Free idempotent generated semigroups over biordered sets

Dandan Yang
The University of York

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Free idempotent generated semigroup IG(E) over biordered set E

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Let IG(E) denote the semigroup defined by the following presentation.

$$\mathrm{IG}(E) = \langle E | e.f = ef \text{ if } (e, f) \text{ is a basic pair} \rangle.$$

IG(E) is called the *free idempotent generated semigroup* on E.

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(IG3) ϕ maps the $\mathcal{R}$-class (respectively $\mathcal{L}$-class) of $e \in E$ onto the corresponding class of e in S'.

(IG4) The restriction of ϕ to the maximal subgroups of IG(E) containing $e \in E$ is a homomorphism onto the maximal subgroup of S' containing e.

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If S is regular semigroup, the *free regular idempotent generated semigroup* RIG(E) on E is defined by adding the relation:

$$ehf = ef(h \in S(e, f))$$

to the presentation of IG(E), where

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The semigroup $\mathrm{RIG}(E)$ also satisfies the properties (IG1), (IG2), (IG3) and (IG4). In addition, $\mathrm{RIG}(E)$ is regular; and the maximal subgroups of any $e \in E$ in $\mathrm{IG}(E)$ and $\mathrm{RIG}(E)$ are isomorphic.

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Brittenham, Margolis and Meakin (2009)

They gave a 72-element semigroup S and proved that IG(E) has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The maximal subgroups of IG(E) for several classes of biordered sets

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Let $\mathcal{T}_n$ be the full transformation semigroup, let E be its biordered set, and let $e \in E$ be an arbitrary idempotent with rank r $(1 \leqslant r \leqslant n-2)$. Then the maximal subgroup H_e of the free idempotent generated semigroup IG(E) containing e is isomorphic to the symmetric group S_r.

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Methods: The authors applied a presentation for H arising from singular squares of $\mathcal{T}_n$ and proved that this presentation is the well known Coxeter presentation for symmetric groups.

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Let $M_n(D)$ be the matrix semigroup of all $n \times n$ matrices over a division ring D. It is well known that the maximal subgroup of $M_n(D)$ with identity $e \in M_n(D)$ of rank r is isomorphic to the general linear group $GL_r(D)$.

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Let E be the biordered set of idempotents of $M_n(D)$, for D a division ring, and let e be an idempotent matrix of rank 1 in $M_n(D)$. For $n \geqslant 3$, the maximal subgroup of IG(E) containing e is isomorphic to D^*, the multiplicative group of units of D.

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Let E be the biordered set of idempotents of $M_n(D)$, for D a division ring, and let e be an idempotent matrix of $M_n(D)$ with rank $r \leqslant n/3$. For $n \geqslant 3$, the maximal subgroup of IG(E) with identity e is isomorphic to the general linear group $GL_r(D)$.

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The maximal subgroups of IG(E) for the biordered set of the endomorphism monoid of free G-act F_n.

I concentrate on the free idempotent generated semigroups IG(E), where E is the biordered set of the endomorphism monoid EndF_n of a n-dimensional independence algebra F_n, where n is finite.

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We proved that if $e \in \text{End}F_n$ is an idempotent of rank m ($m \leqslant n$), then the maximal subgroup H with identity e of EndF_n is isomorphic to the automorphism group AutF_m of a free G-act F_m of rank m.

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In fact, from the paper 'Independence Algebras' by V.Gould, we can also deduce that this result will hold for the endomorphism monoid of finite independence algebra.

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Lemma 1 Let $\alpha, \beta \in D_1$ such that $x_i\alpha = u_i x_l$ and $x_i\beta = v_i x_k$, for some $l, k \in \{1, \cdots, n\}$. Then $\mathrm{Ker}\alpha = \mathrm{Ker}\beta$ if and only if there exists some $g \in G$, such that $u_i = v_i g$ for any $i \in [1, n]$. Furthermore, each $\mathcal{H}$-class in D_1 is a group.

The maximal subgroups of $\mathrm{IG}(E)$ for the biordered set of the endomorphism monoid of free G-act F_n.

Let E be a biordered set. An *E-square* is a sequence (e, f, g, h, e) of elements of E with $e\mathcal{R}f\mathcal{L}g\mathcal{R}h\mathcal{L}e$.

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An E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ is said to be *singular* if there exists $k^2 = k$ such that either of the following conditions holds:

$$ek = e, fk = f, ke = h, kf = g \qquad \text{or}$$

$$ke = e, kh = h, ek = f, hk = g.$$

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Lemma 2 An E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ in D_1 is singular if and only it is a semigroup (i.e. rectangular band).

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Lemma 4 Suppose that f and g are idempotents in EndF_n such that $fg = h$ is a rank 1 idempotent. Then in IG(E), we have $\overline{f}\,\overline{g} = \overline{h}$ if and only if $\overline{f}\,\overline{g}$ is regular in IG(E). (see the board)

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Proposition 5 Suppose e is a rank 1 idempotent, $f, g \in E$ and $\overline{f}\overline{g}$ be an element in the maximal subgroup of RIG(E) with identity $\overline{e}$. Then $fg = e$ implies $\overline{f}\overline{g} = \overline{e}$. Furthermore, if $e = fgh$, then either fg or gh is a rank 1 idempotent implies $\overline{e} = \overline{f}\overline{g}\overline{h}$.

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Recently, we are working on a special case: $n = 3$ and $|G| = 2$ by using elementary methods. (see the board)