Almost and Absolute Pure Acts over Semilattices

Salma Shaheen

19th January 2016

S-acts

Throughout, S is a monoid.

A right S-act is a nonempty set A together with a map

$$A \times S \to A,\ (a, s) \mapsto as$$

such that for all $a \in A$, $s, t \in S$

$$(as)t = a(st) \text{ and } a1 = a.$$

- For any $s \in S$, we have an operation $\boldsymbol{\varrho_s : A \to A}$ given by $\boldsymbol{(a)\varrho_s} = as$. The function $\boldsymbol{\varrho : S \to} T_A$ given by $(s)\boldsymbol{\varrho = \varrho_s}$ is a monoid morphism.

Conversly, if $\boldsymbol{\theta : S \to} T_A$ is monoid morphism, define

$$\boldsymbol{as = (a)\varrho_s}$$

Then A is an S-act.

Examples of S-acts

1. S is an S-act
2. Any right ideal of S is an S-act.
3. Let (K, +, ·) be a field and V be a left vector space. Then V is a left (K, ·)-act but not a (K, +)-act.
4. For any monoid S and a non-empty set A, define as = a for all a ∈ A, then A becomes a right S-act.

Subact of S-acts

Let A_S be an S-act and B ⊆ A, a nonempty subst. Then B is a subact of A if as ∈ B for all a ∈ B and s ∈ S.

Obviously, any right ideal of S is a subact of S_S.

Congruences and Morphisms for S-acts

- Let A be an S-act. An equivalence relation σ on A is called an S-act congruence or a congruence on A, if aσb implies (as) σ(bs) for a,b ∈ A and s ∈ S.
- If X ⊆ A×A, then σ(X) denote the smallest congruence on A containing X.
- A congruence σ is finitely generated if there exists a finite subset X ⊆ A×A such that σ= σ(X).
- The ordered pair (a,b) ∈ σ(X) if and only if either

$$a = b$$

or there exists a natural number n and a sequence

$$\begin{array}{cccccc} a = c_1t_1 & d_2t_2 = c_3t_3 & & \cdots & & d_nt_n = b \\ & d_1t_1 = c_2t_2 & d_3t_3 = c_4t_4 & \cdots & d_{n-1}t_{n-1} = c_nt_n & \end{array}$$

where $t_1, t_2, t_3, \cdots, t_n \in S$ and for each i= 1, · · ·, n either (c_i,d_i) or (d_i,c_i) is in X.

- If σ is a congruence on A, then A/σ is an S-act.
- An S-morphism from A to B is a map f : A → B with (as)f = (af) s for all a ∈ A and s ∈ S.

Free S-acts

- An S-act A is finitely generated if there exists a subset U of such that

$$A = \bigcup_{u \in U} uS \text{ and } |U| < \infty .$$

- An S-act is free if there exists a subset U of A such that $A = \bigcup_{u \in U} uS$ and each element $a \in A$ can be uniquely presented in the form $a = us$, $u \in U$ and $s \in S$.
- Let X be a nonempty set. Then free S-act F(X) on X exists.

Construction for F(X): Let

$$F(X) = X \times S$$

and define

$$(x , s)t = (x , st).$$

Then it is easy to check that F(X) is an S-act. With $x \mapsto (x,1)$, we have F(X) is free on X.

Notice that

$$(x , s) = (x , 1)s \equiv xs.$$

- Free S-acts are disjoint unions of copies of S.

Finitely Presented S-acts

An S-act A_S is cyclic is $A = < \{a\} >$, where $a \in A$.

An S-act A is finitely presented if

$$A \cong F/\sigma$$

for some finitely generated free S-act F and finitely generated congruence σ.

Proposition:

Let A_S be a cyclic S-act. Then A_S is finitely presented if and only if it is isomorphic to a factor act of S_S by a finitely generated right congruence on S, that is,

$$A \cong S/\sigma$$

where σ is finitely generated right congruence.

System of Equations over S-act

- Let A be an S-act. An equation over A has one of the three forms

$$xs = a \quad xs = xt \quad xs=yt$$

where $s,t \in S$, $a \in A$ and x, y are variables

- Let Σ be a system of equtions over A. Then Σ is consistent if Σ has solution in some S-act $B \supseteq A$.

