.

\[ \Gamma(1_X) = 1_{\Gamma(X)} , \quad \Gamma(gf) = \Gamma(g)\Gamma(f) \]

Functors preserve commutativity of diagrams.

\[ \begin{array}{c}
U \xrightarrow{f} V \\
\downarrow{h} \quad \downarrow{g} \\
W \xrightarrow{j} X
\end{array} \]

commutes in \( C \)
Functors must preserve **composition** and **identities**.

\[ \Gamma(1_X) = 1_{\Gamma(X)} \quad \text{and} \quad \Gamma(gf) = \Gamma(g)\Gamma(f) \]

*Functors preserve commutativity of diagrams.*

\[
\begin{array}{ccc}
U & \overset{f}{\longrightarrow} & V \\
\downarrow^{h} & & \downarrow^{g} \\
W & \overset{k}{\longrightarrow} & X \\
\downarrow^{j} & & \downarrow \\
\end{array}
\]

commutes in \( C \)

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Functors preserve commutativity of diagrams.
A functor from **Set** to **Monoid**.

- Take a set $X$.
- Form the *free monoid* $X^*$ (*All finite words over* $X$).

Every function $f : X \rightarrow Y$ induces a homomorphism $map(f) : X^* \rightarrow Y^*$

This is a functor $Free : \textbf{Set} \rightarrow \textbf{Monoid}$. 
Examples of functors (II)

A functor from $\mathbf{Top}_*$ to $\mathbf{Group}$.

- Take a pointed topological space $T$
- Form its fundamental group $\pi_1(T)$

Every continuous map $c : S \to T$
induces a homomorphism

$$\pi(f) : \pi_1(S) \to \pi_1(T)$$

This is a functor $\pi : \mathbf{Top}_* \to \mathbf{Group}$.
In general:

- finding **invariants** (e.g. fundamental group, $K_0$ group, &c.)
- using **constructors** (e.g. monoid semi-ring construction)
- type **re-assignments** (e.g. $\text{Int} \rightarrow \text{Real}$)
- forming **algebraic models**
  (e.g. Brouwer-Heyting-Kolmogorov interpretation)
- ...

are all examples of functors.
Monoidal Categories

and

MacLane’s Theorem
A **tensor** \( X \otimes Y \) on a category is:

a way of combining two objects / arrows to make a new object / arrow of the same category.

- **Objects:** Given \( X, Y \), we can form \( X \otimes Y \).
- **Arrows:** Given \( f, g \), we can form \( f \otimes g \).
Properties of tensors:

A **tensor** is a functor:

\[ _\otimes_ : C \times C \to C \]

Functoriality implies:

1/ Types match:

\[ A \xrightarrow{f} B \]
\[ X \xrightarrow{h} Y \]
Properties of tensors:

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Functoriality implies:

1/ Types match:

\[ A \otimes X \xrightarrow{f \otimes h} B \otimes Y \]
A **tensor** is a functor:

\[ _\otimes_ : C \times C \rightarrow C \]

Functoriality implies:

2/ Composition is preserved:

\[
\begin{align*}
A & \xrightarrow{f} B & C \\
X & \xrightarrow{h} Y & Z
\end{align*}
\]
A tensor is a functor:

\[ \_ \otimes \_ : C \times C \to C \]

Functoriality implies:

2/ Composition is preserved:

\[ \begin{array}{c}
A \\ \downarrow^{gf} \\
C \\
\end{array} \quad \begin{array}{c}
X \\ \downarrow^{kh} \\
Z \\
\end{array} \]
A **tensor** is a functor:

\[\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\]

Functoriality implies:

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Functoriality implies:

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\[ \begin{align*}
A & \xrightarrow{f} B & \xrightarrow{g} C \\
X & \xrightarrow{h} Y & \xrightarrow{k} Z
\end{align*} \]
A **tensor** is a functor:

\[
\_ \otimes \_ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}
\]

Functoriality implies:

2/ Composition is preserved:

\[
A \otimes X \xrightarrow{f \otimes h} B \otimes Y \xrightarrow{g \otimes k} C \otimes Z
\]
Properties of tensors:

A tensor is a functor:

\[
_\_ \otimes _\_ : C \times C \to C
\]

Functoriality implies:

2/ Composition is preserved:

\[
A \otimes X \xrightarrow{(g \otimes k)(f \otimes h) = gf \otimes kh} C \otimes Z
\]
Familiar examples

