INDEPENDENCE ALGEBRAS AND THE DISTRIBUTIVITY CONDITION

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Abstract. The *distributivity condition* arose out of the study of the connection between independence algebras and stable basis algebras. Stable basis algebras were introduced by Fountain and Gould and developed in a series of articles. They form a class of universal algebras, extending that of independence algebras. Gould showed that if a stable basis algebra satisfies the distributivity condition, then it is a reduct of an independence algebra. The distributivity condition is satisfied by independence algebras in the most familiar classes, such as vector spaces, and by all the known additional examples of stable basis algebras. In this note we give the first example of an independence algebra (and hence, of a stable basis algebra) *not* satisfying the distributivity condition.

1. Introduction and Preliminaries

The second author introduced the study of the endomorphism monoids of a class of universal algebras called *v^*-algebras*, which she named *independence algebras*. These appear first in an article of Narkiewicz [15] and were inspired by Marczewski's study of notions of independence, initiated in [14] (see [10] and the survey article [17]). Such algebras may be defined via properties of the closure operator $\langle - \rangle$ which takes a subset of an algebra to the subalgebra it generates. In an independence algebra, $\langle - \rangle$ must satisfy the *exchange property*, which guarantees that we have a well behaved notion of *rank* for subalgebras and hence for endomorphisms, generalising that of the dimension of a vector space. Further, independence algebras are *relatively free*. Precise definitions and further details may be found in [8]. We remark that sets, vector spaces and free acts over any group are examples of independence algebras. A full classification, which we will draw upon for this article, is given by Urbanik in [17].

The study of endomorphism monoids of independence algebras has flourished over the last twenty years (see, for example, [1, 2, 6, 7, 12]), since they provide the framework for understanding the common behaviours of several fundamental examples of monoids, including full transformation monoids and matrix rings over division rings. We denote the monoid of endomorphisms of an algebra $\mathbb{A}$ by $\mathrm{End}(\mathbb{A})$. If $\mathbb{A}$ is an independence algebra of finite rank n, then the set $\mathrm{Sing}(\mathbb{A})$ of endomorphisms of rank strictly less than n forms an idempotent generated ideal [6]. We remark that idempotent generated semigroups are ubiquitous, since every (finite) semigroup embeds into a (finite) idempotent generated semigroup [11].

The endomorphism monoid of an independence algebra $\mathbb{A}$ is regular. Surprisingly, regularity of $\mathrm{End}(\mathbb{A})$ is not necessary for the above results concerning idempotent generation. For example, the results of Laffey [13] show that if $\mathbb{A}$ is a free module of finite rank n over a Euclidean domain, then the set of non-identity idempotents of $\mathrm{End}(\mathbb{A})$ generates the subsemigroup of endomorphisms of rank strictly less than n. Fountain and the second author introduced in [4] a class of algebras called *stable basis algebras* that generalise free modules

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over Euclidean domains, in an attempt to put the results of Laffey, and later work of Fountain [3] and Ruitenberg [16], into a more general setting, an aim achieved in [5]. Stable basis algebras are in particular relatively free algebras in which the closure operator PC (pure closure) satisfies the exchange property. Certainly independence algebras are stable basis algebras. Finitely generated free left modules over left Ore Bezout domains and finitely generated free left $\mathbb{T}$-acts over any cancellative monoid $\mathbb{T}$ such that finitely generated left ideals of $\mathbb{T}$ are principal, are examples of stable basis algebras. We recall that a *Bezout domain* is an integral domain (not necessarily commutative) in which all finitely generated left and right ideals are principal. As for independence algebras, rank is well defined for subalgebras and endomorphisms of basis algebras, where now the rank is defined via the operator PC[1].

If $\mathbb{A}$ and $\mathbb{B}$ are algebras such that the universe (that is, the underlying set) B of $\mathbb{B}$ is contained in the universe of A of $\mathbb{A}$, then $\mathbb{B}$ is a *reduct* of $\mathbb{A}$ if every basic operation of $\mathbb{B}$ is the restriction to B of a basic operation of $\mathbb{A}$. Theorem 4.14 of [9] shows that if $\mathbb{B}$ is a stable basis algebra satisfying the *distributivity condition*, then $\mathbb{B}$ is a reduct of an independence algebra $\mathbb{A}$, having the same rank as $\mathbb{B}$. The distributivity condition is stated precisely in Section 2: essentially it says that unary operations distribute over basic n-ary operations, for $n \geq 2$, and is satisfied for all previously known examples of basis algebras.