Consistency criteria for Σ:

Let $\boldsymbol{\Sigma = \{ xs_i = a_i, xu_i = xv_j : s_i, u_j, v_j \in S, a_i \in A, 1 \leq i \leq n, 1 \leq j \leq m \}}$ and $\sigma = \langle (u_j, v_j) : 1 \leq j \leq m \rangle$. Then Σ is consistent if and only if for all $h, k \in S$ and for all $1 \leq i, i' \leq n$,

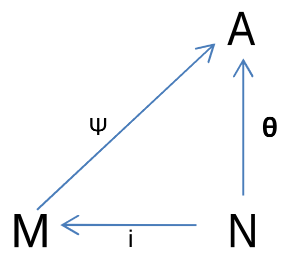
$$s_i h \ \sigma \ s_{i'} k \quad \text{implies } a_i h = a_{i'} k.$$

Almost Pure S-act

An S-system A is almost pure if every finite consistent system of equations in one varaible, with constants from A, has a solution in A.

Proposition: The following conditions are equivalent for as S-act A:

1. A is almost pure;
2. Given any diagram of S-acts and S-homomorphisms



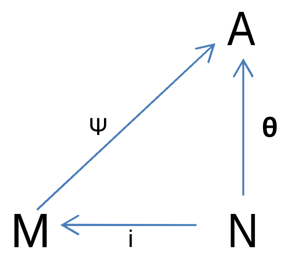
where M is cyclic finitely presented, N is finitely generated and I : N → M is an injection, there exists an S-homomorphism Ψ : M → A such that i Ψ = θ; further, for any $s_1, \cdots, s_n$ in S there is an element a in A with $a = as_1 = \cdots = as_n$.

Absolutely Pure S-acts

An S-system A is absolutely pure if every finite consistent system of equations, with constants from A, has a solution in A.

Proposition: The following conditions are equivalent for as S-act A:

1. A is absolutely pure;
2. Given any diagram of S-acts and S-homomorphisms



where M is finitely presented, N is finitely generated and I : N → M is an injection, there exists an S-homomorphism Ψ : M → A such that i Ψ = θ.

Completely Right Pure Monoids

A monoid S is called completely right pure if all right S-acts are absolutely pure.

Theorem: A monoid S is completely right pure if and only if all S-acts are almost pure.

Absolutely pure S-act ⇒ Almost Pure S-acts

For completely right pure monoids:

Almost Pure S-acts ⇒ Absolutely pure S-act

Does there exists an almost pure S-act which is not absolutely pure???

OR

Does there exists a class of monoids (Other than completely right pure) for which almost pure s-acts are absolutely pure????

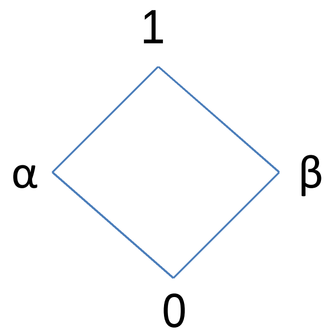
Theorem: A monoid S is completely right pure if and only if S has local left zeros and satisfies (*):
(*) given any finitely generated right congruence **σ** on S and any finitely generated right ideal I of S, there is an element s of I such that for any u,v in S, if u **σ** v then su **σ** sv and for any w I, w **σ** sw.

Corollary: Let I be finitely generated right ideal of a completely right pure monoid. Then I = eS for some idempotent element e of S.

❖ The following conditions are equivalent for a monoid S:

i. S is regular and its principle right ideals are linearly ordered with respect to inclusion.
ii. Every finitely generated right ideal is generated by an idempotent element.

Consider the semilattice S={0, α, β, 1}.



- S is an S-act over itself.
- S is not completely right pure because the principle right ideals {0, α} and {0, β} are not linearly ordered with respect to inclusion.
- S is an almost pure S-act over itself.

For S={0, α, β, 1}, every almost pure S-act is absolutely pure.

Proof: Let A be an arbitrary almost pure S-act.

S(n): All finite consistent system of equations over A in no more than n variables have a solution in A.

Clearly, the assumption is true for n = 1.

Let **Σ** be a finite consistent system of equations in n+1 variables $x_1, x_2, \dots, x_n, x_{n+1}$. Since **Σ** is consistent, **Σ** has a solution $(b_1, b_2, \dots, b_n, b_{n+1})$ in some S-act B ⊇ A.

Case I:

Σ has no equation of the form $x_i s = t$ for all i = 1, 2, … , n+1. Since A is almost pure S-act, so Solution of **Σ** exists in A.

Case II:

Σ has an equation of the form $x_i = t$ for some i = 1, 2, … , n+1. Since **Σ** is consistent. So $b_1 = a \in A$.

