

PROPER RESTRICTION SEMIGROUPS AND PARTIAL ACTIONS

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ABSTRACT. Restriction semigroups and their one-sided versions arise from a number of sources. Attracting a deal of recent interest, they appear under a plethora of names in the literature. It is known that the class of left (right) restriction semigroups admits proper covers, and that proper left (right) restriction semigroups can be described by monoids acting on the right (left) of semilattices. As for restriction semigroups (the two-sided versions), proper covers are known to exist. Here we consider whether proper restriction semigroups can be described in a natural way by monoids acting on both sides of a semilattice.

It transpires that to obtain the full class of proper restriction semigroups, we must use *partial actions* of monoids, thus recovering results of Petrich and Reilly and of Lawson for inverse semigroups and ample semigroups, respectively. We also describe the class of proper restriction semigroups such that the partial actions can be mutually extendable to *actions*. Proper inverse and free restriction semigroups (which are proper) have this form, but we give examples of proper restriction semigroups which do not.

INTRODUCTION

Restriction semigroups and their one-sided versions arise from many sources and have equally many names. The reader can consult [9] or the unpublished notes [8] for history and further details. Restriction semigroups form a variety of semigroups augmented with two unary operations $a \mapsto a^+$ and $a \mapsto a^*$. Every inverse semigroup is restriction with $a^+ = aa^{-1}$ and $a^* = a^{-1}a$, so that, as restriction semigroups form a variety, every subsemigroup of an inverse semigroup that is closed under $+$ and $*$ is restriction. But certainly, not every restriction semigroup is obtained in this way. It is easy to see that any monoid M is restriction, where we declare $a^+ = 1 = a^*$, for every $a \in M$; such restriction semigroups are called *reduced*, so that a reduced inverse semigroup is simply a group. We view restriction semigroups as being natural extensions of inverse semigroups and, indeed, they have many analogous properties. This paper studies the notion of *proper* for a restriction semigroup. There are some remarkable similarities to

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the inverse case - and some curious differences. We outline the picture in this Introduction; further details of undefined terms will be given in Section 1.

The relations additional to associativity that define restriction semigroups are:

$$x^+x = x, x^+y^+ = y^+x^+, (x^+y)^+ = x^+y^+, xy^+ = (xy)^+x,$$

their duals:

$$xx^* = x, x^*y^* = y^*x^*, (xy^*)^* = x^*y^*, x^*y = y(xy)^*,$$

and the connecting relations:

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

A semigroup with a unary operation of $a \mapsto a^+$ ($a \mapsto a^*$) satisfying the first (second) set of identities is called *left (right) restriction*. For any left restriction semigroup S , we put

$$E = \{x^+ : x \in S\},$$

so that if S is restriction, then by the last set of identities, we also have that $E = \{x^* : x \in S\}$. It is easy to see that E is a semilattice under the semigroup multiplication, the *distinguished* semilattice of S . We remark that a restriction semigroup is proper if and only if it is proper as both a left and as a right restriction semigroup.

A classical result of McAlister [12] tells us that for any inverse semigroup S , there is a proper inverse semigroup \widehat{S} (a ‘proper cover’ of S) and an idempotent separating onto morphism $\theta : \widehat{S} \mapsto S$ (a ‘covering morphism’). Correspondingly, from [6, Lemma 6.6] and [1, Theorem 6.4], every (left) restriction semigroup has a proper cover \widehat{S} , where here \widehat{S} is a proper (left) restriction semigroup and now we only insist that θ separate the idempotents of E .

Of course, the power of the McAlister theory is that [12] was followed by [13], in which a structure theorem is given for proper inverse semigroups. Namely, an inverse semigroup is proper if and only if it is isomorphic to a ‘P-semigroup’ $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$, where G is a group acting on a partially ordered set \mathcal{X} containing a semilattice \mathcal{Y} as a sub-partially ordered set, subject to certain conditions. Notice that if S is proper inverse, then S is isomorphic to some $\mathcal{P}(S/\sigma, \mathcal{X}, E(S))$, where $E(S)$ is the set of idempotents of S and σ is the least congruence identifying all the idempotents of $E(S)$. Correspondingly, in [1, Theorem 7.2] it is shown that a left restriction semigroup S is proper if and only if it is isomorphic to a ‘strong M-semigroup’ $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$, where T is a monoid (regarded as a reduced left restriction semigroup) acting by endomorphisms on a semilattice \mathcal{X} with sub-semilattice \mathcal{Y} , again subject to certain conditions. The interested reader should note that although we can take \mathcal{X} to be a semilattice, we have lost the condition ‘ $G\mathcal{X} = \mathcal{Y}$ ’ which appears in McAlister’s result. Further, if S is left restriction, then in the strong M-semigroup isomorphic to S , we can take $T = S/\sigma_E$ and $\mathcal{Y} = E$, where here σ_E is the least congruence identifying all the idempotents of E .

To complete the picture we would, of course, like a structure theorem for proper restriction semigroups, indeed, this is the aim of the current article. But, here is one of those odd situations where one-sided conditions are easier to handle than two-sided. Although it is possible to adapt the one-sided approach to the two-sided case, by adding extra conditions on M-semigroups (see [11] for the proof for the sub-quasi-variety of ample semigroups, and [2] for restriction semigroups), the results are lop-sided and rather artificial.

Since restriction semigroups and monoids form varieties, free objects exist, in particular the free restriction monoid $\mathcal{FRM}(X)$ exists on any non-empty set X . The structure of $\mathcal{FRM}(X)$ has recently been determined [6]; the notable point for this article being that it is obtained from a monoid acting on both sides of a semilattice subject to some *compatibility conditions*. Since $\mathcal{FRM}(X)$ is proper, we were anticipating that a truly two-sided structure theorem for proper restriction semigroups would follow. This is certainly true, but not quite in the way we expected.

In Section 2 we define a *strong M-quadruple* $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ where T is a monoid acting on the left (right) of a semilattice \mathcal{X} (\mathcal{X}') such that \mathcal{X} and \mathcal{X}' both contain \mathcal{Y} as a subsemilattice, subject to certain constraints, including compatibility conditions for the actions. We then construct a semigroup $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ which is proper restriction.

Unfortunately, it is not the case that every proper restriction semigroup S is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. In Section 3 we determine those S that do have this property, calling them *extra proper*. Inverse semigroups and free restriction monoids are extra proper, but we can easily produce examples of proper restriction semigroups that are not. Essentially, extra proper restriction semigroups have an extra amount of left/right symmetry, which is guaranteed by the existence of an involution in the inverse case.

All is not lost, however. Given a proper restriction semigroup S , it is always the case that S/σ_E acts *partially* on the left and right of E , again subject to a variation of the compatibility conditions. From this idea, in Section 4 we develop the notion of a strong M-pair (T, \mathcal{Y}) , where T is a monoid acting partially on the left and right of a semilattice \mathcal{Y} in an analogous way. We then define a semigroup $\mathcal{Q}(T, \mathcal{Y})$ and show that $\mathcal{Q}(T, \mathcal{Y})$ is proper restriction. In Section 5 we show that, conversely, every proper restriction semigroup S is isomorphic to some $\mathcal{Q}(S/\sigma_E, E)$. In fact, this idea is in spirit exactly that of [15] and [11] which consider the inverse and ample cases, respectively. Our proof, however, uses none of their machinery. Sections 4 and 5 give another example of the use of partial actions in understanding the structure of semigroups.

1. PRELIMINARIES

In this section we briefly define the tools needed for the rest of the paper. We refer the reader to [10] for general semigroup background and [8] for further details concerning restriction semigroups and related classes.

We first note that restriction semigroups are algebras with two unary operations, and hence have signature we denote by $(2, 1, 1)$. Similarly, left (right) restriction semigroups are algebras of type $(2, 1)$. In particular, morphisms between (one-sided) restriction semigroups should be considered as morphisms in these augmented signatures.

Just as Green's relations are the major tools for dealing with inverse semigroups, we have relations $\tilde{\mathcal{R}}_E$ and $\tilde{\mathcal{L}}_E$ to help elucidate restriction semigroups.