The aim of this note is to prove the following:

Theorem. *Not all independence algebras and hence, not all stable basis algebras, satisfy the distributivity condition.*

2. The distributivity condition for independence algebras

If $\mathbb{B}$ is a stable basis algebra, then the monoid of non-constant unary term operations will be denoted by $\mathbb{T}$.

Definition 2.1. A stable basis algebra $\mathbb{B}$ satisfies the *distributivity condition* if the clone of $\mathbb{B}$ contains a generating set of basic operations such that for all $a \in T$ and n-ary basic term operations t, where $n \geq 2$, we have

$$a(t(x_1, \dots, x_n)) = t(a(x_1), \dots, a(x_n)).$$

Note that Definition 2.1 is stated here more precisely than in [9], since to show $\mathbb{B}$ does *not* satisfy the distributivity condition, we wish to show it is impossible to choose *any* generating set for the clone that witnesses this.

Certainly vector spaces, and (free) acts over any group satisfy the distributivity condition. As explained at the beginning of Section 4 of [9], all additional examples of independence algebras can also be shown to do so, with the possible exceptions of the $\mathbb{S}$-*homogeneous* algebras or $\mathbb{Q}$-*homogeneous* algebras, where $\mathbb{S}$ is a monoid and $\mathbb{Q}$ a quasifield. We now address these cases.

2.1. All $\mathbb{Q}$-homogeneous independence algebras have the distributivity property.

A *quasifield* $\mathbb{Q}$ is a set Q with at least two elements, together with two binary operations denoted by juxtaposition, and $-$, such that the multiplicative operation has a zero 0, the non-zero elements form a group under multiplication, and four further axioms hold (see 4.3

[1]In earlier articles, rank in a basis algebra was referred to as *PC-rank* but, as there is no ambiguity, we simply use the term *rank* here.

of [17]). In a $\mathbb{Q}$-homogeneous independence algebra $\mathbb{A}$ over a quasifield $\mathbb{Q}$, all fundamental k-ary operations f satisfy

$$f(a - ba_1, \dots, a - ba_k) = a - bf(a_1, \dots, a_k),$$

for all $a, b_1, \dots, b_k \in Q$, where subtraction and multiplication are the operations from $\mathbb{Q}$. Setting $b = 0$ and $a_i = a$ for $1 \le i \le k$, we obtain that $f(a, \dots, a) = a$. An inductive argument gives that the identity is the only non-constant unary term operation of $\mathbb{A}$. The distributivity property now follows trivially.

2.2. **Not all $\mathbb{S}$-homogeneous independence algebras have the distributivity condition.** Let $\mathbb{S}$ be a monoid such that all the non-invertible elements are left zeros. A good example is a group, which is exactly what we will take below. An n-ary operation on S is said to be $\mathbb{S}$-*homogeneous* if for all $s, s_1, \dots, s_n \in S$ we have

$$f(s_1, \dots, s_n)s = f(s_1 s, \dots, s_n s).$$

Since $\mathbb{S}$ is a monoid, the operations $f(x) = sx$ $(s \in S)$ are the only unary $\mathbb{S}$-homogeneous operations. A $\mathbb{S}$-homogeneous independence algebra has underlying set S and the basic operations form a set O of $\mathbb{S}$-homogeneous operations on S containing all the unary $\mathbb{S}$-homogeneous operations. The aim of this subsection is to show that with a careful choice of $\mathbb{S}$ and O, the resulting independence algebra $\mathbb{A}$ does not satisfy the distributivity condition.

Let $Z = \{z_1, z_2, \dots\}$ be a countably infinite set and let $E = \{z_2, z_4, \dots\}$. Let $\mathbb{FG} = \mathbb{FG}(Z)$ be the free group over Z with identity 1 and underlying set which we denote for brevity by $F = FG(Z)$. In the following, concatenation will always refer to the group operation of $\mathbb{FG}$.

Let $F_+ \subseteq F$ be the set of all non-identity elements of F whose normal form does not include any negative exponents, so that F_+ is the underlying set of the copy of the free semigroup $\mathbb{FS}$ on Z sitting inside $\mathbb{FG}$.

