Ehresmann monoids

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Abstract

Ehresmann monoids form a variety of biunary monoids, which includes restriction (hence ample and inverse) monoids. We demonstrate they have a rich structure that is fundamentally different from that of the widely studied restriction monoids. In earlier papers, we developed a theory for *left* Ehresmann monoids, which form a variety of unary monoids. Here we consider the two-sided case. Even for the more tractable class of restriction monoids, the literature shows that the two-sided case cannot be resolved simply by putting together the results in the left and right handed cases. It requires introducing new techniques, and we cannot expect exactly analogous results. In the class of Ehresmann monoids we introduce the notions of *T-normal forms, strongly T-proper* and *T-proper*. Recent work of Jones indicates that, even in the case for restriction monoids, the notion of strongly T-proper yields non-trivial insights that are *not* simple extensions of the approach for inverse monoids.

First, we show that elements of Ehresmann monoids have T-normal forms. Next, we show how to construct a strongly T-proper Ehresmann monoid $\mathcal{P}(T,Y)$ from a semilattice Y acted upon on both sides by a monoid T via order preserving maps. We then prove that any Ehresmann monoid admits a strongly X^*-proper Ehresmann cover and that the free Ehresmann monoid on X is of the form $\mathcal{P}(X^*,E)$. Contrary to the free left Ehresmann monoid, the free Ehresmann monoid does not have uniqueness of X^*-normal forms. In a subsequent article, we characterise monoids of the form $\mathcal{P}(T,Y)$ by showing that they are initial objects in certain categories.

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1 Introduction

Left Ehresmann semigroups, defined in Section 2, form a variety of unary semigroups, that is, of semigroups equipped with an additional basic unary operation of $s \mapsto s^+$. For a left Ehresmann semigroup S, the set $E_S = \{s^+ : s \in S\}$ (or E if S is understood) forms a semilattice under the multiplication in S. We refer to E_S as the semilattice of *projections* or *distinguished semilattice* of S. *Right Ehresmann* semigroups are defined dually, with unary operation denoted by $s \mapsto s^*$. A semigroup is *Ehresmann* if it is both left and right Ehresmann such that the semilattices of projections coincide. It follows that Ehresmann semigroups form a variety of biunary semigroups. Our interest in (left) Ehresmann semigroups comes from several directions, which we now explain.

First, inverse semigroups are Ehresmann where $s^+ = ss^{-1}$ and $s^* = s^{-1}s$ and in this case every idempotent is a projection. In general, however, Ehresmann semigroups need not be regular. Indeed, an Ehresmann semigroup such that every idempotent is a projection is inverse if and only if it is regular. Second, the variety of (left) Ehresmann semigroups is the variety generated by the quasi-variety of (left) adequate semigroups, a result that is a consequence of Kambites's construction of free (left) adequate semigroups [17, 16], and his demonstration that the free (left) adequate semigroup on a given set coincides with the free (left) Ehresmann semigroup on the same set. Left adequate monoids were introduced by Fountain in [5] and have the property that they are precisely the monoids with semilattice of idempotents such that every principal right ideal is projective [5]. Fountain also presented the two-sided case of adequate semigroups in [6]. In fact, the study of (left) adequate semigroups for some time largely focussed on those that were (left) *type A*, later called (left) *ample*. Such semigroups satisfy the 'ample' identities $(xy)^+x = xy^+$ and $x(yx)^* = y^*x$ (or just $(xy)^+x = xy^+$, as appropriate), which are easily seen to hold for inverse semigroups with $^+$ and * defined as above. Indeed, many of the approaches to inverse semigroups, such as the McAlister notion of proper covers [21, 22] have their analogue for (left) ample semigroups and the wider classes of (left) restriction semigroups. One point that one must emphasise is that proving results in the two-sided case usually requires far more than glueing one-sided results together.

As pointed out in [4], the semigroup of binary relations $\mathcal{B}_X$ on any set X is Ehresmann where

$$\rho^+ = \{(x,x) : x \in \operatorname{dom} \rho\} \text{ and } \rho^* = \{(x,x) : x \in \operatorname{im} \rho\}.$$

Certainly $\mathcal{B}_X$ is not regular, although its biunary subsemigroup $\mathcal{PT}_X$ of partial maps is. It is well known that $\mathcal{PT}_X$ is left restriction but not right restriction; consequently, $\mathcal{B}_X$ is neither left nor right restriction.

Our final source of interest in Ehresmann semigroups, whence we obtain the nomenclature, is that they arise as the semigroups associated to Ehresmann's work on ordered categories, as explicated by Lawson in [20]. For further details of the approach to Ehresmann semigroups using pairs of partial orders, we recommend the reader to [20]. We remark that if the ample identities hold, then the two partial orders coincide.

What then can we say about (left) adequate and (left) Ehresmann semigroups? Without the ample identity, almost all of the known approaches fail and new ideas and strategies are required. Largely for ease of expression, but occasionally for technical reasons, we focus here on (left) Ehresmann *monoids*.

To motivate the current work, we briefly outline the approach we took for left Ehresmann monoids in [2, 9]. We say that a left Ehresmann monoid is *T-generated* by a submonoid T if every element of M can be expressed as products of projections and elements of T; for such an M and T, every element has a *T-normal form*. From a monoid T acting on a semilattice Y with identity by order preserving maps, we construct a left Ehresmann monoid $\mathcal{P}_\ell(T, Y)$, containing (an isomorphic copy of) T with semilattice of projections (isomorphic to) Y, which is T-generated and has uniqueness of T-normal forms. Moreover, for a T-generated left Ehresmann monoid M we have a projection separating morphism from some $\mathcal{P}_\ell(T, Y)$ onto M that is an isomorphism if and only if M has uniqueness of T-normal forms. If T is right cancellative, then $\mathcal{P}_\ell(T, Y)$ is left adequate. Kambites describes free left adequate monoids using birooted labelled trees [17], and notes that they are also the free left Ehresmann monoids. In a complimentary way we used our techniques to describe the free left Ehresmann monoid on X as being of the form $\mathcal{P}_\ell(X^*, Y)$; it is therefore left adequate. Consequently, left Ehresmann monoids form the variety generated by the quasi-variety of left adequate monoids (see also [17]). We also showed that uniqueness of T-normal forms implies the property of being *strongly T-proper* which itself implies that of being *T-proper*. The terminology is related to that in the restriction case, but we stress the ideas are new. Our strategy was motivated by, but not directly connected to, the McAlister approach in the inverse case via proper (E-unitary) covers and McAlister P-semigroups [21, 22].

Now we outline the contents of the present article. We aim to study Ehresmann monoids using the same strategy as in the one-sided case. As indicated above, this is rather more than putting results for left and right Ehresmann monoids together. Even for ample semigroups, the development of a two-sided theory of proper covers [19, 3] and the clarification of the structure of free ample semigroups [7] was more taxing than in the one-sided situation. After Section 2 on preliminaries, we begin the work of this article in Section 3 by showing that elements in T-generated Ehresmann monoids have T-normal forms, i.e. an expression that is simultaneously both a left and a right T-normal form. A crucial step in this proof is a rather deeper analysis of the algorithm for achieving normal forms in the one-sided case given in [2].

The key in [7] is the notion of a monoid acting on both sides of a semilattice by morphisms such that the actions are connected via compatibility conditions. Motivated by the way in which an Ehresmann monoid S acts on the left and right of E_S, here we consider a monoid T acting on both sides of a monoid semilattice Y by *order preserving* maps such that the actions are connected by conditions, tailored to this present case, that we again refer to as *compatibility conditions*. In Section 4 we construct an Ehresmann monoid $\mathcal{P}(T, Y)$, containing (an isomorphic copy of) T with semilattice of projections (isomorphic to) Y, which is T-generated. We argue in Section 5 that any T-generated Ehresmann monoid is a projection separating morphic image of $\mathcal{P}(T, Y)$.

Of course, these results would not have much virtue were $\mathcal{P}(T, Y)$ not to possess some distinctive strong properties (and not mere copies of those in the one-sided cases). In fact, we cannot hope for $\mathcal{P}(T, Y)$ to have uniqueness of T-normal forms. We demonstrate this by showing in Section 6 that the free Ehresmann monoid is of the form $\mathcal{P}(X^*, Y)$, but is easily seen not to have uniqueness of X^*-normal forms. However, the uniqueness of T-normal forms of $\mathcal{P}_\ell(T, Y)$ in the one-sided case was something of a surprise, as we now explain. We say that a (left, right) T-generated Ehresmann monoid is *strongly T-proper* if the congruence σ separates T[(a)]. What we had expected was that $\mathcal{P}_\ell(T, Y)$ would be strongly T-proper, a condition that is *implied* by uniqueness of T-normal forms and which implies a condition we called (left) T-proper. We show in Section 4 that $\mathcal{P}(T, Y)$ is strongly T-proper, and hence T-proper. To do so requires careful analysis of the interactions between the corresponding one-sided cases.