Let E be a subset of idempotents of a semigroup S ; we do not assume that E is the set $E(S)$ of all idempotents of S . The relation $\tilde{\mathcal{R}}_E$ is defined on S by the rule that for any $a, b \in S$, $a \tilde{\mathcal{R}}_E b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b.$$

The relation $\tilde{\mathcal{L}}_E$ is defined dually.

Proposition 1.1. [8, Proposition 4.3] *Let S be a semigroup equipped with a unary operation $a \mapsto a^+$. Then S is left restriction with distinguished semilattice E if and only if $E \subseteq E(S)$, E is a commutative subsemigroup, for every $a \in S$ the $\tilde{\mathcal{R}}_E$ -class of a contains a unique idempotent a^+ in E , the relation $\tilde{\mathcal{R}}_E$ is a left congruence, and the 'ample condition' holds, that is, for all $a \in S$ and $e \in E$, $ae = (ae)^+a$.*

Let S be a (left) restriction semigroup. The relation σ_E on S is the least congruence identifying all the elements of E . As explained in [8, Section 8], we can regard σ_E as either a semigroup congruence or a congruence in the augmented signature.

Lemma 1.2. [8, Lemma 8.1] *Let S be a left restriction semigroup. Then for any $a, b \in S$, we have that $a \sigma_E b$ if and only if $ea = eb$ for some $e \in E$.*

It follows by duality that if S is right restriction, then $a \sigma_E b$ if and only if $af = bf$ for some $f \in E$, so that if S is restriction, then either characterisation of σ_E will suffice.

Definition 1.3. A left (right) restriction semigroup is *proper* if $\tilde{\mathcal{R}}_E \cap \sigma_E = \iota$ ($\tilde{\mathcal{L}}_E \cap \sigma_E = \iota$). A restriction semigroup is *proper* if it is proper as both a left and as a right restriction semigroup.

We remark that if S is a proper left restriction semigroup, then E is a σ_E -class, but the converse need not be true [4, Example 3]. However, it is well known that an inverse semigroup (for which we always have $\tilde{\mathcal{R}}_{E(S)} = \mathcal{R}$) is proper if and only if it is E -unitary, that is, if and only if $E(S)$ forms a $\sigma = \sigma_{E(S)}$ -class.

Our aim is to find a structure theorem for proper restriction semigroups: our tools will be actions and partial actions of monoids on partially ordered sets and semilattices.

Definition 1.4. Let T be a monoid and let X be a set. Then T *acts* on X (on the left) if there is a map $T \times X \rightarrow X$, $(t, x) \mapsto t \cdot x$, such that for all $x \in X$ and $s, t \in T$ we have

$$1 \cdot x = x \text{ and } st \cdot x = s \cdot (t \cdot x).$$

Definition 1.5. Let T be a monoid and let X be a set. Then T *acts partially* on X (on the left) if there is a partial map $T \times X \rightarrow X$, $(t, x) \mapsto t \cdot x$, such that for all $s, t \in T$ and $x \in X$,

$$\exists 1 \cdot x \text{ and } 1 \cdot x = x$$

and

$$\text{if } \exists t \cdot x \text{ and } \exists s \cdot (t \cdot x) \text{ then } \exists st \cdot x \text{ and } s \cdot (t \cdot x) = st \cdot x,$$

where we write $\exists u \cdot y$ to indicate that $u \cdot y$ is defined.

Of course, a partial left action of T on X with domain of the action $T \times X$ is an action. Dually, we may define the (partial) right action of T on X .

Definition 1.6. If a monoid T acts on (the left of) a partially ordered set X (semilattice Y), then the action is *order preserving (by morphisms)* if, for any $t \in T$ and $x, y \in X$ with $x \leq y$ ($e, f \in Y$) we have that

$$t \cdot x \leq t \cdot y \quad (t \cdot (e \wedge f) = (t \cdot e) \wedge (t \cdot f)).$$

Notice that if a monoid acts by morphisms on a semilattice Y , then its action is order preserving, but the converse need not be true. If a group G acts by order preserving maps on a partially ordered set, then, as any group action is by bijections, it acts by order automorphisms.

Suppose now that the monoid T acts by morphisms on a semilattice Y . We denote by $Y * T$ the *semidirect product* of Y and T , so that

$$Y * T = Y \times T \text{ and } (e, s)(f, t) = (e \wedge (s \cdot f), st)$$

for all $(e, s), (f, t) \in Y * T$. It is an easy exercise to check that $Y * T$ is proper left restriction with $(e, s)^+ = (e, s)$ and inverse if T is a group. Unfortunately, semidirect products of this kind do not even yield all proper inverse semigroups, which is where the McAlister construction using P-semigroups comes into play. Nevertheless, the ideas underlying all attempts to describe proper semigroups are adaptations of the notion of semidirect product.

There are various approaches to constructing a ‘P-theorem’ for left restriction semigroups and their specialisations (see [4, 11, 7, 1]). The one we now describe is that of [1], since it is this construction that we need in detail for Theorem 3.5.

Definition 1.7. Let T be a monoid acting by morphisms on the left of a semilattice \mathcal{X} having subsemilattice \mathcal{Y} . Suppose that there exists an upper bound ε for \mathcal{Y} in \mathcal{X} such that the following hold:

- (a) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$;
- (b) for all $e, f \in \mathcal{Y}$ and all $t \in T$,

$$e \leq t \cdot \varepsilon \Rightarrow e \wedge t \cdot f \text{ lies in } \mathcal{Y}.$$

Then the triple $(T, \mathcal{X}, \mathcal{Y})$ is called a *strong left \mathcal{M} -triple*.

We note that in [1], strong left \mathcal{M} -triples we referred to for simplicity as *strong \mathcal{M} -triples*. Given a strong left \mathcal{M} -triple $(T, \mathcal{X}, \mathcal{Y})$, we define

$$\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(e, s) \in \mathcal{Y} \times T : e \leq s \cdot \varepsilon\},$$

with binary operation defined by

$$(e, s)(f, t) = (e \wedge s \cdot f, st)$$

for $(e, s), (f, t) \in \mathcal{M}$. We shall call $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ a *strong \mathcal{M} -semigroup*.

Dually, we may define the notion of a *strong right \mathcal{M} -triple* $(T, \mathcal{X}, \mathcal{Y})$, where T acts on the right of \mathcal{X} satisfying the duals of Conditions (a) and (b), and then a semigroup $\mathcal{M}' = \mathcal{M}'(T, \mathcal{X}, \mathcal{Y}) = \{(s, e) \in T \times \mathcal{Y} : e \leq \varepsilon \circ t\}$ under the appropriate semidirect product multiplication.

Proposition 1.8. [1, Lemma 7.1] *Let $(T, \mathcal{X}, \mathcal{Y})$ be a strong left \mathcal{M} -triple. Then $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is a proper left restriction semigroup with*

$$(e, s)^+ = (e, 1), \quad E = \{(e, 1) : e \in \mathcal{Y}\} \cong \mathcal{Y} \text{ and } \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) / \sigma_E \cong T.$$

A left restriction semigroup S with $E = E(S)$ is *weakly left ample*; if, in addition, $\tilde{\mathcal{R}}_{E(S)} = \mathcal{R}^*$, then S is *left ample*. The obvious definitions then apply to give *(weakly) (right) ample* semigroups.

Theorem 1.9. [7, 1] *A semigroup is proper left restriction (weakly left ample, left ample) if and only if it is isomorphic to a strong \mathcal{M} -semigroup $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ for some strong left \mathcal{M} -triple $(T, \mathcal{X}, \mathcal{Y})$ (where T is unipotent, right cancellative).*

We note that the above result in the left ample case can easily be deduced from the given references. The original description of proper left ample semigroups appears in [4] and was re-worked in [11].

2. DOUBLE ACTIONS AND SEMIGROUPS $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$

As explained in the Introduction, our aim is to describe proper restriction semigroups in a way that is genuinely two-sided. Inspiration arose from the definition of a *double action* [6], used to determine the structure of the free ample monoid.