- **Tensor product** of Hilbert spaces / bounded linear maps
- **Cartesian product** (pairing) of Sets / functions
- **Direct sum** of Vector spaces / matrices
- **Disjoint union** of Sets / functions
- **Combining** Binary trees
- …
The final conditions

We also require:

- **Associativity**
  \[ f \otimes (g \otimes h) = (f \otimes g) \otimes h \]

- **A unit object** \( I \in \text{Ob}(\mathcal{C}) \)
  \[ X \otimes I = X = I \otimes X \quad \text{for all objects } X \in \text{Ob}(\mathcal{C}) \]
Monoidal categories usually $^2$ have a unit object $I \in Ob(C)$

$A \otimes I = A = I \otimes A$ for all objects $A \in Ob(C)$

These are *trivial* objects within a category:

- The single-element set.
- The trivial monoid.
- The empty space.
- The underlying scalar field.
- The trivially true proposition.

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$^2$ Part of the original definition. Later shown not to be essential (Saavedra72 / Kock08 / PH13).
A problem, and MacLane’s solution

The problem ...

In real-world examples, the condition

\[ f \otimes (g \otimes h) = (f \otimes g) \otimes h \]

is almost never satisfied.

... and its solution.

MacLane’s theorem lets us pretend that

\[ f \otimes (g \otimes h) = (f \otimes g) \otimes h \]

with no harmful side-effects.
The problem ...

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**MacLane’s theorem** lets us *pretend* that

\[ f \otimes (g \otimes h) = (f \otimes g) \otimes h \]

with no harmful side-effects.
Failure of associativity - an example

Associativity often fails, *in a trivial way!*

The **disjoint union** of sets

Given sets $A, B$,

$$A \uplus B = \{(a, 0)\} \cup \{(b, 1)\}$$

This is not associative ... for ridiculous reasons.
Non-associativity of disjoint union

\[ A \uplus (B \uplus C) = \{(a, 0)\} \cup \{(b, 01)\} \cup \{(c, 11)\} \]

\[ (A \uplus B) \uplus C = \{(a, 00)\} \cup \{(b, 10)\} \cup \{(c, 1)\} \]

These are not the same set – for annoying syntactical reasons.

There is an obvious isomorphism between them ...
Non-associativity of disjoint union

\[ A \uplus (B \uplus C) = \]
\[ \{(a, 0)\} \cup \{(b, 01)\} \cup \{(c, 11)\} \]

\[ (A \uplus B) \uplus C = \]
\[ \{(a, 00)\} \cup \{(b, 10)\} \cup \{(c, 1)\} \]

These are not the same set – for annoying syntactical reasons.

*There is an obvious isomorphism between them ...*
Replacing equality by isomorphism:

- **Strict** associativity:
  \[ A \otimes (B \otimes C) = (A \otimes B) \otimes C \]

- Associativity up to **isomorphism**
  \[ A \otimes (B \otimes C) \xrightarrow{\tau_{ABC}} (A \otimes B) \otimes C \xleftarrow{\tau_{-1}} \]

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How to ignore isomorphisms

Provided the *associativity isomorphisms* satisfy:

1. **naturality**
2. **A coherence condition**

we can *ignore them completely*.

Natural examples generally satisfy these conditions!
We can ‘push arrows through associativity isomorphisms’

\[ \tau(f \otimes (g \otimes h)) = ((f \otimes g) \otimes h)_{\tau} \]
We can ‘push arrows through associativity isomorphisms’

\[ \tau(f \otimes (g \otimes h)) = ((f \otimes g) \otimes h) \tau \]
MacLane’s coherence condition

The two ways of re-arranging

\[ A \otimes (B \otimes (C \otimes D)) \]

into

\[ ((A \otimes B) \otimes C) \otimes D \]

must be *identical*.

Also called MacLane’s Pentagon condition

\[ \tau \tau = (\tau \otimes 1) \tau (1 \otimes \tau) \]
MacLane’s coherence condition

The two ways of re-arranging

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must be \textit{identical}.

Also called \textbf{MacLane’s Pentagon condition}

\[ \tau \tau = (\tau \otimes 1) \tau (1 \otimes \tau) \]
Rebracketing four symbols

\[ A \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes ((B \otimes C) \otimes D) \]

\[ (A \otimes B) \otimes (C \otimes D) \]

\[ (A \otimes (B \otimes C)) \otimes D \]

\[ ((A \otimes B) \otimes C) \otimes D \]
Yes, there are two paths you can go by, but ...