An Ehresmann monoid M that satisfies the ample identities is called *restriction*. In a happy full circle, Jones [15] argues that the notion of strongly T-proper is fundamental even in the restriction case. A restriction monoid M that is proper (in the sense analogous to that in the inverse case) is strongly T-proper for a submonoid T if and only if it is *perfect* [15, Proposition 3.2]. Here M is perfect if σ is a perfect congruence, that is, set products of σ-classes are σ-classes and each σ-class has a greatest element. Jones builds a theory for restriction monoids around the notion of being perfect. An alternative approach to some of Jones's work is given by Kudryatseva in [18].

It is known [17] that the free left Ehresmann monoid is left adequate. In a subsequent article [1] we examine properties of T guaranteeing that $\mathcal{P}(T, Y)$ is left adequate. Further, we characterise monoids of the form $\mathcal{P}(T, Y)$ by showing that they are initial objects in certain categories.

2 Preliminaries

In this section we give the basic definitions and results regarding Ehresmann semigroups and monoids needed for the rest of the article. Further details may be found in the notes [11].

Left Ehresmann semigroups are defined by the associativity identity together with the identities $x^+x = x$, $(x^+y^+)^+ = x^+y^+ = y^+x^+$ and $(xy)^+ = (xy^+)^+$. We observe that, in particular, they also satisfy the identities $x^+ = x^+x^+$ and $(x^+)^+ = x^+$. Dual identities define right Ehresmann semigroups.

Ehresmann semigroups are defined by the identities for left and right Ehresmann semigroups, together with $(x^*)^+ = x^*$ and $(x^+)^* = x^+$. It is these identities which guarantee there is no ambiguity over membership of the semilattice of projections.

There is another approach to (one-sided) Ehresmann semigroups, via the use of relations extending those of Green. For any subset E of idempotents of a semigroup S, let $\widetilde{\mathcal{R}}_E$ be the equivalence relation on S defined by the rule that $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if, for all $e \in E$, $ea = a$ if and only if $eb = b$. If S is a left Ehresmann semigroup with semilattice of

[(a)] It is easy to see that we may regard σ as a semigroup congruence or as a congruence in an augmented signature, whence there is no ambiguity.

projections E, then each $\widetilde{\mathcal{R}}_E$-class of an element a has a unique element of E, which is a^+, and $\widetilde{\mathcal{R}}_E$ is a left congruence. Conversely, if S is a semigroup such that, for some semilattice E of S, each $\widetilde{\mathcal{R}}_E$-class $[a]_{\widetilde{\mathcal{R}}_E}$ has a (necessarily unique) element, say a^+, of E and $\widetilde{\mathcal{R}}_E$ is a left congruence, then the unary semigroup obtained from S by equipping it with the unary operation $a \mapsto a^+$ is left Ehresmann with semilattice of projections E. For a right Ehresmann semigroup, the dual equivalence relation is denoted by $\widetilde{\mathcal{L}}_E$ so that $a \ \widetilde{\mathcal{L}}_E \ b$ if and only if, for all $e \in E$, $ae = a$ if and only if $be = b$. Clearly, in a left (respectively right) Ehresmann semigroup, $a \ \widetilde{\mathcal{R}}_E \ b$ if and only if $a^+ = b^+$ (respectively $a \ \widetilde{\mathcal{L}}_E \ b$ if and only if $a^* = b^*$).

Left (right) Ehresmann and Ehresmann semigroups were first defined in the literature *as semigroups* and by means of these equivalences $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. In this article we regard them as varieties of unary and bi-unary semigroups. Thus, for example, morphisms must additionaly preserve the unary operation(s). Occasionally it helps to stress the signature and in this case we refer to, for example, a morphism of Ehresmann semigroups as a $(2, 1, 1)$-morphism (since the arity of the signature is $(2, 1, 1)$).

A *left* (respectively *right*) *Ehresmann monoid* is a left (respectively right) Ehresmann semigroup together with an identity, hence a (2,1,1,0)-algebra. In such a monoid we necessarily have $1^+ = 1$ (respectively $1^* = 1$). We focus largely on the case for monoids, the existence of a multiplicative identity playing an important role in many of our arguments. A (left, right) Ehresmann monoid is *reduced* if it has trivial semilattice of projections. Monoids, regarded as reduced (left, right) Ehresmann monoids, play a role in this theory that is a distant analogue of that taken by groups in the theory of inverse semigroups. In the sequel, given a (left, right) Ehresmann monoid M the letters E and T denote, respectively, the semilattice of projections of M and a *submonoid* of M.

We now present some technical results, which will be useful in subsequent sections. The relation $\leqslant$ appearing in the next statement is the natural partial order on E.

Lemma 2.1. *Let M be a left Ehresmann monoid. Then for any $a, b \in M$ and $e \in E$, $a^+(ab)^+ = (ab)^+$, $(ea)^+ = ea^+$ and $(aeb)^+ \leqslant (ab)^+ \leqslant a^+$.*

Dually, if M is a right Ehresmann monoid with semilattice of projections E, then for any $a, b \in M$ and $e \in E$, $(ab)^*b^* = (ab)^*$, $(ae)^* = a^*e$ and $(aeb)^* \leqslant (ab)^* \leqslant b^*$.

Proof. We need only give the argument for left Ehresmann semigroups. That $a^+(ab)^+ = (ab)^+$ and $(ea)^+ = ea^+$ is well known and easy to see. Consequently,

$$(aeb)^+ = (aeb^+b)^+ = (ab^+eb)^+ \leqslant (ab^+)^+ = (ab)^+ \leqslant a^+,$$

where we use the fact E is a semilattice and the identity $(xy^+)^+ = (xy)^+$. $\square$

For any monoid U we denote its dual monoid by U^{d}. Let T be a monoid and let Y be a partially ordered set (indeed in this article, Y will always be a semilattice). We say that T *acts on the left of Y via order-preserving maps* if there exists a monoid morphism $\alpha : T \to \mathcal{O}_Y^{\mathrm{d}}$, where $\mathcal{O}_Y$ is the monoid of all order-preserving maps of Y. Normally, we write $t \cdot y$ for $(t\alpha)(y)$. The next lemma is straightforward (see for example [2, Lemma 1.7]).

Lemma 2.2. *Let M be a monoid and let T be a submonoid of M. If M is left Ehresmann then T acts on E on the left via order preserving maps by*

$$(t,e) \mapsto t \cdot e = (te)^+.$$

Dually, if M is right Ehresmann with semilattice of projections E, then T acts on E on the right via order preserving maps by

$$(e,t) \mapsto e \circ t = (et)^*.$$

We recall that a left Ehresmann monoid M is *hedged* if the action of M on E is by morphisms, that is,

$$(mef)^+ = (me)^+(mf)^+.$$

Since the action of E on E by left multiplication is clearly by morphisms, it follows that a T-generated left Ehresmann monoid is hedged if and only if T acts on E by morphisms. The corresponding definitions and remarks hold in the right and two-sided cases.

Lemma 2.3. *Let M be an Ehresmann monoid. For any $a \in M$ and $e, f \in E$, we have*

$$(eaf)^+ = e\big(a(eaf)^*\big)^+.$$

In particular, for any $a \in M$ and $e \in E$,

$$(ea)^+ = e\big(a(ea)^*\big)^+ \quad \textit{and} \quad a \cdot e = a \cdot (ae)^*.$$

Proof. Let $a \in M$ and $e, f \in E$. Then

$$(eaf)^+ = \big(ea(ea)^*f\big)^+ = e\big(a(ea)^*f\big)^+ = e\big(a(eaf)^*\big)^+. \qquad \square$$

Remark 2.4. The rather simple Lemma 2.3 gives in fact a key to Section 4, since its first equality can be rewritten in terms of actions as follows:

$$e(a \cdot f) = e\big(a \cdot \big((e \circ a)f\big)\big).$$

Let S be a semigroup and suppose that $E \subseteq E(S)$. We define the relation σ_E to be the *semigroup congruence on S generated by $E \times E$*. It is clear that, for any $a, b \in S$ we have that $a \,\sigma_E\, b$ if and only if $a = b$ or there exists a sequence

$$a = c_1e_1d_1, c_1f_1d_1 = c_2e_2f_2, \ldots, c_nf_nd_n = b,$$

where $c_1, d_1, \ldots, c_n, d_n \in S^1$ and $(e_1, f_1), \ldots, (e_n, f_n) \in E \times E$. Notice that in an Ehresmann monoid M with semilattice of projections E, we denote σ_E more simply by σ. In this case, for any $a, b \in M$, $a^+ \,\sigma\, b^+$ and $a^* \,\sigma\, b^*$ whether or not $a \,\sigma_E\, b$, giving us the following.