Construct

Σ' = Considering all equations of Σ in $x_2, \dots, x_n, x_{n+1}$ and replacing x_1 by a.

Σ' is consistent. So, by induction Σ' has a solution $(a_2, a_3, \ldots, a_n, a_{n+1})$ in A.

$(a_1 = a, a_2, \ldots, a_n, a_{n+1})$ **is the solution of Σ in A.**

Case III:

Suppose **Σ** has equations of the form :

$x_i s = a$ and $x_i = x_{i'} t$ for some i, i' = 1, 2, … , n+1 and i ≠ i'

Take I = 1 and i' = 2.

Construct

Σ' = Considering all equations of Σ in $x_2, \ldots, x_n, x_{n+1}$ and replacing x_1 by $x_2 t$.

Σ' is consistent because $(b_2, \ldots, b_n, b_{n+1})$ is the solution of Σ in B. So, by induction Σ' has a solution $(a_2, a_3, \ldots, a_n, a_{n+1})$ in A.

$(a_1 = a_2 s, a_2, \ldots, a_n, a_{n+1})$ **is the solution of Σ in A.**

Case IV:

Suppose **Σ** has equations of the form :

$$x_i s = a \quad \text{for } i=1,2,3,\ldots n$$

We prove this case for i = 1, 2 and by the similar argument it would be true for i = 1,2 , ... , n+1.

Rename x_1 by x and x_2 by y and consider

$$\mathbf{\Sigma = \{ x}\alpha = a,\ xs' = xt',\ xs = yt,\ yu = b\}$$

Subcase I:

If s=1, then **Σ** has a solution in A.

Subcase II:

If s = 0, then construct

$$\mathbf{\Sigma' = \{ x}\alpha = a,\ xs' = xt',\ x0 = a.0\}$$

$$\mathbf{\Sigma^* = \{}yu = b,\ yt = a.0\}$$

Σ’ and **Σ*** both are consistent having solution b_1 and b_2 in B. So, by induction, **Σ’** and **Σ*** having solution a_1 and a_2 in A, respectively.

(a_1, a_2) is the solution for **Σ.**

Subcase III:

If s = α, then construct

Σ’ = { **x**α = a, xs’ = xt’ }

Σ* = {yu = b, yt = a}

Σ’ and **Σ*** both are consistent having solution b_1 and b_2 in B. So, by induction, **Σ** and **Σ*** having solution a_1 and a_2 in A, respectively.

(a_1, a_2) is the solution for **Σ.**

Subcase IV:

If $s = \beta$

$$\boldsymbol{\Sigma} = \{ \mathbf{x}\alpha = a,\ xs' = xt',\ x\beta = yt,\ yu = b\}.$$

Consider

$$\boldsymbol{\Sigma'} = \{ \mathbf{x}\alpha = a,\ xs' = xt'\}.$$

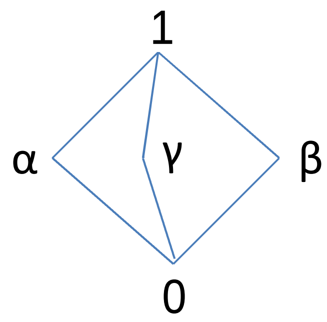
Then corresponds to every $\sigma = < (s', t') >$, **Σ** has solution in A.

So, if

$$\boldsymbol{\Sigma} = \{ \mathbf{x}\alpha = a,\ x\beta = yt,\ yu = b\}.$$

Then, corresponds to every value of t and u, we can either split **Σ** into two system in one variable or (a, b) is the solution of Σ.

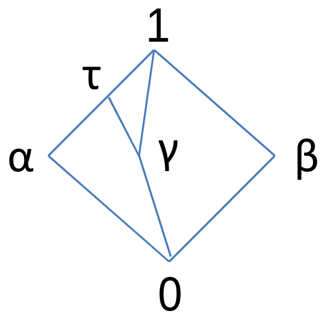
Consider the semilattice S = {0, α, β, γ, 1}.



- S is an S-act over itself.
- S is not completely right pure because the principle right ideals {0, α} and {0, β} are not linearly ordered with respect to inclusion.
- S is not an almost pure S-act. Consider the system of equation

$$\mathbf{\Sigma = \{ x\alpha = \alpha, x\beta = 0, x\gamma = \gamma\}}$$

Σ is consistent because it has solution in extension T = S ∪ {τ}



But **Σ** has no solution in S.

For **S = {0, α, β, γ, 1},** every almost pure S-act is absolutely pure.

For the following semilattices, every almost pure s-act is absolutely pure.