Since F and E are both countably infinite we may choose a function $h : F \to E$, where $w \mapsto h_w$, satisfying the following conditions:

$$\begin{array}{llclll} (h1) & h_{z_1 z_2^{-1}} & = & z_6 & (h2) & h_{z_3 z_2^{-1}} = z_8 \\ (h3) & h_{z_1 z_4^{-1}} & = & z_{10} & (h4) & h \text{ is injective.} \end{array}$$

Now let $\mathbb{A} = \langle F; \{\nu_c^{\mathbb{A}}\}_{c \in F}, g^{\mathbb{A}} \rangle$ where:

1. for each $c \in F$, $\nu_c^{\mathbb{A}}$ is the unary operation given by $\nu_c^{\mathbb{A}}(w) = cw$ (i.e. $\nu_c^{\mathbb{A}}$ acts as left translation by the element c in the group $\mathbb{FG}$);
2. $g^{\mathbb{A}}$ is the binary operation given by $g^{\mathbb{A}}(w_1, w_2) = h_{w_1 w_2^{-1}} w_2$.

Lemma 2.2. *The algebra $\mathbb{A}$ is a monoid independence algebra with underlying monoid $\mathbb{FG}$.*

Proof. We first remark that as $\mathbb{FG}$ is a group, it has no non-invertible elements, and so is a suitable monoid from which to build a monoid independence algebra. We have remarked that $\mathbb{FG}$-homogeneous unary operations are left translations and by construction, all left translations $\nu_c^{\mathbb{A}}, c \in F$ are fundamental in $\mathbb{A}$.

Finally, for all $w_1, w_2, w' \in F$, we have

$$g^{\mathbb{A}}(w_1 w', w_2 w') = h_{w_1 w' (w_2 w')^{-1}} w_2 w' = h_{w_1 w_2^{-1}} w_2 w' = g^{\mathbb{A}}(w_1, w_2) w',$$

so that $g^{\mathbb{A}}$ is $\mathbb{FG}$-homogeneous and hence $\mathbb{A}$ is a monoid independence algebra. $\square$

In order to show that $\mathbb{A}$ does not have the distributivity property, we need to examine the clone of $\mathbb{A}$. We let $L = \{\{\nu_c\}_{c\in F}, g\}$, be the language of $\mathbb{A}$, $X = \{x_1, x_2, \dots,\}$ a countably infinite set of variables (which we may think of as being linearly ordered according to their subscripts), and T the set of terms in the language L over X. The elements of the clone are obtained from the interpretation in $\mathbb{A}$ of elements in T.

For $i, j \in \mathbb{N}$ with $i \leq j$ let $\pi^j_i : F^j \to F^i$ be given by $(w_1, \dots, w_j) \mapsto (w_1, \dots, w_i)$, i.e. π^j_i is the projection to the first i coordinates.

For each term $t \in T$, let $a(t)$ be the largest n such that the variable x_n occurs in t and we define a function $\bar{t} : F^{a(t)} \to F$ by structural induction, as we now describe. We remark that $\bar{t}$ will essentially be the term function associated with t and usually denoted by $t^{\mathbb{A}}$. However, our definition of $\bar{t}$ is needed due to some minor technicalities involving the arities of term functions.

For each $i \in \mathbb{N}$ set $\bar{x}_i(w_1, \dots, w_i) = w_i$. If $t = \nu_c(s)$ for some s, then noting that $a(t) = a(s)$, let $\bar{t} = \nu^{\mathbb{A}}_c \circ \bar{s}$. Finally, for $t = g(t_1, t_2)$, we set

$$\bar{t} = g^{\mathbb{A}} \circ \left(\bar{t}_1 \circ \pi^{a(t)}_{a(t_1)}, \bar{t}_2 \circ \pi^{a(t)}_{a(t_2)} \right),$$

which is well-defined, as $a(t) \geq a(t_1), a(t_2)$. With some abuse of terminology, we will refer to all functions of the form $\bar{t}$ as term functions.

Given a term $t \in T$, let $\nu(t)$ to be the set of $c \in F$ such that ν_c appears in t. We define the *content* of t, denoted C_t, by

$$C_t = \bigcup_{c\in\nu(t)} \{z_i \in Z : z_i \text{ appears in the normal form of } c\}.$$

Note that $C_t \subseteq Z$ and is finite.