Lemma 2.5. *Let M be an Ehresmann monoid. Then E is contained in a σ-class and σ is a $(2,1,1,0)$-congruence.*

As indicated earlier, in this article we require care with signatures. To this end we give a technical but straightforward result, the proof of which we omit. Given a subset A of an Ehresmann monoid M, we denote by $\langle A\rangle_{(2)}$ ($\langle A\rangle_{(2,0)}$, $\langle A\rangle_{(2,1,1,0)}$) the subsemigroup (respectively submonoid, Ehresmann submonoid) generated by A. Of course, the default is $\langle A\rangle_{(2,1,1,0)}$, but it does not hurt clarity to stress the signature, given that we make use of others.

Lemma 2.6. *Let M be an Ehresmann monoid and let X be a subset of M. Put $T = \langle X\rangle_{(2,0)}$. Then*

$$\langle E\cup X\rangle_{(2,1,1,0)} = \langle E\cup X\rangle_{(2)} = \langle E\cup T\rangle_{(2)} = \langle E\cup T\rangle_{(2,1,1,0)}.$$

Moreover, $M = \langle E\cup T\rangle_{(2)}$ when $M = \langle X\rangle_{(2,1,1,0)}$.

We now define the notion of T-proper for an Ehresmann monoid M. Effectively, we say that M is T-proper if it is T-proper regarded as both a left and as a right Ehresmann monoid (see [2, Definition 3.5]).

Definition 2.7. Let M be an Ehresmann monoid with projections E and let T be a submonoid of M such that $M = \langle E\cup T\rangle_{(2)}$. Then M is said *T-proper* if for any $s,t\in T$ and $e\in E$,

$$(se)^+ = (te)^+ \text{ and } se\,\sigma\, te \text{ imply that } se = te$$

and, dually,

$$(es)^* = (et)^* \text{ and } es\,\sigma\, et \text{ imply that } es = et.$$

Of course, the foregoing condition can be phrased using the relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. As noted in [2, Lemma 3.6], proper restriction monoids (and hence proper ample and proper inverse monoids) are examples of T-proper Ehresmann monoids for $T = M$. Clearly a strongly T-proper Ehresmann monoid is T-proper and we will show that the free Ehresmann monoid on X is strongly T-proper, for some T isomorphic to X^*.

A word on notation: for us, 0 is not a natural number and we denote $\mathbb{N}\cup\{0\}$ by $\mathbb{N}_0$.

3 One and two-sided normal factorizations

We recall from [2] the following result concerning factorisations of elements in left Ehresmann monoids, before providing a rather deeper analysis than was necessary in [2], but which will be essential for the exposition in this current article.

Proposition 3.1 ([2]). *Let M be a left Ehresmann monoid and suppose that $M = \langle E\cup T\rangle_{(2)}$. Then any $x\in M$ can be written as*

$$x = t_0e_1t_1\dots e_nt_n,$$

where $n\in\mathbb{N}_0$, $e_1,\dots,e_n\in E$, $t_1,\dots,t_{n-1}\in T\setminus\{1\}$, $t_0,t_n\in T$ and for $1\leqslant i\leqslant n$,

$$e_i < (t_ie_{i+1}t_{i+1}\dots e_nt_n)^+.$$

We point out that in this proposition we necessarily have $e_i \neq 1$, since $e_i < (t_i e_{i+1} t_{i+1} \dots e_n t_n)^+$. We also observe that

$$e_i < (t_i e_{i+1} t_{i+1} \dots e_n t_n)^+ = (t_i e_{i+1} (t_{i+1} \dots e_n t_n)^+)^+ = (t_i e_{i+1})^+.$$

The strategy of the proof of Proposition 3.1 consists of the following procedure over a decomposition $x = s_0 f_1 s_1 \dots f_m s_m$, where $s_0, \dots, s_m \in T$ and $f_1, \dots, f_m \in E$, which can be found, since $M = \langle E \cup T \rangle_{(2)}$ and $1 \in T$:

Step 1. Eliminate all f_i's such that $f_i(s_i f_{i+1} \dots s_{m-1} f_m s_m)^+ = (s_i f_{i+1} \dots s_{m-1} f_m s_m)^+$ to obtain $x = u_0 g_1 u_1 \dots g_k u_k$, where $u_0, \dots, u_k \in T$ and $g_1, \dots g_k \in E$ are such that $g_i(u_i g_{i+1} \dots u_{k-1} g_k u_k)^+ < (u_i g_{i+1} \dots u_{k-1} g_k u_k)^+$.

Step 2. Replace each g_i by g_i', where $g_i' = g_i(u_i g_{i+1} \dots u_{k-1} g_k u_k)^+$. At this stage we have $x = u_0 g_1' u_1 \dots g_k' u_k$ and $g_i' < (u_i g_{i+1}' \dots u_{k-1} g_k' u_k)^+$.

Step 3. Delete any interior u_i's that are 1.

We thus finally obtain a desired form for x, where the elements of T are the remaining u_i's and the elements of E are g_i''s or products of consecutive g_i''s whose u_i's in the middle were deleted.

Let us call this procedure ($\mathrm{P_R}$) and call ($\mathrm{P_L}$) the dual procedure for right Ehresmann monoids.

A factorization $t_0 e_1 t_1 \dots e_n t_n$ as in Proposition 3.1 is said to be *left T-normal* or simply *left normal*, if there is no ambiguity. If in the conditions on the factors, $T \setminus \{1\}$ is replaced by T, we say that the factorization is *weak left T-normal* or simply *weak left normal*. Dual terminology applies to right Ehresmann monoids. Note that in [2] and [9], which deal exlcusively with factorisations in *left* Ehresmann monoids, the adjective 'left' was not needed.

We will be applying both the procedures ($\mathrm{P_R}$) and ($\mathrm{P_L}$) and variations thereof to (two-sided) Ehresmann monoids below. With this in mind it is helpful to pause and point out some subtleties.

Remark 3.2. In a left Ehresmann monoid M, given $a, b \in M$, $e \in E$, we have $aeb = ab$ whenever $eb^+ = b^+$. This simple fact implies that an alternative approach to Step 1 would be that we eliminate idempotents one at a time, in any order, reviewing the conditions on the resulting decomposition of x as an alternating product of elements of T and E - leading to the same factorization of x as $x = u_0 g_1 u_1 \dots g_k u_k$.

Remark 3.3. Notice that in Step 2, $u_i g_{i+1}' \dots g_k' u_k = u_i g_{i+1} \dots g_k u_k$ for any $i \in \{0, \dots, k\}$.

Remark 3.4. Suppose we have a weak left T-normal factorization $t_0 e_1 t_1 \dots e_n t_n$ of x. As indicated by Step 3, eliminating any $t_i = 1$ where $i \in \{1, \dots, n-1\}$ certainly leaves a weak left T-normal factorization (indeed, deleting all such elements reveals a left T-normal factorization). Moreover, if we merely know that $t_0 e_1 t_1 \dots e_n t_n$ has the weaker property that $e_i \leqslant (t_i e_{i+1} \dots e_n t_n)^+$ for $i \in \{1, \dots, n\}$, then again this property is preserved by deleting any interior t_i's that equal 1.

The above remarks, which will be useful in themselves, lead us some way to the following alternative approach to (P_R).

Let M be a left Ehresmann monoid and suppose that $M = \langle E \cup T \rangle_{(2)}$. Suppose that $x \in M$ is written as $x = s_0 f_1 s_1 \dots f_m s_m$ where $s_i \in T$ and $f_j \in E$.

Move A. Eliminate f_i where $f_i\big(s_i f_{i+1} \dots f_m s_m\big)^+ = \big(s_i f_{i+1} \dots f_m s_m\big)^+$.

Move B. Replace f_i where $f_i\big(s_i f_{i+1} \dots f_m s_m\big)^+ < \big(s_i f_{i+1} \dots f_m s_m\big)^+$ by f_i', where $f_i' = f_i\big(s_i f_{i+1} \dots f_m s_m\big)^+$.

Move C. Delete an interior s_i that is 1.

It follows from Remarks 3.2 and 3.3 and the effectiveness of (P_R) that the T-normal form achieved via (P_R) is obtained by applying a sequence of Moves A, then Moves B, then Moves C. Note that the strategy here is a little different from that in [2]: here we are re-labelling the decomposition of x at each stage. We show that any T-normal form achieved by applying Moves A, B and C *in any order* will coincide with that obtained via (P_R) - starting, of course, from *the same* original factorisation of x. In order to show the above uniqueness, we call heavily upon the results of [9].