Definition 2.1. Let T be a monoid and let \mathcal{Y} be a semilattice with identity. Then T acts doubly on \mathcal{Y} , if T acts by morphisms on the left and right of \mathcal{Y} and the compatibility conditions hold, that is, for all $t \in T$ and $e \in \mathcal{Y}$,

$$(t \cdot e) \circ t = (1 \circ t)e \text{ and } t \cdot (e \circ t) = e(t \cdot 1).$$

It is proved in [6, Lemma 6.2] that if a monoid T acts doubly on a semilattice \mathcal{Y} with identity, then the set

$$S = \{(e, s) : e \leq s \cdot 1\} \subseteq \mathcal{Y} * T$$

with

$$(e, s)(f, t) = (e \wedge s \cdot f, st) \text{ and } (e, s)^+ = (e, 1)$$

is a proper restriction monoid.

Moreover, the free restriction monoid is proper and has a structure as above, suggesting that we could use the idea of a double action to produce a structure theorem for proper restriction monoids and semigroups. The natural way is to proceed as follows:

Definition 2.2. Let \mathcal{X} and \mathcal{X}' be semilattices and \mathcal{Y} be a subsemilattice of both \mathcal{X} and \mathcal{X}' . Let $\varepsilon \in \mathcal{X}$ and $\varepsilon' \in \mathcal{X}'$ such that $a \leq \varepsilon, \varepsilon'$ for all $a \in \mathcal{Y}$. Let T be a monoid with identity 1, which acts by morphisms on the left of \mathcal{X} via \cdot and on the right of \mathcal{X}' , via \circ .

Suppose in addition that for all $t \in T$ and $e \in \mathcal{Y}$, the following hold:

- (A) $e \leq t \cdot \varepsilon \Rightarrow e \circ t \in \mathcal{Y}$;
- (B) $e \leq \varepsilon' \circ t \Rightarrow t \cdot e \in \mathcal{Y}$;
- (C) $e \leq t \cdot \varepsilon \Rightarrow t \cdot (e \circ t) = e$;
- (D) $e \leq \varepsilon' \circ t \Rightarrow (t \cdot e) \circ t = e$;
- (E) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$.

We then say that $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is a *strong M-quadruple*.

The above may look a little lopsided, but, in view of the following, it is not.

Lemma 2.3. Let $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ be a strong M-quadruple. Then

(F) for all $t \in T$, there exists $b \in \mathcal{Y}$ such that $b \leq \varepsilon' \circ t$ holds.

Proof. Taking $t \in T$, by (E), there exists $a \in \mathcal{Y}$ such that $a \leq t \cdot \varepsilon$. By (A), $a \circ t \in \mathcal{Y}$ and clearly $a \circ t \leq \varepsilon' \circ t$. \square

Proposition 2.4. Let $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ be a strong M-quadruple. Then $(T, \mathcal{X}, \mathcal{Y})$ is a strong left M-triple. Dually, $(T, \mathcal{X}', \mathcal{Y})$ is a strong right M-triple.

Proof. It only remains to show that if $e, f \in \mathcal{Y}$ and $t \in T$ with $e \leq t \cdot \varepsilon$, then $e \wedge t \cdot f \in \mathcal{Y}$. We have $e \circ t \in \mathcal{Y}$ by Condition (A). Then

$$(e \circ t) \wedge f \leq e \circ t \leq \varepsilon' \circ t.$$

Using Conditions (C) and (B), we now have

$$e \wedge t \cdot f = t \cdot (e \circ t) \wedge t \cdot f = t \cdot ((e \circ t) \wedge f) \in \mathcal{Y}.$$

□

Let $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ be a strong M-quadruple. We define

$$\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y}) = \mathcal{M}(T, \mathcal{X}, \mathcal{Y}) \text{ and } \mathcal{M}' = \mathcal{M}'(T, \mathcal{X}, \mathcal{X}', \mathcal{Y}) = \mathcal{M}'(T, \mathcal{X}', \mathcal{Y}).$$

Proposition 2.5. *Let $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ be a strong M-quadruple as above. Then*

$$\theta : \mathcal{M} \rightarrow \mathcal{M}' \text{ given by } (e, s)\theta = (s, e \circ s)$$

is a semigroup isomorphism.

Proof. First note that if $(e, s) \in \mathcal{M}$ then by Condition (A), $e \circ s \in \mathcal{Y}$ and $e \circ s \leq \varepsilon' \circ s$, so that $(s, e \circ s) \in \mathcal{M}'$.

If $(e, s), (f, t) \in \mathcal{M}$ and $(e, s)\theta = (f, t)\theta$, then clearly $s = t$ and $e \circ t = f \circ t$. As $e, f \leq t \cdot \varepsilon$, we have by (C) that

$$e = t \cdot (e \circ t) = t \cdot (f \circ t) = f.$$

Thus θ is one-one.

Choosing $(u, g) \in \mathcal{M}'$, we have that $g \leq \varepsilon' \circ u$, so that $u \cdot g \in \mathcal{Y}$ and as $u \cdot g \leq u \cdot \varepsilon$, we have that $(u \cdot g, u) \in \mathcal{M}$. Now using Condition (D),

$$(u \cdot g, u)\theta = (u, (u \cdot g) \circ u) = (u, g),$$

so that θ is onto, and hence a bijection.

To see that θ is an isomorphism, let $(e, s), (f, t) \in \mathcal{M}$. Then

$$\begin{aligned} (e, s)\theta(f, t)\theta &= (s, e \circ s)(t, f \circ t) \\ &= (st, ((e \circ s) \circ t) \wedge f \circ t) \\ &= (st, (e \circ s \wedge f) \circ t). \end{aligned}$$

Now, $(e \circ s) \wedge f \leq e \circ s \leq \varepsilon' \circ s$, so that

$$\begin{aligned} (e \circ s) \wedge f &= (s \cdot ((e \circ s) \wedge f)) \circ s \\ &= ((s \cdot (e \circ s)) \wedge s \cdot f) \circ s \\ &= (e \wedge s \cdot f) \circ s. \end{aligned}$$

We can now deduce that θ is an isomorphism, for

$$\begin{aligned} (e, s)\theta(f, t)\theta &= (st, ((e \wedge s \cdot f) \circ s) \circ t) \\ &= (st, (e \wedge s \cdot f) \circ st) \\ &= (e \wedge s \cdot f, st)\theta \\ &= ((e, s)(f, t))\theta. \end{aligned}$$

□

We can now give the main result of this section.

Theorem 2.6. *Let $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ be a strong M -quadruple. Then $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is a proper restriction semigroup such that*

$$(e, t)^+ = (e, 1), (e, t)^* = (e \circ t, 1), E = \{(e, 1) : e \in \mathcal{Y}\} \text{ and } \mathcal{M}/\sigma_E \cong T.$$

Proof. From Proposition 1.8, we know that \mathcal{M} is proper left restriction with $(e, t)^+ = (e, 1)$, $E = \{(e, 1) : e \in \mathcal{Y}\}$ and $\mathcal{M}/\sigma_E \cong T$. Dually, \mathcal{M}' is a proper right restriction with $(t, e)^* = (1, e)$ and distinguished semilattice E' where $E' = \{(1, e) : e \in \mathcal{Y}\}$. Clearly $E\theta = E'$, where θ is the isomorphism from \mathcal{M} to \mathcal{M}' given in Proposition 2.5, so that \mathcal{M} is proper restriction with

$$(e, s)^* = (((e, s)\theta)^*)\theta^{-1} = ((s, e \circ s)^*)\theta^{-1} = (1, e \circ s)\theta^{-1} = (e \circ s, 1).$$

□

In view of Theorem 1.9 we may easily adapt Theorem 2.6 to special cases.

Corollary 2.7. *Let $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ be a strong M -quadruple and let $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. If T is unipotent, then the proper restriction semigroup \mathcal{M} is weakly ample, and if T is left (right) cancellative, then \mathcal{M} is right (left) ample.*

3. EXTRA PROPER RESTRICTION SEMIGROUPS

We would like to be able to say that every proper restriction semigroup S is isomorphic to $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ for some strong M -quadruple $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. Unfortunately, this is not the case.