For each $t \in T$, we define $t^* \in \mathbb{N}$ by structural induction as follows: if $t = x_i$ for some i, then $t^* = i$, if $t = \nu_c(t_1)$ for some $t_1 \in T$, then $t^* = t^*_1$, and if $t = g(t_1, t_2)$ for some $t_1, t_2 \in T$, we set $t^* = t^*_2$. It is easy to see that t^* is the index of the variable that appears syntactically in the "right-most" position of t and clearly, $t^* \leq a(t)$.

The following lemma characterizes behaviour of the functions of the form $\bar{t}$, by connecting them to the group structure on the underlying set F of $\mathbb{A}$. This will be essential to our later arguments.

Lemma 2.3. *Let $t \in T$ such that $a(t) = n$. Then one of the following holds:*

(1) *there exist a $w \in FG(E \cup C_t)$ such that for all $\vec{y} \in F^n$,*

$$\bar{t}(\vec{y}) = w y_{t^*};$$

(2) *there exists a function $f : F^n \to FG(E \cup C_t)$ such that for all $\vec{y} \in F^n$,*

$$\bar{t}(\vec{y}) = f(\vec{y}) y_{t^*}.$$

In addition, there are sequences $z_{j_1}, z_{j_2}, \dots$ on Z, and $\vec{\mu}_1, \vec{\mu}_2, \dots$ on $(F_+)^n$ such that

(a) *$j_i \neq j_{i'}$ for $i \neq i'$, and*

(b) *z_{j_i} appears in the normal form of $f(\vec{\mu}_i)$ with a positive exponent.*

Proof. We remark that if f is as in Condition (2), then, in particular, the image of f is infinite, indeed, the image of f restricted to $(F_+)^n$ is infinite.

We prove the lemma by induction over the structure of t. If $t = x_i$ for some i, then $n = a(t) = t^* = i$ and $\bar{t}(\vec{y}) = \bar{x}_i(y_1, \dots, y_i) = y_i = 1y_{t^*}$ and the result holds with $w = 1$ in Condition (1).

Assume for induction that the result holds for any proper subterm of t.

Case (i) Suppose first that $t = \nu_\sigma(t_1)$ for some term t_1, so that $t^* = t_1^*$. Then $n = a(t) = a(t_1)$ and by induction, t_1 satisfies the conditions of the lemma.

Case (i)(a) If $\bar{t}_1(\vec{y}) = wy_{t_1^*}$ for some $w \in FG(E \cup C_{t_1})$, then $\bar{t}(\vec{y}) = \sigma\bar{t}_1(\vec{y}) = \sigma w y_{t_1^*} = \sigma w y_{t^*}$. Moreover, $\sigma w \subseteq FG(E \cup C_t)$, as $C_t = C_{t_1} \cup C_\sigma$, where C_σ is the set of generators that appear in the normal form of σ. Hence $\bar{t}(\vec{y})$ satisfies Condition (1).

Case (i)(b) Now suppose that Condition (2) holds for t_1, so there exists $f_1 : F^n \to FG(E \cup C_{t_1})$ and sequences $(z_{j_i})_{i\in\mathbb{N}}$ and $(\vec{\mu}_i)_{i\in\mathbb{N}}$ as in (2), such that $\bar{t}_1(\vec{y}) = f_1(\vec{y})y_{t_1^*}$. Then

$$\bar{t}(\vec{y}) = \sigma\bar{t}_1(\vec{y}) = \sigma f_1(\vec{y})y_{t_1^*} = \sigma f_1(\vec{y})y_{t^*}.$$

For each $i \in \mathbb{N}$ let $w_i = f_1(\vec{\mu}_i)$ and put $\sigma w_i =: \tau_i$. The normal form of w_i contains z_{j_i} with a positive exponent, so the normal form of τ_i will do so as well, unless z_{j_i} cancels against a $z_{j_i}^{-1}$. But, considering σ, there are only finitely many indices i for which $z_{j_i}^{-1}$ appears in the normal form of σ. It follows that for infinitely many values i, the element $\tau_i \in F$ contains z_{j_i} in its normal form with a positive exponent.

Define $f : F^n \to F$ by $f = \nu_\sigma^{\mathbb{A}} \circ f_1$. By the assumption on f_1, and as $C_t = C_{t_1} \cup C_\sigma$, we have $f : F^n \to FG(E \cup C_t)$.