Let T be a monoid with identity 1_T and let Y be a semilattice with identity 1_Y. To avoid any ambiguity we assume that $T \cap Y = \emptyset$. Let $T * Y$ be the *semigroup* free product of T and Y, which we consider here as the set of all sequences $(u_1, \dots, u_n)$, that we will usually represent as $u_1 \dots u_n$, with $n \in \mathbb{N}$ and $(u_i, u_{i+1}) \in (T \times Y) \cup (Y \times T)$ for any $i \in \{1, \dots, n-1\}$, endowed with the usual product. We say that an element $u_1 \dots u_n$ has *length* n. The monoids T and Y naturally embed as semigroups in $T * Y$ and with this convention, $E(T * Y) = E(T) \cup Y$.

The left Ehresmann monoid $\mathcal{P}_\ell(T, E) = (T * E)/\sim_\ell$ appearing in Proposition 3.5 is a construction taken from [9]. Note that in [9] the subscript ℓ is not used as [9] deals almost exclusively with *left* Ehresmann monoids. Here we need a subscript to prevent confusion with the left-right duals $\mathcal{P}_r(T, E)$ and $\sim_r$ and the two-sided versions $\mathcal{P}(T, E)$ and $\sim$ introduced in Section 4. The unary monoid $\mathcal{P}_\ell(T, E)$ is defined in [9, Theorem 2.2] and is the free product $T * E$ equipped with a unary operation $u \mapsto u^+$ and factored by the semigroup congruence $\sim_\ell$ generated by

$$H = \{(u^+u, u) : u \in T * E\} \cup \{(1_T, 1_E)\}$$

(we need to label the identity of M separately in T and E to form the semigroup free product). The map $\overline{\psi}_\ell : \mathcal{P}(T, E) \to M$ is given by $[u]_{\sim_\ell}\overline{\psi} = u$, where (with abuse of notation) u is the natural image of u in M. As observed in [9, Theorem 3.1], for $u \in T * E$, the element u^+ in $T * E$ coincides with u^+ in M.

Proposition 3.5. *Let M be a left Ehresmann monoid and suppose that $M = \langle E \cup T \rangle_{(2)}$. Let $x \in M$ have a factorisation $x = s_0 f_1 s_1 \dots f_m s_m$ where $s_i \in T$ and $f_j \in E$, for*

$i \in \{0, \ldots, m\}$, $j \in \{1, \ldots, m\}$. *Then any sequence of applications of Moves A, B and C, in any order, results in the same left T-normal form for x. By earlier comments, this is also the left T-normal form obtained via applying the procedure* $(\mathrm{P_R})$.

Proof. From [9, Corollary 3.4] there is an onto morphism $\overline{\psi}$ from $\mathcal{P}_\ell(T, E)$ to M. Let $x \in M$ be written as $x = s_0f_1s_1 \ldots f_ms_m$ where $s_i \in T$ and $f_j \in E$ and for convenience let $\bar{x}$ denote the element of $T * E$ corresponding to this factorisation. Suppose that applying one of Moves A, B or C to x yields $x = t_0e_1t_1 \ldots me_nt_n$ and let $\bar{y} \in T * E$ correspond to this factorisation. We show that $\bar{x} \sim_\ell \bar{y}$.

Suppose that we apply Move A, so that $y = s_0f_1 \ldots s_{i-1}s_if_{i+1} \ldots f_ms_m$, where we have that $f_i(s_if_{i+1} \ldots f_ms_m)^+ = (s_if_{i+1} \ldots f_ms_m)^+$. Then

$$\begin{array}{rcl} \bar{x} & = & s_0f_1s_1 \ldots f_ms_m \\ & \sim_\ell & s_0f_1 \ldots s_{i-1}f_i(s_if_{i+1} \ldots f_ms_m)^+(s_if_{i+1} \ldots f_ms_m) \\ & = & s_0f_1 \ldots s_{i-1}(s_if_{i+1} \ldots f_ms_m)^+(s_if_{i+1} \ldots f_ms_m) \\ & \sim_\ell & s_0f_1 \ldots s_{i-1}s_if_{i+1} \ldots f_ms_m \\ & = & \bar{y}. \end{array}$$

On the other hand, if we apply Move B, then we replace some f_i such that $f_i(s_if_{i+1} \ldots f_ms_m)^+ < (s_if_{i+1} \ldots f_ms_m)^+$ with $f_i' = f_i(s_if_{i+1} \ldots f_ms_m)^+$. Now $\bar{y} = s_0f_1 \ldots s_{i-1}f_i's_i \ldots f_ms_m$ and

$$\begin{array}{rcl} \bar{x} & = & s_0f_1s_1 \ldots s_{i-1}f_is_i \ldots f_ms_m \\ & \sim_\ell & s_0f_1s_1 \ldots s_{i-1}f_i(s_i \ldots f_ms_m)^+s_i \ldots f_ms_m \\ & = & s_0f_1s_1 \ldots s_{i-1}f_i's_i \ldots f_ms_m \\ & = & \bar{y}. \end{array}$$

Finally, if we apply Move C, deleting $s_i = 1_T$, where $i \in \{1, \ldots, n-1\}$, then $\bar{y} = s_0f_1 \ldots f_if_{i+1}s_{i+1} \ldots f_ms_m$ and

$$\begin{array}{rcl} \bar{x} & = & s_0f_1s_1 \ldots s_{i-1}f_is_if_{i+1} \ldots f_ms_m \\ & = & s_0f_1s_1 \ldots s_{i-1}f_i1_Tf_{i+1} \ldots f_ms_m \\ & \sim_\ell & s_0f_1s_1 \ldots s_{i-1}f_i1_Ef_{i+1} \ldots f_ms_m \\ & = & s_0f_1s_1 \ldots s_{i-1}f_if_{i+1} \ldots f_ms_m \\ & = & \bar{y}. \end{array}$$

It follows that if $x = z = u_0g_1 \ldots g_ku_k$ is any factorisation of x in left T-normal form, where $u_0, \ldots, u_k \in T$ and $g_1, \ldots, g_k \in E$ obtained by applying Moves A, B and C (in any order), then $[\bar{x}] = [\bar{z}]$ where $\bar{z}$ is the element of $T * E$ corresponding to this factorisation. Moreover, it follows from [9, Lemma 2.10] and the discussion at the start of the proof of [9, Theorem 2.2] that $[\bar{z}] = [u_0]_{\sim_\ell}[g_1]_{\sim_\ell} \ldots [g_k]_{\sim_\ell}[u_k]_{\sim_\ell}$ is in left T_ℓ'-normal form, where $t_\ell' = \{[t]_{\sim_\ell} : t \in T\}$. Our result now follows from the fact that $\mathcal{P}_\ell(T, E)$ has uniqueness of left T_ℓ'-normal forms and [9, Theorem 2.2, Lemma 2.10, Corollary 3.4]. $\square$

Remark 3.6. Suppose that M is a left Ehresmann monoid. Then every element of T is a left T-normal factorization, and if $e \in E \backslash \{1\}$, then $1e1$ is a left T-normal factorization.

Assume that $y = s_0f_1s_1 \dots f_ms_m$ and $z = t_0e_1t_1 \dots e_nt_n$, where $s_0, \dots, s_m, t_0, \dots, t_n \in T$ and $f_1, \dots, f_m, e_1, \dots, e_n \in E$, and the factorization $t_0e_1t_1 \dots e_nt_n$ is left T-normal. Let $x = yz = s_0f_1s_1 \dots f_ms_mt_0e_1t_1 \dots e_nt_n$. How might we reduce this to a left T-normal factorization? Clearly we have that $s_mt_0e_1t_1 \dots e_nt_n$ is a left T-normal factorization. In view of Proposition 3.5 the left T-normal factorization obtained from $s_0f_1s_1 \dots f_ms_mt_0e_1t_1 \dots e_nt_n$ can be obtained by first reducing $1f_ms_mt_0e_1t_1 \dots e_nt_n$, obtaining $1(f_me_1)t_1 \dots e_nt_n$ if $s_mt_0 = 1$; $(s_mt_0)e_1t_1 \dots e_nt_n$ if $f_m(s_mt_0e_1t_1 \dots e_nt_n)^+ = (s_mt_0e_1t_1 \dots e_nt_n)^+$; and $1g_m(s_mt_0)e_1t_1 \dots e_nt_n$, where $g_m = f_m(s_mt_0e_1t_1 \dots e_nt_n)^+$ if $f_m(s_mt_0e_1t_1 \dots e_nt_n)^+ < (s_mt_0e_1t_1 \dots e_nt_n)^+$ and $s_mt_0 \neq 1$. Let $u_1h_1u_2 \dots h_pu_p$ be the factorization achieved. Next take $f_{m-1}(s_{m-1}u_1)h_1u_2 \dots h_pu_p$ and proceed as previously (notice that the factorization $(s_{m-1}u_1)h_1u_2 \dots h_pu_p$ is left T-normal).

Continue in this (finite) manner to obtain a left T-normal factorization.