Lemma 3.1. *Let the proper restriction semigroup S be isomorphic to $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ for some strong M -quadruple $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. Then*

$$E \cong \mathcal{Y} \text{ and } S/\sigma_E \cong T.$$

Proof. Since the isomorphism preserves $+$ and $*$,

$$E \cong \{(e, 1) : e \in \mathcal{Y}\} = \mathcal{Y}' \cong \mathcal{Y}$$

and so

$$S/\sigma_E \cong \mathcal{M}/\sigma_{\mathcal{Y}'} \cong T.$$

□

Proposition 3.2. *Let S be a finite proper ample semigroup. Suppose that S is isomorphic to $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ for some strong M -quadruple $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. Then S is inverse.*

Proof. From [4, Lemma 1.3] (adjusted to the semigroup case) we have that S/σ_E is cancellative. By Lemma 3.1, T is cancellative and hence a group by finiteness. If we let $(e, t) \in \mathcal{M}$, then $e \leq t \cdot \varepsilon$, so that $e \circ t \in \mathcal{Y}$ and $e = t \cdot (e \circ t)$. Hence $e = t \cdot (t^{-1} \cdot e) = t \cdot (e \circ t)$ so that $t^{-1} \cdot e = e \circ t$. As $t^{-1} \cdot e \leq t^{-1} \cdot \varepsilon$, we see that $(t^{-1} \cdot e, t^{-1}) \in \mathcal{M}$ and

$$(e, t)(t^{-1} \cdot e, t^{-1})(e, t) = (e, 1)(e, t) = (e, t),$$

giving that \mathcal{M} is regular. Since $E(\mathcal{M})$ is a semilattice, \mathcal{M} is inverse. □

We remark that finite proper ample semigroups that are *not* inverse certainly exist. From [5, Theorem 3.2 and Corollary 3.3], every finite ample semigroup has a finite proper ample cover, so that if all finite proper restriction semigroups were inverse, so would all finite ample semigroups be inverse. Let \mathcal{I}_2 be the symmetric inverse semigroup on $\{1, 2\}$ and let $\alpha \in \mathcal{I}_2$ be defined by $\text{dom } \alpha = \{1\}$, $1\alpha = 2$. Then $S = \{\alpha, \alpha^+, \alpha^*, \emptyset\}$ is a subsemigroup of \mathcal{I}_2 closed under $+$ and $*$, which is ample but not inverse.

In order to isolate those proper restriction semigroups that *are* isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$, we introduce the following notion.

Let S be a restriction semigroup. Then S *satisfies Condition (EP)* if it satisfies $(\text{EP})^r$ and its dual $(\text{EP})^l$.

$(\text{EP})^r$: for all $s, t, u \in S$, if $s \sigma_E t u$ then there exists $v \in S$ with $t^+ s = t v$ and $u \sigma_E v$.

Lemma 3.3. *Let S be a restriction semigroup satisfying Condition (EP) such that E is a σ_E -class. Then S is proper.*

Proof. Let $a, b \in S$ and suppose that $a (\widetilde{\mathcal{R}}_E \cap \sigma_E) b$. Then $a \sigma_E b b^*$ so that with $a = s, b = t$ and $b^* = u$ in $(\text{EP})^r$ we have that $b^+ a = b v$ for some $v \in S$ with $b^* \sigma_E v$, giving $v \in E$. But $b^+ = a^+$ and so $a = b v = (b v)^+ b = a^+ b = b$. Dually, $\widetilde{\mathcal{L}}_E \cap \sigma_E$ is trivial. \square

Definition 3.4. Let S be a proper restriction semigroup. Then S is *extra proper* if it satisfies Condition (EP).

Theorem 3.5. *Let S be a proper restriction semigroup. Then S is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ if and only if S is extra proper.*

Proof. Let $\mathcal{M} = \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ for some strong M-quadruple $(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$. We show that \mathcal{M} is extra proper.

Let $\alpha, \beta, \gamma \in \mathcal{M}$ be such that

$$\alpha \sigma_E \beta \gamma.$$

Then we must have that $\beta = (e, s), \gamma = (f, t)$ and $\alpha = (g, st)$ for some $e, f, g \in \mathcal{Y}$ and $s, t \in T$.

We have that

$$e \wedge g \leq e \leq s \cdot \varepsilon$$

so that by (A), $(e \wedge g) \circ s \in \mathcal{Y}$ and so $(e \wedge g) \circ s \leq \varepsilon'$. Since the action of t is order preserving, this gives us that $(e \wedge g) \circ st \leq \varepsilon' \circ t$. Also, as

$$e \wedge g \leq g \leq st \cdot \varepsilon$$

we have that $(e \wedge g) \circ st \in \mathcal{Y}$. Since also $(e \wedge g) \circ st \leq \varepsilon' \circ t$, (B) gives that $t \cdot ((e \wedge g) \circ st) \in \mathcal{Y}$. Now $(e \wedge g) \circ st \leq \varepsilon$ and so $t \cdot ((e \wedge g) \circ st) \leq t \cdot \varepsilon$, yielding that

$$\nu = (t \cdot ((e \wedge g) \circ st), t) \in \mathcal{M}.$$

Clearly, $\nu \sigma_E \gamma$ and

$$\begin{aligned} \beta\nu &= (e, s)(t \cdot ((e \wedge g) \circ st), t) = (e \wedge (s \cdot (t \cdot ((e \wedge g) \circ st))), st) = \\ &(e \wedge (st \cdot ((e \wedge g) \circ st)), st) = (e \wedge (e \wedge g), st) = (e \wedge g, st) = (e, 1)(g, st) = \beta^+ \alpha. \end{aligned}$$

We have shown that \mathcal{M} satisfies $(EP)^r$. From Lemma 3.1, $\mathcal{M} \cong \mathcal{M}' = \mathcal{M}'(T, \mathcal{X}', \mathcal{Y})$ and by duality, we must have that \mathcal{M}' satisfies $(EP)^l$. As the isomorphism between \mathcal{M} and \mathcal{M}' preserves the distinguished semilattices, we must have that \mathcal{M} satisfies $(EP)^l$ also.

To prove the converse, we use the construction of the strong M-triple associated with a proper restriction semigroup S given in [1, Theorem 7.2]. Before doing so, we make the following remark, that will help us over an awkward point in our argument.

Suppose we have disjoint semilattices \mathcal{X} and \mathcal{X}' containing subsemilattices \mathcal{Y} and \mathcal{Y}' respectively, such that there is an isomorphism $\theta : \mathcal{Y} \rightarrow \mathcal{Y}'$. Suppose that there are upper bounds $\varepsilon \in \mathcal{X}$ and $\varepsilon' \in \mathcal{X}'$ of \mathcal{Y} and \mathcal{Y}' , respectively. Let T be a monoid with identity 1, which acts by morphisms on the left of \mathcal{X} via \cdot and on the right of \mathcal{X}' , via \circ .

Suppose in addition that for all $t \in T$ and $e \in \mathcal{Y}$, the following hold:

- (A)' $e \leq t \cdot \varepsilon \Rightarrow e\theta \circ t \in \mathcal{Y}'$;
- (B)' $e\theta \leq \varepsilon' \circ t \Rightarrow t \cdot e \in \mathcal{Y}$;
- (C)' $e \leq t \cdot \varepsilon \Rightarrow t \cdot (e\theta \circ t)\theta^{-1} = e$;
- (D)' $e\theta \leq \varepsilon' \circ t \Rightarrow (t \cdot e)\theta \circ t = e\theta$;
- (E) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $e \leq t \cdot \varepsilon$.

Then, by suitable relabelling, it is possible to construct a strong M-quadruple $(T, \mathcal{X}, \mathcal{X}'', \mathcal{Y})$ where $\mathcal{X}'' = (\mathcal{X}' \setminus \mathcal{Y}') \cup \mathcal{Y}$.

Suppose now that S is extra proper. From [1, Theorem 7.2], S is isomorphic to $\mathcal{M} = \mathcal{M}(T, \mathcal{B}^T, \mathcal{Y})$ for some strong M-triple $(T, \mathcal{B}^T, \mathcal{Y})$ constructed as below.

First, $T = S/\sigma_E$. We then let \mathcal{B} be the semilattice of ideals of E with a zero adjoined. Notice that if I, J are ideals of \mathcal{B} , then $IJ = I \cap J$ and $I \leq J$ if and only if $I \subseteq J$. We have that \mathcal{B}^T is the semilattice of all maps from T into \mathcal{B} , with operation defined by $\alpha(fg) = (\alpha f)(\alpha g)$ for all $\alpha \in T$ and for all $f, g \in \mathcal{B}^T$. Moreover, T acts on the left of \mathcal{B}^T via $\alpha(\beta \cdot f) = (\alpha\beta)f$, for all $\alpha, \beta \in T$ and $f \in \mathcal{B}^T$.