We obtain that $\bar{t}(\vec{y}) = f(\vec{y})y_{t^*}$. It is easy to see that $\bar{t}$ satisfies Condition (2), with the sequences $(z_{j_i})_{i\in\mathbb{N}}$ and $(\vec{\mu}_i)_{i\in\mathbb{N}}$ obtained from the corresponding sequences for $\bar{t}_1$ by removing finitely many elements.

Case (ii) We now consider the case that $t = g(t_1, t_2)$ for some terms t_1, t_2. By induction the lemma holds for t_1 and t_2. Let $n_1 = a(t_1), n_2 = a(t_2)$, so that n is the maximum of n_1 and n_2, and $t^* = t_2^*$. Notice that $C_t = C_{t_1} \cup C_{t_2}$.

Case (ii)(a) Assume first that Condition (2) holds for t_2, so that

$$\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right) = f_2\left(\pi^n_{n_2}(\vec{y})\right) y_{t_2^*}$$

for some $f_2 : F^{n_2} \to FG(E \cup C_{t_2})$, such that there are sequences $(z_{j_i})_{i\in\mathbb{N}}$ and $(\vec{\mu}_i)_{i\in\mathbb{N}}$ as in (2).

Since $t^* = t_2^*$ we have

$$\bar{t}(\vec{y}) = h_{\bar{t}_1\left(\pi^n_{n_1}(\vec{y})\right)\left(\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right)\right)^{-1}}\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right) = h_{\bar{t}_1\left(\pi^n_{n_1}(\vec{y})\right)\left(\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right)\right)^{-1}} f_2\left(\pi^n_{n_2}(\vec{y})\right) y_{t^*}.$$

Define $f : F^n \to F$ by

$$f(\vec{y}) = h_{\bar{t}_1\left(\pi^n_{n_1}(\vec{y})\right)\left(\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right)\right)^{-1}} f_2\left(\pi^n_{n_2}(\vec{y})\right).$$

We have that $\bar{t}(\vec{y}) = f(\vec{y})y_{t^*}$, as required. Moreover, $f : F^n \to FG(E \cup C_t)$, by the conditions on f_2 and since the image of h lies in E. Let $\vec{\mu}'_i \in F_+^n$ be obtained by extending $\vec{\mu}_i$ to arity n with $n - n_2$ arbitrary elements from F_+. By Condition (2) for t_2, we have that z_{j_i} appears in the normal form of $f_2(\vec{\mu}_i) = w_i$ with a positive exponent. Now

$$f(\vec{\mu}'_i) = h_{\bar{t}_1\left(\pi^n_{n_1}(\vec{\mu}'_i)\right)(\bar{t}_2(\vec{\mu}_i))^{-1}} w_i.$$

By definition of h, the first factor is just an element of E, so in particular an element of F_+. It follows that the generator z_{j_i} in w_i cannot cancel, and hence appears in the normal form of $f(\vec{\mu}'_i)$.

Thus $\bar{t}$ satisfies Condition (2) with f and the sequences $(\vec{\mu}'_i)_{i\in\mathbb{N}}$ and $(z_{j_i})_{i\in\mathbb{N}}$.

Case (ii)(b) For our final case we assume that $\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right) = w_2 y_{t_2^*}$, for some $w_2 \in FG(E \cup C_{t_2})$. We make four further case distinctions.

(1) $\bar{t}_1$ satisfies Condition (1) and $t_1^* = t_2^*$. We have for $\vec{u} \in F^{n_1}$ that $t_1(\vec{u}) = w_1 u_{t_1^*}$ for some w_1, and then for $\vec{y} \in F^n$ we see that $\bar{t}_1\left(\pi^n_{n_1}(\vec{y})\right) = w_1 y_{t_1^*}$, and

$$\bar{t}(\vec{y}) = h_{\bar{t}_1\left(\pi^n_{n_1}(\vec{y})\right)\left(\bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right)\right)^{-1}} \bar{t}_2\left(\pi^n_{n_2}(\vec{y})\right) = h_{w_1 y_{t_1^*}(w_2 y^*_{t_2})^{-1}} w_2 y_{t_2^*} = h_{w_1 w_2^{-1}} w_2 y_{t^*}$$

as $t_1^* = t_2^*$. Thus $\bar{t}$ also satisfies Condition (1), as $h_{w_1 w_2^{-1}} w_2 \in FG(E \cup C_t)$.