**MacLane’s pentagon**

\[
\begin{array}{c}
A \otimes (B \otimes (C \otimes D)) \\
\downarrow \tau_{-,-,-} \\
(A \otimes B) \otimes (C \otimes D) \\
\downarrow \tau_{-,-,-} \\
((A \otimes B) \otimes C) \otimes D \\
\downarrow \tau_{-,-,-} \otimes 1_-
\end{array}
\]

\[
\begin{array}{c}
1_\otimes \tau_{-,-,-} \\
\downarrow \tau_{-,-,-} \\
A \otimes ((B \otimes C) \otimes D) \\
\downarrow \tau_{-,-,-} \\
(A \otimes (B \otimes C)) \otimes D
\end{array}
\]

commutes!
Mac Lane’s coherence theorem:

When we have

1. Naturality
2. Coherence

every **canonical diagram** – built up using

\[ \tau_{\cdot,\cdot,\cdot}, \ _{\otimes} \ _ \text{ and } 1_ \]

is *guaranteed* to commute.
A consequence:

Given a tensor that is \textit{associative up to isomorphism},

\[
A \otimes (B \otimes C) \xrightarrow{\tau_{ABC}} A \otimes (B \otimes C) \xleftarrow{\tau_{ABC}^{-1}} A \otimes (B \otimes C)
\]

We can ‘pretend it is \textit{strictly associative’}

\[
A \otimes (B \otimes C) = A \otimes (B \otimes C)
\]

with no “harmful side-effects”.

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The theory of coherence has written itself out of existence!

By appealing to MacLane’s theorem ...

We can completely ignore questions of coherence, naturality, pentagons, canonical diagrams, &c.
Two common descriptions of MacLane’s theorem:

1. Every canonical diagram commutes.

2. We can treat

\[ A \otimes (B \otimes C) \overset{\tau_{A,B,C}}{\longrightarrow} (A \otimes B) \otimes C \]

\[ (A \otimes B) \otimes C \overset{\tau^{-1}_{A,B,C}}{\longrightarrow} A \otimes (B \otimes C) \]

as a strict identity

\[ A \otimes B \otimes C = A \otimes B \otimes C \]

with no ‘harmful side-effects’.

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Two inaccurate descriptions of MacLane’s theorem:

1. Every canonical diagram commutes.

2. We can treat

\[ A \otimes (B \otimes C) \]  \quad \xrightarrow{\tau_{A,B,C}} \quad  \quad (A \otimes B) \otimes C \]

\[ (A \otimes B, C) \]  \quad \xleftarrow{\tau_{A,B,C}} \quad  \quad (A \otimes B) \otimes C \]

as a strict identity

\[ A \otimes B \otimes C \quad = \quad A \otimes B \otimes C \]

with no ‘harmful side-effects’.

Coherence in Hilbert’s hotel

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Two contrary claims:

- Not every canonical diagram commutes.  
  (Claim 1)

- Treating associativity **isomorphisms** as **strict identities** can have major consequences.\(^3\)  
  (Claim 2)

\(^3\)everything collapses to a triviality ...
A simple example:

The **Cantor monoid** $\mathcal{U}$ (single-object category).

- Single object $\mathbb{N}$.
- Arrows: all bijections on $\mathbb{N}$.

The tensor

We have a tensor $(\_ \star \_): \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{U}$.

$$(f \star g)(n) = \begin{cases} 
2.f \left( \frac{n}{2} \right) & \text{n even,} \\
2.g \left( \frac{n-1}{2} \right) + 1 & \text{n odd.}
\end{cases}$$
The Cantor monoid has only one object —

\[ N \star (N \star N) = N = (N \star N) \star N \]

\((\_ \star \_): U \times U \rightarrow U\) is associative up to a natural isomorphism

\[
\tau(n) = \begin{cases} 
2n & n \pmod{2} = 0, \\
 n + 1 & n \pmod{4} = 1, \\
 \frac{n-1}{2} & n \pmod{4} = 3.
\end{cases}
\]

that satisfies MacLane's pentagon condition.

This is not the identity map!
Not all canonical diagrams commute:

This diagram does \textit{not} commute.
Using an actual number:

On the upper path, $1 \mapsto 2$. 
Taking the right hand path:

\[ 1 \not= 2, \text{ so this diagram does not commute.} \]
What does MacLane’s thm. actually say?
A recent (May 2013) report

“Hines uses MacLane’s theorem – the fact that all canonical diagrams commute – to construct a large class of examples where . . . ”

— Anonymous Referee

(Category Theory / Theoretical Computing journal).
If in doubt ...