The goal now is to prove a two-sided version of Proposition 3.1 for Ehresmann monoids. The statement is the following.

Proposition 3.7. *Let M be an Ehresmann monoid such that $M = \langle E \cup T \rangle_{(2)}$. Then any $x \in M$ can be written as*

$$x = t_0e_1t_1 \dots e_nt_n,$$

where $n \in \mathbb{N}_0$, $e_1, \dots, e_n \in E$, $t_1, \dots, t_{n-1} \in T \setminus \{1\}$, $t_0, t_n \in T$ and for $1 \leqslant i \leqslant n$,

$$e_i < (t_ie_{i+1}t_{i+1} \dots e_nt_n)^+$$

and

$$e_i < (t_0e_1t_1 \dots e_{i-1}t_{i-1})^*.$$

A factorization $x = t_0e_1t_1 \dots e_nt_n$ as in this proposition is said to be *T-normal*.

Proof. Let $x \in M$. By Proposition 3.1, x has a left normal factorization (we lose no clarity by dropping the 'T' in this proof)

$$x = t_0e_1t_1 \dots e_nt_n,$$

where $n \geqslant 0$, $e_1, \dots, e_n \in E$, $t_1, \dots, t_{n-1} \in T \setminus \{1\}$, $t_0, t_n \in T$ and for $1 \leqslant i \leqslant n$,

$$e_i < (t_ie_{i+1}t_{i+1} \dots e_nt_n)^+.$$

We now apply (the dual of) Move A to this factorisation. Let $i_1 \in \{1, \dots, n\}$ be such that $\big(t_0e_1t_1 \dots e_{i_1-1}t_{i_1-1}\big)^*e_{i_1} = \big(t_0e_1t_1 \dots e_{i_1-1}t_{i_1-1}\big)^*$. Then

$$x = t_0e_1t_1 \dots e_{i_1-1}(t_{i_1-1}t_{i_1})\, e_{i_1+1}t_{i_1+1} \dots e_nt_n.$$

This factorization of x is weak left normal, since for $i \in \{1, \dots, i_1 - 1\}$, we have $e_i < \big(t_ie_{i+1}t_{i+1} \dots e_{i_1-1}t_{i_1-1}e_{i_1}t_{i_1} \dots e_nt_n\big)^+$ and so, by Lemma 2.1,

$$e_i < \big(t_ie_{i+1}t_{i+1} \dots e_{i_1-1}(t_{i_1-1}t_{i_1})\, e_{i_1+1}t_{i_1+1} \dots e_nt_n\big)^+.$$

Continuing this process of applying (the duals of) Moves A, after a finite number of steps we obtain a weak left normal factorization of x

$$x = s_0 e_{r_1} s_1 \dots e_{r_k} s_k,$$

where $1 \leqslant r_1 < r_2 < \dots < r_k \leqslant n$, $s_0, \dots, s_k \in T$, and, for any $i \in \{1, \dots, k\}$,

$$\big(s_0 e_{r_1} s_1 \dots e_{r_{i-1}} s_{i-1}\big)^* e_{r_i} < \big(s_0 e_{r_1} s_1 \dots e_{r_{i-1}} s_{i-1}\big)^*.$$

Next we use (the duals of) Moves B. We replace in the previous factorization of x each e_{r_i} by

$$f_i = \big(s_0 e_{r_1} s_1 \dots e_{r_{i-1}} s_{i-1}\big)^* e_{r_i},$$

thus obtaining

$$x = s_0 f_1 s_1 \dots f_k s_k.$$

By definition, for any $i \in \{1, \dots, k\}$ we have $f_i < \big(s_0 e_{r_1} s_1 \dots e_{r_{i-1}} s_{i-1}\big)^*$; from the dual of Remark 3.3 we have that $s_0 e_{r_1} s_1 \dots e_{r_{i-1}} s_{i-1} = s_0 f_1 s_1 \dots f_{i-1} s_{i-1}$.

Now we aim to show that $f_i \leqslant \big(s_i f_{i+1} s_{i+1} \dots f_k s_k\big)^+$ for any $i \in \{1, \dots, k\}$. We certainly have $f_k \leqslant e_{r_k} < s_k^+$. Let $i \in \{2, \dots, k\}$ and assume that $f_i \leqslant \big(s_i f_{i+1} s_{i+1} \dots f_k s_k\big)^+$. Set $z = s_0 e_{r_1} s_1 \dots e_{r_{i-2}} s_{i-2}$ $(= s_0 f_1 s_1 \dots f_{i-2} s_{i-2})$. Then

$$\begin{aligned}
\big(s_{i-1} f_i s_i f_{i+1} s_{i+1} \dots f_k s_k\big)^+ &= \big(s_{i-1} f_i (s_i f_{i+1} s_{i+1} \dots f_k s_k)^+\big)^+ \\
&= (s_{i-1} f_i)^+ \\
&= \big(s_{i-1} (z\, e_{r_{i-1}} s_{i-1})^* e_{r_i}\big)^+
\end{aligned}$$

and

$$\begin{aligned}
f_{i-1} &= z^* e_{r_{i-1}} \\
&= z^* e_{r_{i-1}} (s_{i-1}\, e_{r_i})^+ \text{ (by observation after Proposition 3.1)} \\
&= \big(z^* e_{r_{i-1}} s_{i-1}\, e_{r_i}\big)^+ \\
&= z^* e_{r_{i-1}} \big(s_{i-1} (z^* e_{r_{i-1}} s_{i-1}\, e_{r_i})^*\big)^+ \qquad \text{(by Lemma 2.3)} \\
&= z^* e_{r_{i-1}} \big(s_{i-1} (z\, e_{r_{i-1}} s_{i-1}\, e_{r_i})^*\big)^+ \\
&= z^* e_{r_{i-1}} \big(s_{i-1} (z\, e_{r_{i-1}} s_{i-1})^* e_{r_i}\big)^+.
\end{aligned}$$

It follows now

$$f_{i-1} \leqslant \big(s_{i-1} f_i s_i f_{i+1} s_{i+1} \dots f_k s_k\big)^+.$$

Hence $f_i \leqslant \big(s_i f_{i+1} s_{i+1} \dots f_k s_k\big)^+$ for any $i \in \{1, \dots, k\}$.

In view of Remark 3.4 and its dual we can, at this stage, assume that no interior s_i is 1. For, we can remove any that are, and obtain a new factorisation with the same properties.

If for all $i \in \{1, \dots, k\}$, we have $f_i < (s_i f_{i+1} \dots f_k s_k)^+$, then we are done, as the expression of x is a weak normal factorization and, as proved below, it is in fact a normal factorization. Otherwise, let $j_1 \in \{1, \ldots, k\}$ be such that $f_{j_1} = \big(s_{j_1} f_{j_1+1} s_{j_1+1} \ldots f_k s_k\big)^+$. Then

$$f_{j_1} s_{j_1} f_{j_1+1} s_{j_1+1} \ldots f_k s_k = s_{j_1} f_{j_1+1} s_{j_1+1} \ldots f_k s_k$$

and we may eliminate f_{j_1} to get

$$x = s_0 f_1 s_1 \ldots f_{j_1-1}(s_{j_1-1} s_{j_1}) f_{j_1+1} s_{j_1+1} \ldots f_k s_k.$$

Also, for $i \in \{1, \ldots, j_1 - 1\}$,

$$f_i < \big(s_0 f_1 s_1 \ldots f_{i-1} s_{i-1}\big)^*$$

and for $i \in \{j_1 + 1, \ldots, k\}$, by Lemma 2.1,

$$\begin{aligned} f_i &< \big(s_0 f_1 s_1 \ldots f_{j_1-1} s_{j_1-1} f_{j_1} s_{j_1} f_{j_1+1} s_{j_1+1} \ldots f_{i-1} s_{i-1}\big)^* \\ &\leqslant \big(s_0 f_1 s_1 \ldots f_{j_1-1}(s_{j_1-1} s_{j_1}) f_{j_1+1} s_{j_1+1} \ldots f_{i-1} s_{i-1}\big)^*. \end{aligned}$$

Hence the new factorization of x is weak right normal.