(2) $\bar{t}_1$ satisfies Condition (2) with respect to $f_1 : F^{n_1} \to FG(E \cup C_{t_1})$ and $t_1^* = t_2^*$.
In this case

$$\bar{t}(\vec{y}) = h_{f_1\left(\pi^n_{n_1}(\vec{y})\right) w_2^{-1}} w_2 y_{t^*}.$$

We claim that $\bar{t}$ satisfies Condition (2). Let f be given by

$$f(\vec{y}) = h_{f_1\left(\pi^n_{n_1}(\vec{y})\right) w_2^{-1}} w_2.$$

Then $f : F^n \to FG(E \cup C_t)$ by the same argument as above, so it remains to construct appropriate sequences $(\vec{\mu}_i)_{i\in\mathbb{N}}$ and $(z_{j_i})_{i\in\mathbb{N}}$.

By the remark at the beginning of this proof, $f_1\left(\pi^n_{n_1}\left(F^n_+\right)\right)$ is infinite and hence so is $f_1\left(\pi^n_{n_1}\left(F^n_+\right)\right) w_2^{-1}$. The function h is injective and maps into E, so it follows that $h_{f_1(\pi^n_{n_1}(\vec{x}))w_2^{-1}}$ takes on infinitely many values z_{j_i} in E as $\vec{x}$ runs over F^n_+. Only finitely many of these values can cancel against a generator in the normal form of w_2. The existence of $(\vec{\mu}_i)_{i\in\mathbb{N}}$ and $(z_{j_i})_{i\in\mathbb{N}}$ follows.

(3) $\bar{t}_1$ satisfies Condition (1) and $t_1^* \neq t_2^*$. In this case we have that $\bar{t}_1(\vec{y}) = w_1 y_{t_1^*}$ for some $w_1 \in FG(E \cup C_{t_1})$, for all $\vec{y} \in F^{n_1}$, and thus

$$\bar{t}(\vec{y}) = h_{w_1 y_{t_1^*}(y_{t_2^*})^{-1} w_2^{-1}} w_2 y_{t^*}.$$

Consider the set $P \subset Z^2$ of pairs (u_1, u_2) for which $u_1 \neq u_2$, and neither u_1, u_2 nor their inverses appear in the normal forms of w_1 or w_2; clearly P is infinite. For any $(u_1, u_2) \in P$, the normal form of $w_1 u_1 u_2^{-1} w_2^{-1}$ contains the subexpression $u_1 u_2^{-1}$, and these are the only occurrences of u_1 and u_2 in the normal form. It follows that $w_1 u_1 u_2^{-1} w_2^{-1}$ takes on only distinct and hence infinitely many elements of F as (u_1, u_2) runs through P. As h is injective, $h_{w_1 u_1 u_2^{-1} w_2^{-1}}$ also takes on infinitely many values from E as (u_1, u_2) runs through P. Only finitely many of those values can cancel against generators from the normal form of w_2. Removing the corresponding pairs from P we see that $\bar{t}$ satisfies Condition (2) with respect to $f : F^n \to FG(E \cup C_t)$, where

$$f(\vec{y}) = h_{w_1 y_{t_1^*} y^{-1}_{t_2^*} w_2^{-1}} w_2,$$

with the $\vec{\mu}_i \in F^n_+$ being chosen so that $\vec{\mu}_i(t_1^*) = u_1^{(i)}$, $\vec{\mu}_i(t_2^*) = u_2^{(i)}$ and with arbitrary elements of F_+ in all other coordinates, where $(u_1^{(i)}, u_2^{(i)})$ runs over a cofinite subset of P and $(z_{j_i})_{i\in\mathbb{N}} = (h_{w_1 u_1^{(i)}(u_2^{(i)})^{-1} w_2^{-1}})_{i\in\mathbb{N}}$.