... ask the experts:


“It follows that **any diagram** whose morphisms are built using [canonical isomorphisms], identities and tensor product commutes.”
Do not as a rule rely on Wikipedia as your sole source of information.

The best thing about Wikipedia are the external links from entries.
Moreover all diagrams involving [canonical iso.s] must commute. (p. 158)

These three [coherence] diagrams imply that "all" such diagrams commute. (p. 159)

We can only prove that every "formal" diagram commutes. (p. 161)
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MacLane, on MacLane’s theorem

Moreover all diagrams involving [canonical iso.s] must commute. (p. 158)

These three [coherence] diagrams imply that “all” such diagrams commute. (p. 159)

We can only prove that every “formal” diagram commutes. (p. 161)
MacLane’s coherence theorem for associativity

All diagrams within the image of a certain functor are guaranteed to commute.

This usually means all canonical diagrams.

In some circumstances, this is not the case.
Dissecting MacLane’s theorem
— a closer look

A technicality:

In common with MacLane, we study \textit{monogenic categories}.

Objects are generated by:

- Some object $S$,
- The tensor $(\_, \_)$.

This is not a restriction — $S$ is thought of as a ‘variable symbol’.
Dissecting MacLane’s theorem
— a closer look

A technicality:
In common with MacLane, we study *monogenic categories*. Objects are generated by:

- Some object $S$,
- The tensor $(\_ \otimes \_)$.

This is not a restriction — $S$ is thought of as a ‘variable symbol’.
The source of the functor

This is based on (non-empty) *binary trees*.

- **Leaves** labelled by $x$,
- **Branchings** labelled by $\square$.

The **rank** of a tree is the number of leaves.
A posetal category of trees

MacLane's category $\mathcal{W}$.

- **(Objects)** All non-empty binary trees.
- **(Arrows)** A unique arrow between any two trees of the same rank.

— write this as $(v \leftarrow u) \in \mathcal{W}(u, v)$.

Key points:

1. $(\square\_\_)$ is a tensor on $\mathcal{W}$.
2. $\mathcal{W}$ is posetal — all diagrams over $\mathcal{W}$ commute.
MacLane’s theorem relies on a monoidal (i.e. tensor-preserving) functor

\[ \mathcal{W} \text{Sub} : (\mathcal{W}, \square) \rightarrow (\mathcal{C}, \otimes) \]

This is based on a notion of substitution.

i.e. mapping formal symbols to concrete objects & arrows.
The functor itself

On objects:

- \( \mathcal{W} \text{Sub}(x) = S \),
- \( \mathcal{W} \text{Sub}(u \Box v) = \mathcal{W} \text{Sub}(u) \otimes \mathcal{W} \text{Sub}(v) \).

An object of \( \mathcal{W} \):
An inductively defined functor (I)

On objects:

- $\mathcal{W} \text{Sub}(x) = S$,
- $\mathcal{W} \text{Sub}(u \Box v) = \mathcal{W} \text{Sub}(u) \otimes \mathcal{W} \text{Sub}(v)$.

An object of $\mathcal{C}$:

![Diagram of objects connected by symbols representing tensor products and inclusions.](image-url)
On arrows:

- \( \mathcal{W} \text{Sub}(u \leftarrow u) = 1_1. \)

- \( \mathcal{W} \text{Sub}(a \square v \leftarrow a \square u) = 1_1 \otimes \mathcal{W} \text{Sub}(v \leftarrow u). \)

- \( \mathcal{W} \text{Sub}(v \square b \leftarrow u \square b) = \mathcal{W} \text{Sub}(v \leftarrow u) \otimes 1_1. \)

- \( \mathcal{W} \text{Sub}((a \square b) \square c \leftarrow a \square (b \square c)) = \tau_{1,1,1}. \)

The role of the Pentagon

The Pentagon condition \( \implies \mathcal{W} \text{Sub} \) is a monoidal functor.
On arrows:

- \( W\text{Sub}(u \leftarrow u) = 1 \).

- \( W\text{Sub}(a □ v \leftarrow a □ u) = 1 \otimes W\text{Sub}(v \leftarrow u) \).

- \( W\text{Sub}(v □ b \leftarrow u □ b) = W\text{Sub}(v \leftarrow u) \otimes 1 \).

- \( W\text{Sub}((a □ b) □ c \leftarrow a □ (b □ c)) = \tau_{-,-,-} \).