Proceeding with this process of applying Moves A, after a finite number of steps we arrive to a weak normal factorization of x

$$x = u_0 f_{\ell_1} u_1 \ldots f_{\ell_p} u_p\,.$$

This is in fact a normal factorization, since if, for some $i \in \{1, \ldots, p-1\}$, we have $u_i = 1$, then

$$\begin{aligned} f_{\ell_i} &< \big(u_i f_{\ell_{i+1}} u_{i+1} \ldots f_{\ell_p} u_p\big)^+ \\ &= \big(f_{\ell_{i+1}} u_{i+1} \ldots f_{\ell_p} u_p\big)^+ \\ &= f_{\ell_{i+1}} \big(u_{i+1} f_{\ell_{i+2}} u_{i+2} \ldots f_{\ell_p} u_p\big)^+ \\ &= f_{\ell_{i+1}} \end{aligned}$$

and

$$\begin{aligned} f_{\ell_{i+1}} &< \big(u_0 f_{\ell_1} u_1 \ldots f_{\ell_{i-1}} u_{i-1} f_{\ell_i} u_i\big)^* \\ &= \big(u_0 f_{\ell_1} u_1 \ldots f_{\ell_{i-1}} u_{i-1} f_{\ell_i}\big)^* \\ &= \big(u_0 f_{\ell_1} u_1 \ldots f_{\ell_{i-1}} u_{i-1}\big)^* f_{\ell_i} \\ &= f_{\ell_i}, \end{aligned}$$

which is a contradiction. Hence $u_i \neq 1$ for every $i \in \{1, \ldots, p-1\}$, and this completes the proof. □

Remark 3.8. It follows from the proof of Proposition 3.7 that a factorization of an element x of an Ehresmann monoid that is both left and right T-normal can be reached by taking any factorization $x = s_0 f_1 s_1 \ldots f_m s_m$, where $s_0, \ldots, s_m \in T$ and $f_1, \ldots, f_m \in E$, applying procedure $(\mathrm{P_R})$, then procedure $(\mathrm{P_L})$ and, finally, procedure $(\mathrm{P_R})$ again.

We say that an Ehresmann (respectively left Ehresmann, right Ehresmann) monoid has *uniqueness of T-normal* (respectively *left T-normal, right T-normal*) *factorizations* if each of its elements has a unique T-normal (respectively left T-normal, right T-normal) factorization.

4 A construction

In this section we show how to construct Ehresmann monoids from actions of monoids on the left and on the right on semilattices. Such constructions are inspired by those of [9, Section 3] for left Ehresmann monoids, which provide a skeleton for our approach. However, our results here for the two-sided case require different and more delicate proofs.

Let T be a monoid with identity 1_T and let Y be a semilattice with identity 1_Y; we assume that $T \cap Y = \emptyset$. Suppose that T acts on the left of Y by order preserving maps. The multiplication of the semilattice also induces a left action of Y on itself by order preserving maps (even by morphisms). By definition of semigroup free product, there exists a semigroup morphism $\phi_\ell \colon T * Y \to \mathcal{O}_Y^{\mathrm{d}}$, i.e a left action of $T * Y$ on Y, that extends both previous left actions of T on Y and of Y on Y. Following [9], given $u \in T * Y$ and $e \in Y$, we denote $(u\phi_\ell)(e)$ by $u \cdot e$ and define u^+ as

$$u^+ = u \cdot 1_Y \in Y.$$

Note. Clearly, $e^+ = e$ for all $e \in Y$, and $(u^+)^+ = u^+$ for all $u \in T * Y$. In addition, given $u \in T * Y$, if w is obtained from u via insertion or deletion of 1_Y's and 1_T's, then $u^+ = w^+$.

Lemma 4.1. *For any $u, v \in T * Y$ and $e \in Y$,*

(a) $(uv)^+ = u \cdot v^+ = (uv^+)^+$, $(eu)^+ = eu^+$, $(uv)^+ \leqslant u^+$ and $(uev)^+ \leqslant (uv)^+$. In particular, $(v^+v)^+ = v^+ = (v^+)^+$.

(b) if $e \leqslant v^+$, then $(uev)^+ = (ue)^+$.

Proof. (a) Given $u, v \in T*Y$, we have $(uv)^+ = (uv)\cdot 1_Y = u\cdot(v\cdot 1_Y) = u\cdot v^+ = u\cdot(v^+\cdot 1_Y) = (uv^+)\cdot 1_Y = (uv^+)^+$. It follows that if $e \in Y$ and $u, v \in T * Y$, then $(eu)^+ = e \cdot u^+ = eu^+$. Moreover, since $v^+ \leqslant 1_Y$, the fact that the left action of $T * Y$ on Y is via order preserving maps ensures that $u \cdot v^+ \leqslant u \cdot 1_Y$, which is equivalent to $(uv)^+ \leqslant u^+$. Also as $ev^+ \leqslant v^+$, we have $u \cdot (ev^+) \leqslant u \cdot v^+$, whence $(uev)^+ \leqslant (uv)^+$.

(b) If $u, v \in T * Y$ and $e \in Y$ are such that $e \leqslant v^+$, then, by (a), we obtain $(uev)^+ = (uev^+)^+ = (ue)^+$. □

Now, dually, suppose that T also acts on the right of Y by order preserving maps, which means that there is a monoid morphism $\phi_r \colon T \to \mathcal{O}_Y$. As for the left action of T on Y, this morphism can also be naturally extended to a semigroup morphism $\phi_r \colon T * Y \to \mathcal{O}_Y$, such that its restriction to Y is the monoid morphism induced by the multiplication in Y. For $u \in T * Y$ and $e \in Y$, we represent $e(u\phi_r)$ by $e \circ u$. We now define, for $u \in T * Y$,

$$u^* = 1_Y \circ u.$$

The above considerations for the left action of $T * Y$ on Y have their analogue for the right action of $T * Y$ on Y. Thus, $e^* = e$ for all $e \in Y$. Also, for any $u \in T * Y$, $(u^*)^* = u^*$ and if w is obtained from u via insertion or deletion of 1_Y's and 1_T's, then $u^* = w^*$. In particular, for any $u \in T * Y$,

$$(1_Y u)^+ = u^+ = (u1_Y)^+ \qquad \text{and} \qquad (1_Y u)^* = u^* = (u1_Y)^*.$$

The following lemma is the dual of Lemma 4.1.

Lemma 4.2. *For any $u, v \in T * Y$ and $e \in Y$,*

(a) $(uv)^* = u^* \circ v = (u^*v)^*$, $(ue)^* = u^*e$, $(uv)^* \leqslant v^*$ *and* $(uev)^* \leqslant (uv)^*$. *In particular,* $(vv^*)^* = v^* = (v^*)^*$.

(b) if $e \leqslant u^*$, *then* $(uev)^* = (ev)^*$.

We observe that from Lemmas 4.1(a) and 4.2(a), for any $u \in T * Y$ and $e \in Y$ we have

$$u \cdot e = (ue)^+ \quad \text{and} \quad e \circ u = (eu)^*.$$

By Remark 2.4 and its dual, consider the following two conditions connecting the left and the right actions of $T * Y$ on Y: for any $t \in T$, $e, f \in Y$,

$$e(t \cdot f) = e\big(t \cdot ((e \circ t)f)\big) \tag{CC1}$$

and

$$(e \circ t)f = \big((e(t \cdot f)) \circ t\big)f. \tag{CC2}$$

We refer to these as the *compatibility conditions* and observe that they generalize the weak compatibility conditions of [9]. The above observation together with Lemmas 4.1(a) and 4.2(a) imply that the conditions (CC1) and (CC2) can be rewritten in terms of the unary operations $^+$ and * as follows, respectively:

$$(etf)^+ = e\big(t(etf)^*\big)^+ \tag{CC1}$$

and

$$(etf)^* = \big((etf)^+t\big)^*f. \tag{CC2}$$

Although the semigroup $T * Y$ is not an Ehresmann monoid with distinguished semilattice Y, we borrow from Section 3 the following terminology relative to $T * Y$. We say that an element of $T * Y$

$$u = t_0e_1t_1 \dots e_nt_n,$$

where $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in Y$, is

- *left normal* if $t_i \neq 1_T$ for any $i \in \{1, \dots, n-1\}$, and $e_i < (t_ie_{i+1}t_{i+1} \dots e_nt_n)^+$ for any $i \in \{1, \dots, n\}$.
- *right normal* if $t_i \neq 1_T$ for any $i \in \{1, \dots, n-1\}$, and $e_i < (t_0e_1t_1 \dots e_{i-1}t_{i-1})^*$ for any $i \in \{1, \dots, n\}$.
- *normal* if it is both left normal and right normal.

If in any of these definitions we do not require that $t_i \neq 1_T$ for any $i \in \{1, \dots, n-1\}$, the element u is said to be *weak left normal*, *weak right normal* and *weak normal*, respectively. We denote by $\mathcal{N}_\ell$ the set of left normal elements of $T * Y$ and by $\mathcal{N}_r$ the set of right normal elements of $T * Y$. Thus $\mathcal{N}_\ell \cap \mathcal{N}_r$ is the set of normal elements of $T * Y$, which we denote by $\mathcal{N}$. Recall that if u is weak left normal or weak right normal, then $e_i \neq 1_Y$ for every $i \in \{1, \ldots, n\}$. We also say that an element of $T * Y$

$$u = t_0 e_1 t_1 \ldots e_n t_n,$$

where $t_0, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in Y$, is

- *quasi left normal* if $e_i \neq 1_Y$ and $e_i \leqslant (t_i e_{i+1} t_{i+1} \ldots e_n t_n)^+$ for any $i \in \{1, \ldots, n\}$.
- *quasi right normal* if $e_i \neq 1_Y$ and $e_i \leqslant (t_0 e_1 t_1 \ldots e_{i-1} t_{i-1})^*$ for any $i \in \{1, \ldots, n\}$.