For any $e \in E$, the map $f_e \in \mathcal{B}^T$ is defined by

$$(t\sigma_E)f_e = \{(ne)^+ : n\sigma_E t\}$$

and then

$$\mathcal{Y} = \{f_e : e \in E\}$$

is a subsemilattice of \mathcal{B}^T isomorphic to E via $e \mapsto f_e$. Defining $\varepsilon \in \mathcal{B}^T$ by

$$(t\sigma_E)\varepsilon = \{m^+ : m\sigma_E t\}$$

we have that ε is an upper bound for \mathcal{Y} in \mathcal{B}^T . Moreover, $(T, \mathcal{B}^T, \mathcal{Y})$ is a strong M-triple and S is isomorphic to $\mathcal{M}(T, \mathcal{B}^T, \mathcal{Y})$ via a $(2, 1)$ -isomorphism ψ , where $s\psi = (f_{s^+}, s\sigma_E)$.

We show that the strong M-triple $(T, \mathcal{B}^T, \mathcal{Y})$ can be extended to a strong M-quadruple $(T, \mathcal{B}^T, \mathcal{X}', \mathcal{Y})$, for some partially ordered set \mathcal{X}' .

By the dual of [1, Theorem 7.2], we can construct a strong right M-triple $(T, {}^T\mathcal{B}, \mathcal{Y}')$, where ${}^T\mathcal{B}$ is the semilattice of functions from T to \mathcal{B} *written on the left of their arguments*, and T acts on ${}^T\mathcal{B}$ on the right via $(f \circ \alpha)(\beta) = f(\alpha\beta)$ for all $f \in {}^T\mathcal{B}$ and $\alpha, \beta \in T$. For any $e \in E$ we define $g_e \in {}^T\mathcal{B}$ by

$$g_e(t\sigma_E) = \{(en)^* : n \sigma_E t\}$$

and put $\mathcal{Y}' = \{g_e : e \in E\}$. Then \mathcal{Y}' is a subsemilattice of ${}^T\mathcal{B}$ isomorphic to E via $e \mapsto g_e$. It follows that $\theta : \mathcal{Y} \rightarrow \mathcal{Y}'$ given by $f_e\theta = g_e$ is an isomorphism. Finally, $\varepsilon' \in {}^T\mathcal{B}$ defined by

$$\varepsilon'(t\sigma) = \{m^* : m \sigma_E t\}$$

is an upper bound for \mathcal{Y}' in ${}^T\mathcal{B}$ which enables the conditions for $(T, {}^T\mathcal{B}, \mathcal{Y}')$ to be a strong right M-triple to be satisfied.

We need to show that Conditions (A)'-(D)' are satisfied. We show that (A)' and (C)' hold; (B)' and (D)' then follow by duality.

We first show that for any $s \in S$, $g_{s^+} \circ s\sigma_E = g_{s^*}$.

Let $r\sigma \in T$. Then

$$\begin{aligned} g_{s^*}(r\sigma_E) &= \{(s^*h)^* : h \sigma_E r\} \\ &= \{(sh)^* : h \sigma_E r\} \\ &= \{(s^+sh)^* : h \sigma_E r\} \\ &\subseteq \{(s^+k)^* : k \sigma sr\} \\ &= (g_{s^+} \circ s\sigma_E)(r\sigma_E). \end{aligned}$$

For the converse, we need (EP)^r. Let $(s^+k)^* \in (g_{s^+} \circ s\sigma_E)(r\sigma_E)$ where $k \sigma_E sr$. By (EP)^r, there exists $v \in S$ with $v \sigma_E r$ such that $s^+k = sv$. Then

$$(s^+k)^* = (sv)^* \in g_{s^*}(r\sigma_E)$$

and it follows that

$$g_{s^*}(r\sigma_E) = (g_{s^+} \circ s\sigma_E)(r\sigma_E).$$

Since $r\sigma_E$ was any element of T , $g_{s^*} = g_{s^+} \circ s\sigma_E$ as required. By duality, $s\sigma_E \cdot f_{s^*} = f_{s^+}$.

Let $f_e \in \mathcal{Y}$, let $t\sigma_E \in T$ and suppose that $f_e \leq t\sigma_E \cdot \varepsilon$. From the proof of [1, Theorem 7.2], there is an $s \in S$ such that $s^+ = e$ and $s \sigma_E t$. By the above, we have that

$$f_e\theta \circ t\sigma_E = g_e \circ t\sigma_E = g_{s^+} \circ s\sigma_E = g_{s^*} \in \mathcal{Y}'$$

so that Condition (A)' holds. Further,

$$s\sigma_E \cdot (f_{s^+}\theta \circ s\sigma_E)\theta^{-1} = s\sigma_E \cdot g_{s^*}\theta^{-1} = s\sigma_E \cdot f_{s^*} = f_{s^+}$$

so that Condition (C)' holds.

From the remarks at the beginning of this direction of the proof, relabelling will produce a strong M-triple $(T, \mathcal{B}^T, {}^T\mathcal{B}, \mathcal{Y})$ as required. It remains to show that ψ preserves $*$. Let $s \in S$ then (bearing in mind we have identified \mathcal{Y} and \mathcal{Y}'), we have

$$(s\psi)^* = (f_{s^+}, s\sigma_E)^* = (f_{s^+} \circ s\sigma_E, 1) = (f_{s^*}, 1) = s^*\psi,$$

so that ψ is an isomorphism in the signature $(2, 1, 1)$, as required. \square

Example 3.6. Every inverse semigroup has (EP). For, if s, t, u are elements of an inverse semigroup S with $s\sigma tu$, then $t^+s = tt^{-1}s$ and $t^{-1}s\sigma tt^{-1}u\sigma u$.

Example 3.7. Every reduced restriction semigroup has (EP). For, if s, t, u are elements of a reduced restriction semigroup S with $s\sigma tu$, then $s = tu$ and $t^+s = s = tu$.

Less trivially, free restriction monoids have (EP).

Example 3.8. Let $\mathcal{FRM}(X)$ be the free restriction monoid on a non-empty set X . We use the characterisation of $\mathcal{FRM}(X)$ as a submonoid of the free inverse monoid $\mathcal{FIM}(X)$ on X , given in [6].

Let $\mathcal{FG}(X)$ be the free group on X , and regard elements of $\mathcal{FG}(X)$ as reduced words over X . Let

$$\mathcal{Y} = \{A \subseteq \mathcal{FG}(X) : 1 \leq |A| < \infty, A \text{ is prefix closed}\}.$$

Then

$$\mathcal{FIM}(X) = \{(A, w) : A \in \mathcal{Y}, w \in A\}$$

with

$$(A, w)(B, v) = (A \cup wB, wv) \text{ and } (A, w)^{-1} = (w^{-1}A, w^{-1}).$$

From [6], $\mathcal{FRM}(X)$ is the submonoid of $\mathcal{FIM}(X)$ given by

$$\mathcal{FRM}(X) = \{(A, w) \in \mathcal{FIM}(X) : w \in X^*\}$$

and for any $(A, w), (B, v) \in \mathcal{FRM}(X)$, we have that

$$(A, w)^+ = (A, 1) \text{ and } (A, w)\sigma_E(B, v) \text{ if and only if } w = v.$$

Suppose that $(A, w), (B, v), (C, u) \in \mathcal{FRM}(X)$ with

$$(A, w)\sigma(B, v)(C, u).$$

Then $w = vu$ and

$$\begin{aligned} (B, v)^+(A, w) &= (B, v)(B, v)^{-1}(A, w) = (B, v)(v^{-1}B, v^{-1})(A, w) = \\ &= (B, v)(v^{-1}B \cup v^{-1}A, v^{-1}w) = (B, v)(v^{-1}B \cup v^{-1}A, u) \end{aligned}$$

and as $(v^{-1}B \cup v^{-1}A, u) \in \mathcal{FRM}(X)$, Condition (EP)^r holds. Dually, (EP)^l holds.