The role of the Pentagon

The Pentagon condition \( \implies \) \( W\text{Sub} \) is a monoidal functor.
An inductively defined functor (II)

On arrows:

- $\mathcal{W}Sub(u \leftarrow u) = 1$.
- $\mathcal{W}Sub(a □ v \leftarrow a □ u) = 1 \otimes \mathcal{W}Sub(v \leftarrow u)$.
- $\mathcal{W}Sub(v □ b \leftarrow u □ b) = \mathcal{W}Sub(v \leftarrow u) \otimes 1$.
- $\mathcal{W}Sub((a □ b) □ c \leftarrow a □ (b □ c)) = \tau_{\cdot,\cdot,\cdot}$.

The role of the Pentagon

The Pentagon condition $\implies \mathcal{W}Sub$ is a monoidal functor.
An inductively defined functor (II)

On arrows:

- \( \mathcal{W} \text{Sub}(u \leftarrow u) = 1_\_ \).
- \( \mathcal{W} \text{Sub}(a\square v \leftarrow a\square u) = 1_\_ \otimes \mathcal{W} \text{Sub}(v \leftarrow u) \).
- \( \mathcal{W} \text{Sub}(v\square b \leftarrow u\square b) = \mathcal{W} \text{Sub}(v \leftarrow u) \otimes 1_\_ \).
- \( \mathcal{W} \text{Sub}((a\square b)\square c \leftarrow a\square (b\square c)) = \tau_{\_,\_,\_} \).

The role of the Pentagon

The Pentagon condition \( \implies \mathcal{W} \text{Sub} \) is a monoidal functor.
The story so far ... 

We have a functor $\mathcal{W}_{\text{Sub}} : (\mathcal{W}, \square) \to (\mathcal{C}, \otimes)$.

- Every object of $\mathcal{C}$ is the image of an object of $\mathcal{W}$
- Every canonical arrow of $\mathcal{C}$ is the image of an arrow of $\mathcal{W}$
- Every diagram over $\mathcal{W}$ commutes.

As a corollary:

The image of every diagram in $(\mathcal{W}, \square)$ commutes in $(\mathcal{C}, \otimes)$.

Question: Are all canonical diagrams in the image of $\mathcal{W}_{\text{Sub}}$?

- This is only the case when $\mathcal{W}_{\text{Sub}}$ is an embedding!

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The story so far ...

We have a functor $\mathcal{W}_{Sub} : (\mathcal{W}, \square) \to (\mathcal{C}, \otimes)$.

- Every **object** of $\mathcal{C}$ is the image of an object of $\mathcal{W}$
- Every **canonical arrow** of $\mathcal{C}$ is the image of an arrow of $\mathcal{W}$
- Every **diagram** over $\mathcal{W}$ commutes.

As a corollary:

The image of every diagram in $(\mathcal{W}, \square)$ commutes in $(\mathcal{C}, \otimes)$.

**Question:** Are all canonical diagrams in the image of $\mathcal{W}_{Sub}$?

- This is only the case when $\mathcal{W}_{Sub}$ is an **embedding**!
We have a functor $\mathcal{W}_{Sub} : (\mathcal{W}, \Box) \to (\mathcal{C}, \otimes)$.

- Every object of $\mathcal{C}$ is the image of an object of $\mathcal{W}$
- Every canonical arrow of $\mathcal{C}$ is the image of an arrow of $\mathcal{W}$
- Every diagram over $\mathcal{W}$ commutes.

As a corollary:

The image of every diagram in $(\mathcal{W}, \Box)$ commutes in $(\mathcal{C}, \otimes)$.

Question: Are all canonical diagrams in the image of $\mathcal{W}_{Sub}$?

- This is only the case when $\mathcal{W}_{Sub}$ is an embedding!
We have a functor $\mathcal{W}_{\text{Sub}} : (\mathcal{W}, \Box) \rightarrow (\mathcal{C}, \otimes)$.

- Every **object** of $\mathcal{C}$ is the image of an object of $\mathcal{W}$
- Every **canonical arrow** of $\mathcal{C}$ is the image of an arrow of $\mathcal{W}$
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The story so far ...

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“A beautiful (useful) theory slain by an ugly counterexample”?

A full theory of coherence for associativity is:

- more *mathematically* elegant,
- *much more* practically useful!
single-object categories

\[ \mathcal{W} \text{Sub} : (\mathcal{W}, \square) \rightarrow (\mathcal{C}, \otimes) \text{ can never be an} \]

embedding when \( \mathcal{C} \) has a \textbf{finite} set of objects.

