1. Semigroup basics

- \( S = \) finite semigroup.

(Three running) Egs: \( [n] = \{1, \ldots, n\} \)

(i). \( S_n = \) all bijections \( [n] \rightarrow [n] \) under composition.

(ii). \( I_n = \) all partial bijections \( X \rightrightarrows Y \), \( X, Y \subseteq [n] \) under composition:

\[
\begin{array}{ccc}
& a & \\
\rightarrow & ab & \\
& b & \\
\end{array}
\]

(all fun, actions, etc.
on the right)

(ii). \( T_n = \) all maps \( [n] \rightarrow [n] \) under composition.

- Inverses in semi-groups: an inverse of \( a \in S \) is a \( b \) s.t.

\[(*) \quad ab = a \quad \quad ba = b \]

(i). In a monoid s.t. \( \forall a \exists! b \) with \( ab = 1 = ba \) \((\Rightarrow (*))\)

\[ i.e.: \ a \text{gp!} \]

(ii). In a monoid s.t. \( \forall a \exists! b \) satisfying \((*)\) \(< \Rightarrow ab = id_X \ \text{idempotents}\) \( ba = id_Y \)

\[ i.e.: \ \text{in an inverse monoid} \]

(iii). In a monoid s.t. \( \forall a \exists (\text{many}) b \) satisfying \((*)\)
fibres of $a$ (equiv. classes of $a$)

e.g.:

In a regular monoid

from now on $S = \text{finite regular monoid}$

- Structure: Green's relations in $S_n$, $T_n$
  1. $a \mathcal{L} b \iff \text{im}(a) = \text{im}(b)$
  2. $a \mathcal{R} b \iff \text{fibres of } a = \text{fibres of } b$
  3. $a \mathcal{J} b \iff |\text{im}(a)| = |\text{im}(b)|$
  4. $a \mathcal{H} b \iff 1 + 2$

$S_n$: these are all trivial! (any $a, b$ are related)

In: (by 1, 2)

$X \rightarrow Y$

$X \rightarrow Z$

$W \rightarrow Y$

$W \rightarrow Z$

$\text{dom} = X$

$\text{dom} = W$

$\text{Ga} \text{im} = Y$

$\text{Gb} \text{im} = Z$

by 3: $a \mathcal{J} b \iff$

belong in same eggbox

$i.e.: J_a = J:\text{-class of } a:$

H-classes

R-classes

L-classes

R and L commute
to give nice
"eggbox" picture
These pictures hold for all $S$ (finite regular monoids).

In $T_n / T_n$ the $J$-classes not totally ordered:

- Idempotents/subgroups:
  - Idempotent $e = e^2$

Fix domain $X$: only one such map $\xrightarrow{id_X} X$ with $X \leq \{n\}$

Every $R$-class has exactly one idempotent (similarly every $L$-class).

Similarly for every inverse or regular semigroup.

$H$-class in $T_n$ of contains all bijections $\{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_n\}$ i.e. a copy of $S_n$ in $T_n$

In general the a subgp. of $S$ (with identity $e$)

If $G$ a subgroup of $S$ then $G \leq H_e$ for some $e$.

(hereditary or maximal subgroups)
$R$-class: 

\[ \xymatrix{ e & a \ar@{<-}[u] \ar@{->}[r] & H \ar@{<-}[lu] \ar@{->}[u] } \]

$\text{gp. } H_e$ \hspace{1cm} $H_a$ (clear in $\text{In}$)

$\Rightarrow$ every element of $H_a$ has unique expression $ga (g \in H_e)$ (and similarly in an $L$-class).

Two gp. $H$-classes in a $J$-class isomorphic.