Finally in this section we give an example of an infinite proper ample semigroup without (EP), also showing that a proper ample semigroup can be a $(2, 1, 1)$ -subalgebra of a proper inverse semigroup, yet not itself be extra proper.

Example 3.9. Let X be a set with at least two elements, and let $X_i = \{x_i : x \in X\}$ for $i \in \{0, 1\}$ be sets in one-one correspondence with X . Let S be a strong semilattice $Y = \{1, 0\}$ of cancellative monoids $S_1 = X_1^*$ and $S_2 = \mathcal{FG}(X_0)$, with connecting morphism $\phi_{1,0}$ given by $x_1\phi_{1,0} = x_0$.

It follows from [3, Theorem 1], that S is ample, with $\mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$ -classes S_1 and S_0 . As the connecting homomorphism is one-one, it is easy to see that S is proper.

Let x, y be distinct elements of X . Then

$$e_0x_1 = x_0 = y_0(y_0^{-1}x_0) = e_0(y_1y_0^{-1}x_0)$$

so that $x_1\sigma_E y_1(y_0^{-1}x_0)$. If $y_1^+x_1 = y_1w$ for some $w \in S$ we would have that $x_1 = y_1w$, which is impossible.

4. PARTIAL ACTIONS AND SEMIGROUPS $\mathcal{M}(T, \mathcal{Y})$

In this section we use partial actions to define the notion of a strong M-pair (T, \mathcal{Y}) , where T is a monoid acting partially on both sides of a semilattice \mathcal{Y} , based on strong M-triples and quadruples. From a strong M-pair (T, \mathcal{Y}) we can define a semigroup $\mathcal{M}(T, \mathcal{Y})$ which is proper restriction. In Section 5 we show that, conversely, every proper restriction semigroup is isomorphic to some $\mathcal{M}(T, \mathcal{Y})$. Our construction is analogous to that of Petrich and Reilly in the inverse case [15] and Lawson in the ample case [11]. However, our proofs are new and direct.

Let T be a monoid, acting partially on the left and right of a semilattice \mathcal{Y} , via \cdot and \circ respectively. Suppose that both actions preserve the partial order and the domains of each $t \in T$ are order ideals, that is, for each $t \in T$ and $e, f \in \mathcal{Y}$ with $e \leq f$, if $\exists t \cdot f$ ($\exists f \circ t$), then $\exists t \cdot e$ ($\exists e \circ t$) and $t \cdot e \leq t \cdot f$ ($e \circ t \leq f \circ t$). Suppose in addition that for $e \in \mathcal{Y}$ and $t \in T$, the following hold:

- (A) if $\exists e \circ t$, then $\exists t \cdot (e \circ t)$ and $t \cdot (e \circ t) = e$;
- (B) if $\exists t \cdot e$, then $\exists (t \cdot e) \circ t$ and $(t \cdot e) \circ t = e$;
- (C) for all $t \in T$, there exists $e \in \mathcal{Y}$ such that $\exists e \circ t$.

We then say that the pair (T, \mathcal{Y}) is a *strong M-pair*. It is clear from Conditions (A) and (C) that a strong M-pair also satisfies the dual of Condition (C). Notice that the partial actions of an element t of T on the left and right of \mathcal{Y} are mutually inverse on their respective domains.

For a strong M-pair (T, \mathcal{Y}) we define

$$\mathcal{M} = \mathcal{M}(T, \mathcal{Y}) = \{(e, s) \in \mathcal{Y} \times T : \exists e \circ s\}$$

with binary operation given by

$$(e, s)(f, t) = (s \cdot ((e \circ s) \wedge f), st).$$

Dually, we can define $\mathcal{M}' = \mathcal{M}'(T, \mathcal{Y})$.

To proceed to show that \mathcal{M} is a semigroup, we require a technical result.

Proposition 4.1. *Let (T, \mathcal{Y}) be a strong M-pair. Then*

(1) if $\exists e \circ a$ and $\exists f \circ a$, then $\exists(e \wedge f) \circ a$ and

$$e \circ a \wedge f \circ a = (e \wedge f) \circ a.$$

(2) if $\exists a \cdot e$ and $\exists a \cdot f$, then $\exists a \cdot (e \wedge f)$ and

$$a \cdot e \wedge a \cdot f = a \cdot (e \wedge f).$$

Proof. (1) Suppose $\exists e \circ a$ and $\exists f \circ a$. As $\exists e \circ a$ and $e \wedge f \leq e$, $\exists(e \wedge f) \circ a$ since the domain of $\circ a$ is an order ideal. It follows from \circ being order preserving that $(e \wedge f) \circ a \leq e \circ a$ and similarly we have $(e \wedge f) \circ a \leq f \circ a$. Therefore

$$(e \wedge f) \circ a \leq e \circ a \wedge f \circ a.$$

Conversely, as $\exists e \circ a$, Condition (A) gives that $\exists a \cdot (e \circ a) = e$. As $e \circ a \wedge f \circ a \leq e \circ a$ we must have $\exists a \cdot (e \circ a \wedge f \circ a)$ since the domain of $a \cdot$ is an order ideal. Since \cdot is order preserving,

$$a \cdot (e \circ a \wedge f \circ a) \leq a \cdot (e \circ a) = e.$$

Similarly, $a \cdot (e \circ a \wedge f \circ a) \leq f$ and so $a \cdot (e \circ a \wedge f \circ a) \leq e \wedge f$.

From Condition (B), $\exists[a \cdot (e \circ a \wedge f \circ a)] \circ a$ and

$$e \circ a \wedge f \circ a = [a \cdot (e \circ a \wedge f \circ a)] \circ a \leq (e \wedge f) \circ a.$$

Hence $e \circ a \wedge f \circ a = (e \wedge f) \circ a$ and so (1) holds. The proof of (2) is dual. \square

Theorem 4.2. *Let (T, \mathcal{Y}) be a strong M -pair. Then $M = \mathcal{M}(T, \mathcal{Y})$ is a proper restriction semigroup with*

$$(e, a)^+ = (e, 1), (e, a)^* = (e \circ a, 1), E = \{(e, 1) : e \in \mathcal{Y}\} \cong \mathcal{Y} \text{ and } M/\sigma_E \cong T.$$

If T is unipotent, (right, left) cancellative, then \mathcal{M} is weakly ample, (left, right) ample, respectively.

Proof. To see that the binary operation in \mathcal{M} is well defined, let $(e, a), (f, b) \in \mathcal{M}$. We wish to show $(a \cdot ((e \circ a) \wedge f), ab) \in M$. By Condition (A), $\exists a \cdot (e \circ a)$ since $\exists e \circ a$. As $(e \circ a) \wedge f \leq e \circ a$, certainly $\exists a \cdot ((e \circ a) \wedge f)$.

We wish to show that $\exists[a \cdot ((e \circ a) \wedge f)] \circ ab$. We have $\exists[a \cdot ((e \circ a) \wedge f)] \circ a$ and $[a \cdot ((e \circ a) \wedge f)] \circ a = (e \circ a) \wedge f$. Also, $\exists f \circ b$ so that $\exists((e \circ a) \wedge f) \circ b$ and hence

$$\exists([a \cdot ((e \circ a) \wedge f)] \circ a) \circ b.$$

From Definition 1.5, we deduce $\exists[a \cdot ((e \circ a) \wedge f)] \circ ab$. Therefore the binary operation is closed.