The \textit{Cantor monoid} has precisely one object

Where did this come from?
Hilbert’s Hotel

A children’s story about infinity.
Hilbert’s “Grand Hotel”

An infinite corridor, with rooms numbered $0, 1, 2, 3, \ldots$

\[ \mathbb{N} \leftrightarrow \mathbb{N} \quad \text{the successor function.} \]

\[ \mathbb{N} \simeq \mathbb{N} \cup \mathbb{N} \quad \text{the Cantor pairing.} \]

\[ \mathbb{N} \simeq \mathbb{N} \times \mathbb{N} \quad \text{an exercise!} \]

\[ [\mathbb{N} \to \{0, 1\}] \quad \text{is not isomorphic to } \mathbb{N} \]
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The categorical identity $S \cong S \otimes S$

Exhibited by two canonical isomorphisms:

- (Code) $\triangleleft : S \otimes S \to S$
- (Decode) $\triangleright : S \to S \otimes S$

These are unique (up to unique isomorphism).

Examples

- The natural numbers $\mathbb{N}$, Separable Hilbert spaces, Infinite matrices, Cantor set & other fractals, &c.
- C-monoids, and other untyped (single-object) categories with tensors
- Any unit object $I$ of a monoidal category . . .
Self-similarity

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Examples

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- Any unit object $I$ of a monoidal category . . .
At a self-similar object $S$, we may define a tensor by

$$( _\star _ ) \text{ makes } C(S, S) \text{ a single-object monoidal category!}$$
The tensor $(_*_{})$ is associative \textit{up to isomorphism}.

Claim: This is the identity arrow \textit{precisely when} the object $S$ is trivial.
The tensor \((\_ \star \_\) \) is associative \textit{up to isomorphism}. 

\[ \begin{array}{c}
S \xrightarrow{\triangleright} S \otimes S \xrightarrow{1_S \otimes \triangleright} S \otimes (S \otimes S) \\
\Downarrow \tau \\
S \xleftarrow{\triangleleft} S \otimes S \xleftarrow{\triangleleft \otimes 1_S} (S \otimes S) \otimes S \\
\Downarrow \tau_{S,S,S}
\end{array} \]

\textbf{Claim}: This is the identity arrow \textit{precisely when} the object \(S\) is trivial.
constructing categories where all canonical diagrams commute
Given a **badly-behaved** category \((\mathcal{C}, \otimes)\), we can **build a well-behaved (non-strict) version**.

Think of this as the **Platonic Ideal** of \((\mathcal{C}, \otimes)\).

We (still) assume \(\mathcal{C}\) is *monogenic*, with objects generated by \(\{S, _\otimes _\}\).
We will construct $\text{Plat}_C$

A version of $C$ for which $\mathcal{WSub}$ is an embedding.
Constructing $\text{Plat}_C$

**Objects** are free binary trees

![Binary Tree Diagram]

*Leaves* labelled by $S \in \text{Ob}(C)$,

*Branchings* labelled by $\square$.

There is an **instantiation map** $\text{Inst} : \text{Ob}(\text{Plat}_C) \rightarrow \text{Ob}(C)$

$S\square((S\square S)\square S) \mapsto S \otimes ((S \otimes S) \otimes S)$

This is not just a matter of syntax!

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What about arrows?

Homsets are copies of homsets of $\mathcal{C}$

Given trees $T_1, T_2$,

$$\text{Plat}_\mathcal{C}(T_1, T_2) = \mathcal{C}(\text{Inst}(T_1), \text{Inst}(T_2))$$

Composition is inherited from $\mathcal{C}$ in the obvious way.
The tensor \( (\square) : Plat_C \times Plat_C \to Plat_C \) is

- **(Objects)** A free formal pairing, \( A \square B \),
- **(Arrows)** Inherited from \((C, \otimes)\), so \( f \square g \overset{\text{def.}}{=} f \otimes g \).
The functor

\[ \mathcal{W} \text{Sub} : (\mathcal{W}, \square) \rightarrow (\text{Plat}_C, \square) \]

is always monic.

As a corollary:

All canonical diagrams of \((\text{Plat}_C, \square)\) commute.

Instantiation defines an epic monoidal functor

\[ \text{Inst} : (\text{Plat}_C, \square) \rightarrow (\mathcal{C}, \otimes) \]

through which McL’\text{’}s substitution functor always factors.
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All canonical diagrams of \((\text{Plat}_C, \square)\) commute.