We denote by $\mathcal{N}'_\ell$ the set of quasi left normal elements of $T * Y$ and by $\mathcal{N}'_r$ the set of quasi right normal elements of $T * Y$.

Lemma 4.3. *(a) If the left and right actions of T on Y satisfy (CC1), then, for any $u \in \mathcal{N}'_r$, $v \in T * Y$, we have $(uv)^+ = (uu^*v)^+$. In particular, $u^+ = (uu^*)^+$.*

(b) If the left and right actions of T on Y satisfy (CC2), then, for any $u \in T * Y$, $v \in \mathcal{N}'_\ell$, we have $(uv)^* = (uv^+v)^*$. In particular, $v^* = (v^+v)^*$.

Proof. (a) Suppose that the left and right actions of T on Y satisfy (CC1). Let $u \in \mathcal{N}'_r$ and $v \in T * Y$. If $u \in T$, then

$$\begin{aligned}
(uv)^+ &= (uv^+)^+ && \text{(by Lemma 4.1(a))}\\
&= (1_Y uv^+)^+ \\
&= 1_Y\big(u(1_Y uv^+)^*\big)^+ && \text{(by (CC1) as } u \in T)\\
&= \big(u(uv^+)^*\big)^+ \\
&= (uu^*v^+)^+ && \text{(by Lemma 4.2(a))}\\
&= (uu^*v)^+ && \text{(by Lemma 4.1(a)).}
\end{aligned}$$

Suppose that $u \notin T$. Then $u = t_0 e_1 t_1 \ldots e_n t_n$, where $n \in \mathbb{N}$, $t_0, \ldots, t_n \in T$, $e_1, \ldots, e_n \in Y \setminus \{1_Y\}$ and $e_n \leqslant (t_0 e_1 t_1 \ldots e_{n-1} t_{n-1})^*$. It follows that

$$\begin{aligned}
(uv)^+ &= (t_0 e_1 t_1 \ldots e_{n-1} t_{n-1} e_n t_n v^+)^+ && \text{(by Lemma 4.1(a))}\\
&= \big(t_0 e_1 t_1 \ldots e_{n-1} t_{n-1} (e_n t_n v^+)^+\big)^+ && \text{(by Lemma 4.1(a))}\\
&= \big(t_0 e_1 t_1 \ldots e_{n-1} t_{n-1} e_n (t_n (e_n t_n v^+)^*)^+\big)^+ && \text{(by (CC1)) as } t_n \in T\\
&= \big(t_0 e_1 t_1 \ldots e_{n-1} t_{n-1} e_n t_n (e_n t_n v^+)^*\big)^+ && \text{(by Lemma 4.1(a))}\\
&= \big(u(e_n t_n v^+)^*\big)^+ \\
&= \big(u(e_n t_n)^* v^+\big)^+ && \text{(by Lemma 4.2(a))}\\
&= \big(u((t_0 e_1 t_1 \ldots e_{n-1} t_{n-1})^* e_n t_n)^* v^+\big)^+ \\
&= \big(u(t_0 e_1 t_1 \ldots e_{n-1} t_{n-1} e_n t_n)^* v^+\big)^+ && \text{(by Lemma 4.2(a))}\\
&= \big(uu^*v^+\big)^+ \\
&= \big(uu^*v\big)^+ && \text{(by Lemma 4.1(a)).}
\end{aligned}$$

(b) It is the dual of (a). $\square$

For any set X we denote by $\mathcal{T}(X)$ the full transformation monoid on X. Let us recall the definition of the semigroup morphism $\psi\colon T*Y\to\mathcal{T}^{\mathrm{d}}(\mathcal{N}_\ell)$ of [9, Section 2], that we denote here by ψ_ℓ. This morphism mimics the process described in Remark 3.6 for a left Ehresmann monoid. It is defined as the semigroup morphism $\psi_\ell\colon T*Y\to\mathcal{T}^{\mathrm{d}}(\mathcal{N}_\ell)$ whose restrictions to T and E are defined as follows:

– for all $t\in T$ and $t_0e_1t_1\dots e_nt_n\in\mathcal{N}_\ell$, where $t_0,\dots,t_n\in T$ and $e_1,\dots,e_n\in E$,

$$(t\psi_\ell)(t_0e_1t_1\dots e_nt_n)=(tt_0)e_1t_1\dots e_nt_n;$$

– $(1_Y)\psi_\ell$ is the identity map of $\mathcal{N}_\ell$;
if $e\in Y\setminus\{1_Y\}$, then $(e\psi_\ell)(1_T)=1_Te1_T$;
and for any $v=t_0e_1t_1\dots e_nt_n\in\mathcal{N}_\ell$, where $t_0,\dots,t_n\in T$ and $e_1,\dots,e_n\in E$,

$$(e\psi_\ell)(v)=\begin{cases}1_T(ee_1)t_1e_2t_2\dots e_nt_n & \text{if } t_0=1_T\\ v=\ t_0e_1t_1\dots e_nt_n & \text{if } t_0\neq 1_T \text{ and } v^+=v^+e\\ 1_T(ev^+)t_0e_1t_1\dots e_nt_n & \text{if } t_0\neq 1_T \text{ and } v^+e<v^+.\end{cases}$$

We note that it follows from the definition that $(1_T)\psi_\ell$ is the identity map of $\mathcal{T}(\mathcal{N}_r)$. Let $\psi_r\colon T*Y\to\mathcal{T}(\mathcal{N}_r)$ be the dual of ψ_ℓ.

Lemma 4.4. *For any $u\in T*Y$ and $v\in\mathcal{N}_\ell$,*

(a) $\big((u\psi_\ell)(v)\big)^+=(uv)^+$; and

(b) if the left and right actions of T on Y satisfy (CC2), then $\big((u\psi_\ell)(v)\big)^*=(uv)^*$.

Proof. Both (a) and (b) will be proved by induction on the length of u.

Let $u\in T\cup Y$ and $v\in\mathcal{N}_\ell$. If $u\in T$, then $(u\psi_\ell)(v)=uv$, whence $\big((u\psi_\ell)(v)\big)^+=(uv)^+$ and $\big((u\psi_\ell)(v)\big)^*=(uv)^*$. If $u=1_Y$, then $(u\psi_\ell)(v)=v$, whence $\big((u\psi_\ell)(v)\big)^+=v^+=(uv)^+$ and $\big((u\psi_\ell)(v)\big)^*=v^*=(uv)^*$. Suppose now that $u\in Y\setminus\{1_Y\}$. If $v=1_T$, then $(u\psi_\ell)(v)=1_Tu1_T=1_Tuv$, whence $\big((u\psi_\ell)(v)\big)^+=(1_Tuv)^+=(uv)^+$ and $\big((u\psi_\ell)(v)\big)^*=(1_Tuv)^*=(uv)^*$. Suppose that $v\neq 1_T$. Put $v=t_0e_1t_1\dots e_nt_n$, where $n\in\mathbb{N}_0$, $t_0,\dots,t_n\in T$ and $e_1,\dots,e_n\in Y$. In view of the definition of ψ_ℓ, we will analyze three cases.

<u>Case 1</u>. $t_0=1_T$. Then $(u\psi_\ell)(v)=t_0ue_1t_1\dots e_nt_n$, and therefore $\big((u\psi_\ell)(v)\big)^+=(ue_1t_1\dots e_nt_n)^+=(ut_0e_1t_1\dots e_nt_n)^+=(uv)^+$ and similarly $\big((u\psi_\ell)(v)\big)^*=(uv)^*$.

<u>Case 2</u>. $t_0\neq 1_T$ and $uv^+=v^+$. Then $(u\psi_\ell)(v)=v$. Thus by Lemma 4.1(a) we have $\big((u\psi_\ell)(v)\big)^+=v^+=uv^+=(uv)^+$. If (CC2) holds, by Lemma 4.3(b) we have $\big((u\psi_\ell)(v)\big)^*=v^*=(v^+v)^*=(uv^+v)^*=(uv)^*$.

<u>Case 3</u>. $t_0\neq 1_T$ and $uv^+<v^+$. Then $(u\psi_\ell)(v)=1_Tuv^+t_0e_1t_1\dots e_nt_n=1_Tuv^+v$. Therefore by Lemma 4.1(a) we have $\big((u\psi_\ell)(v)\big)^+=(1_Tuv^+v)^+=(1_Tuv^+v^+)^+=(uv^+)^+=(uv)^+$ and under Condition (CC2) by Lemma 4.3(b) we have $\big((u\psi_\ell)(v)\big)^*=(1_Tuv^+v)^*=(uv^+v)^*=(uv)^*$.