(4) $\bar{t}_1$ satisfies Condition (2) with respect to $f_1 : F^{n_1} \to FG(E \cup C_{t_1})$, and $t_1^* \neq t_2^*$.
This case is similar to the previous one. We have that

$$\bar{t}(\vec{y}) = h_{f_1\left(\pi^n_{n_1}(\vec{y})\right) y_{t_1^*} y^{-1}_{t_2^*} w_2^{-1}} w_2 y_{t^*}.$$

Let $P \subset Z^2$ be the set of pairs (u_1, u_2) for which $u_1, u_2 \notin E \cup C_t$, $u_1 \neq u_2$ and neither u_1, u_2 nor their inverses appear in the normal form of w_2; clearly P is infinite. If

$\vec{z} \in F_+^n$ and $(u_1, u_2) \in P$, then in the expression

$$f_1\left(\pi^n_{n_1}(\vec{z})\right) u_1 u_2^{-1} w_2^{-1},$$

u_1 and u_2^{-1} cannot cancel against any generators from the normal forms of w_2^{-1} and $f_1\left(\pi^n_{n_1}(\vec{z})\right)$, in the latter case because f_1 maps into $F(E \cup C_t)$. Arguing as previously we see that $\bar{t}$ satisfies Condition 2.

By structural induction, the lemma holds for all terms $t \in T$. □

We are ready show our main result.

Theorem 2.4. *The independence algebra $\mathbb{A}$ does not satisfy the distributivity property.*

Proof. By way of contradiction, assume that W is set of functions that generate the clone of $\mathbb{A}$ and witness the distributivity property. The clone of $\mathbb{A}$ contains the function $g^{\mathbb{A}}$. By the first three conditions of our choice of h, calculation shows that $g(z_1, z_2) = z_6 z_2, g(z_3, z_2) = z_8 z_2$, and $g(z_1, z_4) = z_{10} z_4$. Combined, these results show that g depends on both of its arguments.

It follows that W must contain an operation v that depends on more than one argument, for otherwise the entire clone of $\mathbb{A}$ would consist of functions that are essentially unary. As v is in the clone of $\mathbb{A}$, it is a composition of $\mathbb{A}$-operations and projections, and it is easy to see that such v must have the form $\bar{t} \circ \pi^n_m$ for some $t \in T$. Moreover, $W \setminus \{v\} \cup \{\bar{t}\}$ also generates the clone of $\mathbb{A}$ and witnesses the distributivity property. Thus, we may assume that $v = \bar{t}$ for a term $t \in T$; since v depends on at least two variables, so does $\bar{t}$.

Such a $\bar{t}$ must satisfy one of the two conditions from Lemma 2.3. As the first condition implies that $\bar{t}$ only depends on one variable, we must have instead that $\bar{t}(\vec{y}) = f(\vec{y}) y_{t^*}$, where f is as in Condition (2) of Lemma 2.3. Let $(\vec{\mu}_i)_{i\in\mathbb{N}}$ and $(z_{j_i})_{i\in\mathbb{N}}$ be the sequences associated to f satisfying (a) and (b) of Lemma 2.3, and set $w_1 = f(\vec{\mu}_1)$.

We have that $\bar{t}(\vec{\mu}_1) = f(\vec{\mu}_1)\mu_1^*$ - where μ_1^* is the t^*-th entry of $\vec{\mu}_1$. Choose $a \in Z \setminus (E \cup C_t)$. As we assume that $\mathbb{A}$ satisfies the distributivity property, and W is a witness of it, we have that $\nu_a(\bar{t}(\vec{\mu}_1)) = \bar{t}(\nu_a(\vec{\mu}_1))$, where, with abuse of notation, $\nu_a(\vec{\mu}_1) = a\vec{\mu}_1$ is the element of F^n obtained by multiplying every coordinate of $\vec{\mu}_1$ by a on the left. Hence $af(\vec{\mu}_1)\mu_1^* = f(a\vec{\mu}_1)a\mu_1^*$ and so $af(\vec{\mu}_1) = f(a\vec{\mu}_1)a$. Now, $f(a\vec{\mu}_1)$ is in the image of f which is contained in $FG(E \cup C_t)$ by Lemma 2.3. As $a \notin FG(E \cup C_t)$, the normal form of $f(a\vec{\mu}_1)a$, and hence the normal form of $af(\vec{\mu}_1)$, will end with a. However, $f(\vec{\mu}_1)$ is also an element of $FG(E \cup C_t)$, and so its normal form does not contain a. It follows that $f(\vec{\mu}_1) = 1$. However, by Lemma 2.3, the normal form of $f(\vec{\mu}_1)$ contains the generator z_{j_1}, a contradiction. □

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