We now show the multiplication is associative. Suppose $(e, a), (f, b), (g, c) \in \mathcal{M}(T, \mathcal{Y})$. Then

$$\begin{aligned} (e, a)[(f, b)(g, c)] &= (e, a)(b \cdot ((f \circ b) \wedge g), bc) \\ &= (a \cdot ((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))), abc). \end{aligned}$$

As $\exists b \cdot ((f \circ b) \wedge g)$, Condition (B) gives $\exists(b \cdot ((f \circ b) \wedge g)) \circ b$ and so

$$\exists((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))) \circ b.$$

Then by Condition (A),

$$\exists b \cdot (((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))) \circ b)$$

and

$$b \cdot (((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))) \circ b) = (e \circ a) \wedge (b \cdot ((f \circ b) \wedge g)).$$

So,

$$\begin{aligned} (e, a)[(f, b)(g, c)] &= (a \cdot (b \cdot (((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))) \circ b)), abc) \\ &= (ab \cdot (((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))) \circ b), abc). \end{aligned}$$

We also have $(f \circ b) \wedge g \leq f \circ b$ and so by Condition (A),

$$b \cdot ((f \circ b) \wedge g) \leq b \cdot (f \circ b) = f.$$

Using Proposition 4.1,

$$\begin{aligned} ((e \circ a) \wedge (b \cdot ((f \circ b) \wedge g))) \circ b &= ((e \circ a) \wedge f \wedge (b \cdot ((f \circ b) \wedge g))) \circ b \\ &= (((e \circ a) \wedge f) \circ b) \wedge ((b \cdot ((f \circ b) \wedge g)) \circ b) \\ &= (((e \circ a) \wedge f) \circ b) \wedge ((f \circ b) \wedge g). \end{aligned}$$

So,

$$\begin{aligned} (e, a)[(f, b)(g, c)] &= (ab \cdot (((e \circ a) \wedge f) \circ b) \wedge ((f \circ b) \wedge g), abc) \\ &= (ab \cdot (((e \circ a) \wedge f) \circ b) \wedge g, abc) \end{aligned}$$

as \circ is order-preserving.

We have $(e, a)(f, b) \in \mathcal{M}(T, \mathcal{Y})$, $\exists a \cdot ((e \circ a) \wedge f)$ and so by Condition (B), $\exists (a \cdot ((e \circ a) \wedge f)) \circ a$ and $(a \cdot ((e \circ a) \wedge f)) \circ a = (e \circ a) \wedge f$. So,

$$\begin{aligned} (((e \circ a) \wedge f) \circ b) \wedge g &= (((a \cdot ((e \circ a) \wedge f)) \circ a) \circ b) \wedge g \\ &= ((a \cdot ((e \circ a) \wedge f)) \circ ab) \wedge g. \end{aligned}$$

Hence

$$\begin{aligned} (e, a)[(f, b)(g, c)] &= (ab \cdot (((a \cdot ((e \circ a) \wedge f)) \circ ab) \wedge g), abc) \\ &= (a \cdot ((e \circ a) \wedge f), ab)(g, c) \\ &= [(e, a)(f, b)](g, c). \end{aligned}$$

Therefore $\mathcal{M}(T, \mathcal{Y})$ is a semigroup.

It is easy to see that

$$E = \{(e, 1) : e \in \mathcal{Y}\}$$

is a semilattice isomorphic to \mathcal{Y} .

We define unary operations of $+$ and $*$ on \mathcal{M} by

$$(e, a)^+ = (e, 1) \text{ and } (e, a)^* = (e \circ a, 1).$$

Clearly \mathcal{M} satisfies the identities

$$x^+x = x, x^+y^+ = y^+x^+, x^*y^* = y^*x^*, (x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

Let $(e, a), (f, b) \in \mathcal{M}$. Then

$$((e, a)^+(f, b))^+ = ((e, 1)(f, b))^+ = (e \wedge f, b)^+ = (e \wedge f, 1) = (e, 1)(f, 1) = (e, a)^+(f, b)^+,$$

and

$$\begin{aligned} ((e, a)(f, b))^+(e, a) &= (a \cdot (e \circ a \wedge f), ab)^+(e, a) \\ &= (a \cdot (e \circ a \wedge f), 1)(e, a) \\ &= (a \cdot ((e \circ a) \wedge f) \wedge e, a) \\ &= (a \cdot ((e \circ a) \wedge f), a) \quad \text{as } a \cdot ((e \circ a) \wedge f) \leq a \cdot (e \circ a) = e \\ &= (e, a)(f, 1) \\ &= (e, a)(f, b)^+ \end{aligned}$$

so that \mathcal{M} satisfies $(x^+y)^+ = x^+y^+$ and $xy^+ = (xy)^+x$.

Further,

$$(e, a)(e, a)^* = (e, a)(e \circ a, 1) = (a \cdot (e \circ a \wedge e \circ a), a) = (e, a),$$

so that $xx^* = x$ holds, and

$$((e, a)(f, b)^*)^* = ((e, a)(f \circ b, 1))^* = (a \cdot (e \circ a \wedge f \circ b), a)^* = (e \circ a \wedge f \circ b, 1) = (e, a)^*(f, b)^*,$$

so that $(xy^*)^* = x^*y^*$ holds. Finally,

$$\begin{aligned} (f, b)((e, a)(f, b))^* &= (f, b)(a \cdot (e \circ a \wedge f), ab)^* \\ &= (f, b)([a \cdot (e \circ a \wedge f)] \circ ab, 1) \\ &= (b \cdot (f \circ b \wedge [a \cdot (e \circ a \wedge f)] \circ ab), b) \\ &= (b \cdot (f \circ b \wedge [a \cdot (b \cdot ((e \circ a) \wedge f) \circ b)] \circ ab), b) \\ &= (b \cdot (f \circ b \wedge [ab \cdot ((e \circ a \wedge f) \circ b)] \circ ab), b) \\ &= (b \cdot (f \circ b \wedge (e \circ a \wedge f) \circ b), b) \\ &= (b \cdot ((f \wedge e \circ a \wedge f) \circ b), b) \\ &= (e \circ a \wedge f, b) \\ &= (e \circ a, 1)(f, b) \\ &= (e, a)^*(f, b) \end{aligned}$$

so that $x^*y = y(xy)^*$ is satisfied and \mathcal{M} is a restriction semigroup with $^+, ^*$ and E as given.

Again, let $(e, a), (f, b) \in \mathcal{M}$. If $a = b$, then clearly

$$(e \wedge f, 1)(e, a) = (e \wedge f, 1)(f, b)$$

so that $(e, a)\sigma_E(f, b)$; conversely, if we are given that $(e, a)\sigma_E(f, b)$, then as $(g, 1)(e, a) = (g, 1)(f, b)$ for some $(g, 1) \in E$, we must have that $a = b$. It follows that

$$(e, a)\sigma_E(f, b) \text{ if and only if } a = b$$

and hence $\mathcal{M}/\sigma_E \cong T$.

Suppose now that $(e, a), (f, a) \in \mathcal{M}$. If in addition we have that $(e, a) \widetilde{\mathcal{R}}_E(f, a)$, then $(e, 1) = (e, a)^+ = (f, a)^+ = (f, 1)$, so that $e = f$ and $(e, a) = (f, a)$. On the other hand, if we are given that $(e, a) \widetilde{\mathcal{L}}_E(f, a)$, then $(e \circ a, 1) = (e, a)^* = (f, a)^* = (f \circ a, 1)$ so that $e \circ a = f \circ a$. But then

$$e = a \cdot (e \circ a) = a \cdot (f \circ a) = f$$

and again, $(e, a) = (f, a)$. It follows that \mathcal{M} is proper as required.

It is clear that if T is unipotent, then $E = E(\mathcal{M})$ so that \mathcal{M} is weakly ample. If in addition T is left (right) cancellative, then it is an easy exercise to show that for any element $(e, a) \in \mathcal{M}$, we have that $(e, a)^* \mathcal{L}^*(e, a) ((e, a)^+ \mathcal{R}^*(e, a))$, so that \mathcal{M} is right (left) ample. \square

As we claim our approach is symmetric, we finish this section with our justification.

Proposition 4.3. *Let (T, \mathcal{Y}) be a strong M -pair. Then the map $\theta : \mathcal{M} \rightarrow \mathcal{M}'$ given by $(e, a)\theta = (a, e \circ a)$ is an isomorphism.*

Proof. It is straightforward to show that θ is a well defined bijection, and preserves $+$ and $*$. To show that θ preserves the binary operation, let $(e, a), (f, b) \in \mathcal{M}$. Then

$$\begin{aligned} (e, a)\theta(f, b)\theta &= (a, e \circ a)(b, f \circ b) \\ &= (ab, (e \circ a \wedge b \cdot (f \circ b)) \circ b) \\ &= (ab, (e \circ a \wedge f) \circ b) \\ &= (ab, ((a \cdot (e \circ a \wedge f)) \circ a) \circ b) \\ &= (ab, (a \cdot (e \circ a \wedge f)) \circ ab) \\ &= (a \cdot (e \circ a \wedge f), ab)\theta \\ &= ((e, a)(f, b))\theta. \end{aligned}$$

\square

We end this section with a brief word on the case for proper inverse semigroups. A group G acts *partially* on the left of a set X if it acts partially as a monoid and if, in addition, for any $g \in G$ and $x \in X$, if $\exists g \cdot x$, then $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$. Whenever we talk explicitly of groups acting partially, we will assume the partial action is subject to this extra condition.