3. Instantiation defines an epic monoidal functor

\[ \text{Inst} : (\text{Plat}_C, \square) \to (\mathcal{C}, \otimes) \]

through which McL’.s substitution functor always factors.
MacLane’s substitution functor always factors through the platonic ideal:

\[
(W, \Box) \xrightarrow{W_{Sub}} (\text{Plat}_C, \Box) \xrightarrow{\text{Inst}} (C, \otimes)
\]

This gives a monic / epic decomposition of his functor.
What does the Platonic Ideal of a **single-object** category actually look like?

The simplest possible case:

The trivial monoidal category \((\mathcal{I}, \otimes)\).

- **Objects:** \(\text{Ob}(\mathcal{I}) = \{x\}\).
- **Arrows:** \(\mathcal{I}(x, x) = \{1_x\}\).
- **Tensor:**
  \[ x \otimes x = x , \quad 1_x \otimes 1_x = 1_x \]
What is the platonic ideal of $\mathcal{I}$?

(Objects) All non-empty binary trees:

(Arrows) For all trees $T_1, T_2$,

$Plat_\mathcal{I}(T_1, T_2)$ is a single-element set.

There is a unique arrow between any two trees!
A Ph.D. Thesis: The prototypical self-similar category $(\mathcal{X}, \square)$

- **Objects:** All non-empty binary trees.
- **Arrows:** A unique arrow between any two objects.

This monoidal category:

1. was introduced to study self-similarity $S \simeq S \otimes S$,
2. contains MacLane's $(\mathcal{W}, \square)$ as a subcategory.
Coherence for Self-Similarity

(a special case of a much more general theory)
A straightforward coherence theorem

We base this on the category \((X, □)\)

- **Objects** All non-empty binary trees.
- **Arrows** A unique arrow between any two trees.

This category is posetal — all diagrams over \(X\) commute.

We will define a monoidal substitution functor:

\[ X \text{Sub} : (X, □) \to (C, ⊗) \]
The self-similarity substitution functor

An inductive definition of \( \mathcal{X}_{\text{Sub}} : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes) \)

**On objects:**

\[
\begin{align*}
x & \mapsto S \\
u \Box v & \mapsto \mathcal{X}_{\text{Sub}}(u) \otimes \mathcal{X}_{\text{Sub}}(v)
\end{align*}
\]

**On arrows:**

\[
\begin{align*}
(x \leftarrow x) & \mapsto 1_S \in \mathcal{C}(S, S) \\
(x \leftarrow x \Box x) & \mapsto \triangleleft \in \mathcal{C}(S \otimes S, S) \\
(x \Box x \leftarrow x) & \mapsto \triangleright \in \mathcal{C}(S, S \otimes S) \\
(b \Box v \leftarrow a \Box u) & \mapsto \mathcal{X}_{\text{Sub}}(b \leftarrow a) \otimes \mathcal{X}_{\text{Sub}}(v \leftarrow u)
\end{align*}
\]
Interesting properties:

1. \( \mathcal{X}_{Sub} : (\mathcal{X}, \square) \to (\mathcal{C}, \otimes) \) is always functorial.

2. Every arrow built up from \( \{\triangleleft, \triangle, 1_S, \_ \otimes \_\} \) is the image of an arrow in \( \mathcal{X} \).

3. The image of every diagram in \( \mathcal{X} \) commutes.

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Interesting properties:

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   is the image of an arrow in $\mathcal{X}$.

3. The image of every diagram in $\mathcal{X}$ commutes.
There is a monic-epic decomposition of $\mathfrak{X}_{Sub}$.

Every canonical (for self-similarity) diagram in $(\text{Plat}_C, \square)$ commutes.
Relating associativity and self-similarity
A tale of two functors

Comparing the *associativity* and *self-similarity* categories.

<table>
<thead>
<tr>
<th>MacLane’s $(\mathcal{W}, \square)$</th>
<th>The category $(\mathcal{X}, \square)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Objects:</strong> Binary trees.</td>
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</tr>
<tr>
<td><strong>Arrows:</strong> Unique arrow between two trees <em>of the same rank.</em></td>
<td><strong>Arrows:</strong> Unique arrow between any two trees.</td>
</tr>
</tbody>
</table>

There is an obvious inclusion $(\mathcal{W}, \square) \hookrightarrow (\mathcal{X}, \square)$
Is associativity a restriction of self-similarity?

Does the following diagram commute?

\[ (\mathcal{W}, \Box) \xrightarrow{\mathcal{W}_{\text{Sub}}} (\mathcal{X}, \Box) \xrightarrow{\mathcal{X}_{\text{Sub}}} (\mathcal{C}, \otimes) \]

Does the **associativity** functor factor through the **self-similarity** functor?
Proof by contradiction:

Let’s assume this is the case.

