3. Reduction and induction

* Recall: $S = \text{finite regular monoid}$

- poset of $J$-classes:
- $G_e$-representations $\xrightarrow{\text{induction}} S$-representations $\xleftarrow{\text{reduction}}$

(1). Reduction (everyone else seems to say "restriction")

$E_g: S = \text{In}, V = \text{partial permutation rep.}$

(irreducible with $\dim V = n$)

\[ e = \text{id}_X: X \to X, \quad |X| = n \]

\[ G_e = \{ \text{bijections } X \to X \} \]

\[ \cong S_2 \]

\[ V_e \text{ (:= } \{ v.e \mid v \in V \}) = \text{basis}\]

\[ \{ v_i \mid i \in X \} \]

\[ (\Rightarrow \dim V_e = 2) \]

For $g \in G_e$ define

\[ (v.e).g = v.(g.e) \]

\[ = v.(g.e) = (v.g).e \]

$\Rightarrow V_e \text{ a } G_e \text{ representation (}\cong \text{ permutation rep. of } S_2)\]
$E_g: S = T_n$, $V =$ mapping rep. (reducible with basis $\{v_1, \ldots, v_n\}$)

$W \subset V$ hyperplane $\Sigma x_i = 0$ (irred. with $\dim W = n - 1$)

J-class poset:

- $n$
- $n-1$
- $\vdots$
- $1$

\[ e = \begin{array}{ccccccc}
1 & 2 & \cdots & l-1 & l & \cdots & n \\
\downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
1 & 2 & \cdots & l-1 & l & \cdots & n \\
\end{array} \]

$G_e = \text{all bijections } \{\text{fibres}\} \to \text{im } (e)$

$\cong S_e$

$V_e = \{e:\text{-space with basis } \{v_1, \ldots, v_e\}\} \quad \text{and } W \subset V_e$ a hyperplane $\Sigma x_i = 0$

$V_e \cong$ permutation rep. of $S_e$

i.e.: $W_{\text{irred.}} \quad \text{We } S_e$-rep $\{1 \leq l \leq n\}$

$T_n$-rep

$\dim = n - 1$
Thus if $V$ an irreducible $S$-representation and $e$ the apex of $V$ then $V \downarrow Ge := Ve$ an irreducible $Ge$-representation.

(2). Induction $V \alpha Ge$-representation

\[ \text{Choose transversal } A = \{ a_j \} \]
\[ \exists \text{! expressions for } H \text{-classes in } Re_{a_j}(ge \in Ge) \text{ with } e \in A \]

For $a_j \in A$ let $V_j = \{ v \circ a_j : v \in V \}$

(a $k$-space $\cong V$ with $\lambda (v \circ a_j) + \mu (u \circ a_j) = (\lambda v + \mu u) \circ a_j$)

Define $S$-action on $\bigoplus_{a_j \in A} V_j$ by

\[ (v \circ a_j) \cdot b = \begin{cases} v \cdot g \circ a_k, & a_j b \in Re \Rightarrow a_j b = g a_k \\ 0, & a_j b \notin Re. \end{cases} \]
Example: $S = \text{In}$

\[ \begin{array}{c}
\begin{array}{c}
\text{bijection } i \mapsto j \\
\lim l = |\text{dom } l|
\end{array}
\end{array} \]

$V = \text{trivial } G_e\text{-rep. (irred.)}$

$1$-dim. with basis \{v\}$ and $v \cdot e = v$

$\bigoplus V_j \text{ has basis } \{v_j = v \otimes a_j\}$ with

$\forall j \in E \implies \text{dom}(a_j b) = \{1\} \iff j \in \text{dom}(b)$

(in which case $a_j b = a_j$)

$\Rightarrow V_j \cdot b = \begin{cases} v \otimes a_j b = v_j b, & j \in \text{dom}(b) \\ 0, \text{ else.} \end{cases}$

the partial permutation rep. of In (irreducible)

Example: $S = T_n$

\[ \begin{array}{c}
\begin{array}{c}
\text{one } R\text{-class} \\
\text{n } L\text{-classes (all idempotents)}
\end{array}
\end{array} \]

$V = \text{trivial rep. of } G_e_1 \text{ basis } \{v\}$ (irreducible)
\[ \Theta \] \text{basis } \mathcal{V}_v = v \otimes e_j \text{? with } e^b_i = b_{ij} \]
\[ \Rightarrow vi \cdot b = v \cdot e \otimes e^b_j = v_{ib} \]

The mapping rep. of \( T_n \), reducible

with sub-rep. \( W = \{ \sum \lambda_i v_i : \sum \lambda_i = 0 \} \)

notice: \( L_e = \{ e_j \} \) with \( v_j \cdot e_j = v_1 \) for all \( j \)

\[ v = \sum \lambda_i v_i \text{ with } v \cdot e_j = 0 \iff (\sum \lambda_i) v_i = 0 \]
\[ \iff \sum \lambda_i = 0 \iff v \in W. \]

In general: \( V \) an \( S \)-representation and \( UCV \) a subrepresentation \( \Rightarrow \) quotient space \( V/U \) an \( S \)-representation via \( (v + U) \cdot a = v \cdot a + U. \)

\( UCV \) maximal sub-representation \( \iff \) given \( UCV \)

we have \( W = U \) or \( W = V. \)

Then \( U \) maximal \( \iff V/U \) irreducible.

\[ \text{V an irreducible } \]
\[ \text{Ge-representation } \]
\[ A = \{ a_j \} \]

and \( V_j \) as before.
If \( \text{Ann}(L_e) = \{ v \in \bigoplus_{A} V_j : v \cdot a = 0 \text{ for all } a \in L_e \} \)
then \( \text{Ann}(L_e) \) (the unique) maximal subrepresentation of \( \bigoplus_{A} V_j \):
\[ V \uparrow S := \bigoplus_{A} V_j / \text{Ann}(L_e) \] irreducible \( S \)-rep.

**Ex:** (upto \( \cong \) of \( S \)-reps.) \( V \uparrow S \) does not depend on choice of \( e \in J_j \) choice of transversal \( A \).

**Ex:** \( S = \text{In} \) and \( V \) a \( G \)-rep

- **Diagram**:
- \( e = X \rightarrow X \)
- \( q_j = X \rightarrow Y \)
- \( v \in \bigoplus_{A} V_j \) with \( v \cdot a = 0 \text{ for all } a \in L_e \)
- \( v = \sum_{i} v_i \otimes a_j \)
- If \( a_j^* \) is inverse of \( a_j \) then
  \[ a_i^* a_j^* \in \text{Re} \iff \text{dom}(a_i^* a_j^*) = X \]
  \[ \iff \text{im} a_i = \text{dom} a_j^* = Y \iff i = j \]
- \( 0 = v \cdot a_j^* = (v_j \otimes a_j) \cdot a_j^* = v_j \otimes e \)
  \[ a_j a_j^* = e \]
- \( \Rightarrow v_j = 0 \Rightarrow v_j \otimes a_j = 0 \text{ (for all } i) \Rightarrow v = 0 \Rightarrow \text{Ann}(L_e) = 0 \)
Ex: if $S$ is an inverse monoid and $V$ a $G_e$-rep then $\text{Ann}(le) = 0$.

Eg: $S = T_n$, $V =$ trivial rep. of $G_e =$ trivial group

$V^S = \text{mapping rep.}/W$ (1-dimensional)

with basis $v_i + W$ ($v_i - v_j \in W \Rightarrow v_i + W = v_j + W$)

and $(v_i + W).b = v_{i b} + W = v_i + W$

= trivial rep. of $T_n$