Corollary 4.4. *Let (G, \mathcal{Y}) is a strong M -pair where G is a group. Then $\mathcal{M}(G, \mathcal{Y})$ is a proper inverse semigroup.*

Proof. We know from Theorem 4.2 that $\mathcal{M} = \mathcal{M}(G, \mathcal{Y})$ is ample. If $\exists e \circ g$, we have from the above that $(e \circ g, g^{-1}) \in \mathcal{M}$ and then

$$(e, g)(e \circ g, g^{-1}) = (g \cdot (e \circ g \wedge e \circ g), 1) = (e, 1) = (e, g)^+.$$

It follows that \mathcal{M} is inverse. \square

5. A STRUCTURE THEOREM FOR PROPER RESTRICTION SEMIGROUPS

We now show that any proper restriction semigroup is isomorphic to one constructed as in the previous section. The directness of our proof is influenced by Munn's approach [14] to the proof of the P-theorem.

Theorem 5.1. *Every proper restriction semigroup S is isomorphic to some $\mathcal{M}(S/\sigma_E, E)$.*

Proof. Let $T = S/\sigma_E$. We shall define a partial action of T on the right of E by

$$\exists e \circ m\sigma_E \Leftrightarrow \exists s \in S \text{ with } e = s^+ \text{ and } m\sigma_E = s\sigma_E,$$

in which case

$$e \circ m\sigma_E = s^+ \circ s\sigma_E = s^*.$$

This is clearly well-defined, since S is proper.

For any $e \in E$, we have that $e = e^+$ and $e\sigma_E = 1_T$, so that $\exists e \circ 1_T$ and $e \circ 1_T = e$.

Suppose $\exists s^+ \circ s\sigma_E$ and $\exists (s^+ \circ s\sigma_E) \circ t\sigma_E$. As $\exists s^* \circ t\sigma_E$, there must be a $u \in S$ with $s^* = u^+$ and $u\sigma_E = t\sigma_E$. So

$$(s^+ \circ s\sigma_E) \circ t\sigma_E = s^* \circ t\sigma_E = u^+ \circ u\sigma_E = u^*.$$

We wish to show that $\exists s^+ \circ (st)\sigma_E$ and $u^* = s^+ \circ (st)\sigma_E$. We have $(su)^+ = (su^+)^+ = (ss^*)^+ = s^+$ and similarly $(su)^* = u^*$. Clearly, $su \sigma_E st$, so $\exists s^+ \circ (st)\sigma_E$ and

$$s^+ \circ (st)\sigma_E = (su)^+ \circ (su)\sigma_E = (su)^* = u^*$$

as required. Therefore \circ is a partial right action.

We shall show that the domain of each $z\sigma_E \in T$ is an order ideal. Suppose $e, f \in E$ with $e \leq f$ and $\exists f \circ z\sigma_E$. Then there exists $s \in S$ with $s^+ = f$ and $s\sigma_E z$. Now $(es)^+ = es^+ = ef = e$ and $es \sigma_E s \sigma_E z$, so that $\exists e \circ z\sigma_E$. Further, from the third identity for $*$, $(es)^* s^* = (ess^*)^* = (es)^*$, so that

$$e \circ z\sigma_E = (es)^* \leq s^* = f \circ z\sigma_E,$$

and the action is order-preserving.

Dually, we can define a partial left action of T on \mathcal{Y} by

$$\exists m\sigma_E \cdot e \Leftrightarrow \exists s \in S \text{ with } e = s^* \text{ and } m\sigma_E = s\sigma_E,$$

in which case

$$m\sigma_E \cdot e = s\sigma_E \cdot s^* = s^+.$$

Then \cdot is a partial left action by order-preserving partial maps, such that the domain of each $t \in T$ is an order ideal.

Suppose $\exists e \circ m\sigma_E$. Then $e = s^+$ and $m\sigma_E = s\sigma_E$ for some $s \in S$ and $e \circ m\sigma_E = s^+ \circ s\sigma_E = s^*$. Certainly then $\exists s\sigma_E \cdot s^*$ and

$$m\sigma_E \cdot (e \circ m\sigma_E) = s\sigma_E \cdot (s^+ \circ s\sigma_E) = s\sigma_E \cdot s^* = s^+ = e.$$

Thus Condition (B) holds, and dually, Condition (A) in the definition of strong M-pair, also holds. Finally, if $m\sigma_E \in T$, then $\exists m^+ \circ m\sigma_E$, so that (C) holds.

We have shown that $(S/\sigma_E, E)$ is a strong M-pair. Let $\mathcal{M} = \mathcal{M}(S/\sigma_E, E)$ and let $\theta : S \rightarrow \mathcal{M}$ be defined by

$$s\theta = (s^+, s\sigma_E).$$

As S is proper, θ is one-one, and by definition of \circ , θ is onto.

We must show that θ is a morphism. Let $s, t \in S$. Then

$$\begin{aligned} s\theta t\theta &= (s^+, s\sigma_E)(t^+, t\sigma_E) \\ &= (s\sigma_E \cdot (s^+ \circ s\sigma_E \wedge t^+), (st)\sigma_E) \\ &= (s\sigma_E \cdot (s^*t^+), (st)\sigma_E) \\ &= ((st^+)\sigma_E \cdot (st^+)^*, (st)\sigma_E) \\ &= (st^+)^+, (st)\sigma_E \\ &= ((st)^+, (st)\sigma_E) \\ &= (st)\theta. \end{aligned}$$

Finally, for any $s \in S$,

$$s^*\theta = (s^*, 1) = (s^+, s\sigma_E)^* = (s\theta)^*$$

and

$$s^+\theta = (s^+, 1) = (s^+, s\sigma_E)^+ = (s\theta)^+$$

so that θ is an isomorphism as required. \square

For a different presentation of the following corollary in the ample case we refer the reader to [11, Theorem 4.3]. We note that if (T, \mathcal{Y}) is a strong M-pair, then, as commented earlier, the partial left and right action of $t \in T$ on \mathcal{Y} produces mutually inverse elements of $\mathcal{I}_{\mathcal{Y}}$. This fact appears explicitly in the statement of [11, Theorem 4.3].

Corollary 5.2. *Cf [11] A semigroup proper restriction (weakly ample, left ample, right ample, ample) if and only if it is isomorphic to $\mathcal{M}(T, \mathcal{Y})$ for some strong M-pair (T, \mathcal{Y}) (where T is unipotent, right cancellative, left cancellative, cancellative, respectively).*

Our final result is, again couched in other language, Corollary 3.3 of [15].

Corollary 5.3. *Cf [15] A semigroup is proper inverse if and only if it is isomorphic to some $\mathcal{M}(G, \mathcal{Y})$ where G is a group.*

Proof. From Corollary 4.4, if G is a group, then $\mathcal{M} = \mathcal{M}(G, \mathcal{Y})$ is proper inverse.

Conversely, let S be proper inverse. From Theorem 5.1, we know that S is isomorphic to $\mathcal{M} = \mathcal{M}(S/\sigma, E)$ where $E = E(S)$. We must show that $G = S/\sigma$ acts partially as a group on E .

Notice that if $\exists t\sigma \cdot e$, then $t\sigma = s\sigma$ and $e = s^* = s^{-1}s$ for some $s \in S$. Now $t\sigma \cdot e = s\sigma \cdot s^* = s^+ = ss^{-1}$. We have $(t\sigma)^{-1} = (s\sigma)^{-1} = s^{-1}\sigma$, and

$(s^{-1})^* = ss^{-1} = s^+$, so $\exists(t\sigma)^{-1} \cdot (t\sigma \cdot e) = s^{-1}\sigma \cdot (s^{-1})^* = (s^{-1})^+ = s^{-1}s = e$. The dual argument finishes the proof. \square

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