Special arrows of \((\mathcal{X}, \square)\)

For arbitrary trees \(u, e, v\),

\[
\begin{align*}
t_{uev} &= ((u \square e) \square v \leftarrow u \square (e \square v)) \\
l_v &= (v \leftarrow e \square v) \\
r_u &= (u \leftarrow u \square e)
\end{align*}
\]
Since all diagrams over $X$ commute:

The following diagram over $(X, □)$ commutes:

Let's apply $\mathcal{X} \text{Sub}$ to this diagram.

By Assumption: $t_{uev} \mapsto \tau_{U,E,V}$ (assoc. iso.)

Notation: $u \mapsto U$, $v \mapsto V$, $e \mapsto E$, $l_v \mapsto \lambda_V$, $r_u \mapsto \rho_U$
Since all diagrams over $X$ commute:

The following diagram over $(X, □)$ commutes:

$$
\begin{array}{ccc}
  u □ (e □ v) & \xrightarrow{t_{uev}} & (u □ e) □ v \\
  \downarrow 1_u □ l_v & & \downarrow r_u □ 1_v \\
  u □ v & &
\end{array}
$$

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Since all diagrams over $X$ commute:

The following diagram over $(C, \otimes)$ commutes:

$$U \otimes (E \otimes V) \xrightarrow{\tau_{UEV}} (U \otimes E) \otimes V$$

$$\downarrow 1_U \otimes \lambda_U \quad \downarrow \rho_U \otimes 1_V$$

$$U \otimes V$$

This is MacLane’s **units triangle** — the defining equation for a unit (trivial) object.

The choice of $e$ was *arbitrary* — every object is trivial!
Since all diagrams over $X$ commute:

The following diagram over $(\mathcal{C}, \otimes)$ commutes:

$$U \otimes (E \otimes V) \xrightarrow{\tau_{UEV}} (U \otimes E) \otimes V$$

This is MacLane's **units triangle** — the defining equation for a unit (trivial) object.

The choice of $e$ was *arbitrary* — every object is trivial!
A general result

The following diagram commutes

\[ (\mathcal{W}, \Box) \rightarrow (\mathcal{X}, \Box) \]
\[ \mathcal{W}_{\text{Sub}} \rightarrow (\mathcal{C}, \otimes) \]
\[ \mathcal{W}_{\text{Sub}} \rightarrow (\mathcal{C}, \otimes) \]

exactly when \( (\mathcal{C}, \otimes) \) is degenerate —

i.e. all objects are trivial.
An important special case:
What is **strict self-similarity**?

Can the code / decode maps

\[ \langle : S \otimes S \rightarrow S , \rangle : S \rightarrow S \otimes S \]

be **strict identities**?

In **single-object** monoidal categories:

We only have one object, so \( S \otimes S = S \).

Take the identity as both the **code** and **decode** arrows.

Untyped \( \equiv \) Strictly Self-Similar.
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\[ \text{Untyped } \equiv \text{ Strictly Self-Similar.} \]
Generalising Isbell’s argument

1. **Strict associativity:** \( A \otimes (B \otimes C) = (A \otimes B) \otimes C \)
   
   All arrows of \((\mathbb{W}, \Box)\) are mapped to identities of \((\mathcal{C}, \otimes)\).

2. **Strict self-similarity:** \( S \otimes S = S \).
   
   All arrows of \((\mathcal{X}, \Box)\) are mapped to the identity of \((\mathcal{C}, \otimes)\).

\( \mathbb{W}_{\text{Sub}} \) trivially factors through \( \mathcal{X}_{\text{Sub}} \).

The conclusion

Strictly associative untyped monoidal categories are **degenerate**.
We see special cases of this in many areas:

- **(Monoid Theory)**
  Congruence-freeness (e.g. the polycyclic monoids).

- **(Group Theory)**
  No normal subgroups (e.g. Thompson’s group $\mathcal{F}$).

- **($\lambda$ calculus / Logic)**
  Hilbert-Post completeness / Girard’s dynamical algebra.

- **(Linguistics)**
  Recently (re)discovered ... not yet named!
Another perspective ...

Another way of looking at things:

**The ‘No Simultaneous Strictness’ Theorem**

One cannot have both

(I) **Associativity** \( A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \)

(II) **Self-Similarity** \( S \cong S \otimes S \)

as strict equalities.