Now we will complete the proofs of (a) and (b) by induction on the length of u.

(a) Let $k \in \mathbb{N}$ and assume, as induction hypothesis, that $\big((u\psi_\ell)(v)\big)^+ = (uv)^+$ for any $u \in T * Y$ of length k and any $v \in \mathcal{N}_\ell$. Let $u \in T * Y$ of length $k+1$ and let $v \in \mathcal{N}_\ell$. Then $u = wz$, where $w \in T * Y$ and $z \in T \cup Y$ such that w has length k. It follows that

$$\begin{aligned}
\big((u\psi_\ell)(v)\big)^+ &= \big((w\psi_\ell)\big((z\psi_\ell)(v)\big)\big)^+ \\
&= \big(w\big((z\psi_\ell)(v)\big)\big)^+ && \text{(by induction hypothesis as } (z\psi_\ell)(v) \in \mathcal{N}_\ell) \\
&= \big(w\big((z\psi_\ell)(v)\big)^+\big)^+ && \text{(by Lemma 4.1(a))} \\
&= \big(w(zv)^+\big)^+ && \text{(as proved before since } z \in T \cup Y) \\
&= (wzv)^+ && \text{(by Lemma 4.1(a))} \\
&= (uv)^+.
\end{aligned}$$

(b) Assume that (CC2) holds. Let $k \in \mathbb{N}$ and suppose, as induction hypothesis, that $\big((u\psi_\ell)(v)\big)^* = (uv)^*$ for any $u \in T * Y$ of length k and any $v \in \mathcal{N}_\ell$. Let $u \in T * Y$ of length $k+1$ and let $v \in \mathcal{N}_\ell$. We have $u = wz$, where $w \in T * Y$ and $z \in T \cup Y$ such that w has length k. Then $\big((u\psi_\ell)(v)\big)^* = \big((w\psi_\ell)\big((z\psi_\ell)(v)\big)\big)^* = \big(w\big((z\psi_\ell)(v)\big)\big)^*$. If $z \in T$, then $(z\psi_\ell)(v) = zv$, and therefore $\big((u\psi_\ell)(v)\big)^* = (wzv)^* = (uv)^*$. If $z = 1_Y$, then $(z\psi_\ell)(v) = v$, whence $\big((u\psi_\ell)(v)\big)^* = (wv)^* = (wzv)^* = (uv)^*$. Now suppose that $z \in Y \setminus \{1_Y\}$. In the case that $v = 1_T$, we have $(z\psi_\ell)(v) = 1_T z 1_T = 1_T zv$, and so $\big((u\psi_\ell)(v)\big)^* = (w1_T zv)^* = (wzv)^* = (uv)^*$. Suppose that $v \neq 1_T$. Put $v = t_0e_1t_1 \dots e_nt_n$, where $n \in \mathbb{N}_0$, $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in Y$. Once more, in view of the definition of ψ_ℓ, let us consider three cases.

<u>*Case 1*</u>. $t_0 = 1_T$. Then $(z\psi_\ell)(v) = t_0ze_1t_1 \dots e_nt_n$, whence $\big((u\psi_\ell)(v)\big)^* = (wt_0ze_1t_1 \dots e_nt_n)^* = (wzt_0e_1t_1 \dots e_nt_n)^* = (uv)^*$.

<u>*Case 2*</u>. $t_0 \neq 1_T$ and $zv^+ = v^+$. Then $(z\psi_\ell)(v) = v$. Taking into account Lemma 4.3(b) we have $\big((u\psi_\ell)(v)\big)^* = (wv)^* = (wv^+v)^* = (wzv^+v)^* = (wzv)^* = (uv)^*$.

<u>*Case 3*</u>. $t_0 \neq 1_T$ and $zv^+ < v^+$. Then $(z\psi_\ell)(v) = 1_T zv^+ t_0e_1t_1 \dots e_nt_n = 1_T zv^+v$. Again Lemma 4.3(b) guarantees that $\big((u\psi_\ell)(v)\big)^* = (w1_T zv^+v)^* = (wzv^+v)^* = (uv^+v)^* = (uv)^*$, which completes the proof. □

The next result may be found in the proof of Lemma 2.5 of [9], however, we prove it here for completeness.

Lemma 4.5. *If $e \in Y$ and $u \in \mathcal{N}_\ell$ are such that $u^+ \leqslant e$, then $(e\psi_\ell)(u) = u$.*

Proof. Let $e \in Y$ and $u \in \mathcal{N}_\ell$ such that $u^+ \leqslant e$. Clearly the result is true if $e = 1_Y$. We suppose therefore that $e < 1_Y$. Observe that since $u^+ \leqslant e$ and $1_T^+ = 1_T \cdot 1_Y = 1_Y$ it follows that $u \neq 1_T$. Set $u = t_0e_1t_1 \dots e_nt_n$, where $n \in \mathbb{N}_0$, $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in Y$. Let us consider two cases.

Case 1. $t_0 = 1_T$. Then $(e\psi_\ell)(u) = 1_T e e_1 t_1 e_2 t_2 \dots e_n t_n$. As $u \in \mathcal{N}_\ell$, we have $e_1 \leqslant (t_1 e_2 t_2 \dots e_n t_n)^+$ and it follows that

$$\begin{aligned} e_1 &= e_1(t_1 e_2 t_2 \dots e_n t_n)^+ \\ &= (e_1 t_1 e_2 t_2 \dots e_n t_n)^+ && \text{(by Lemma 4.1(a))} \\ &= (1_T e_1 t_1 e_2 t_2 \dots e_n t_n)^+ \\ &= u^+. \end{aligned}$$

Then $ee_1 = eu^+ = u^+ = e_1$, and hence $(e\psi_\ell)(u) = u$.

Case 2. $t_0 \neq 1_T$. Since $eu^+ = u^+$, the definition of ψ_ℓ assures that $(e\psi_\ell)(u) = u$. $\square$

The next result is Lemma 2.5 of [9], but we opt to present here a much simplified proof. Let

$$H_\ell = \big\{(u^+u, u)\colon\ u \in T * Y\big\} \cup \{(1_T, 1_Y)\}$$

and

$$H_r = \big\{(uu^*, u)\colon\ u \in T * Y\big\} \cup \{(1_T, 1_Y)\}.$$

Let $\sim_\ell$, $\sim_r$ and $\sim$ be the congruences on $T*Y$ generated by H_ℓ, H_r and $H_\ell \cup H_r$, respectively.

Lemma 4.6. *We have $\sim_\ell \subseteq \ker\psi_\ell$.*

Proof. It suffices to show that $(u^+u)\psi_\ell = u\psi_\ell$ for any $u \in T * Y$. Let $u \in T * Y$ and let $v \in \mathcal{N}_\ell$. Then $\big((u^+u)\psi_\ell\big)(v) = (u^+\psi_\ell)\big((u\psi_\ell)(v)\big)$. By Lemmas 4.4(a) and 4.1(a), we have $\big((u\psi_\ell)(v)\big)^+ = (uv)^+ \leqslant u^+$. Lemma 4.5 now implies that $(u^+\psi_\ell)\big((u\psi_\ell)(v)\big) = (u\psi_\ell)(v)$. Therefore $\big((u^+u)\psi_\ell\big)(v) = (u\psi_\ell)(v)$ for any $v \in \mathcal{N}_\ell$, and hence $(u^+u)\psi_\ell = u\psi_\ell$. $\square$

It was proved in [9, Section 2] that for each $u \in T * Y$, there exists a unique $v \in \mathcal{N}_\ell$ such that $u \sim_\ell v$. By [9, Lemmas 2.5 and 2.6], we have $v = (u\psi_\ell)(1_T)$. Thus the morphism ψ_ℓ simulates a process of "left normalize" a word. Moreover, we have the following result.

Lemma 4.7. *Let $u \in T * Y$, $v \in \mathcal{N}_\ell$, $v', v'' \in T * Y$ and $t \in T$ such that $v = v'tv''$. Then $tv'' \in \mathcal{N}_\ell$ and $(u\psi_\ell)(v) = \big((uv')\psi_\ell\big)(tv'') = \big((uv)\psi_\ell\big)(1_T)$.*

Proof. The fact that $tv'' \in \mathcal{N}_\ell$ is clear. By [9, Lemma 2.6], we have $tv'' = \big((tv'')\psi_\ell\big)(1_T)$, and since ψ_ℓ is a morphism, $\big((uv)\psi_\ell\big)(1_T) = \big((uv')\psi_\ell\big)\Big(\big((tv'')\psi_\ell\big)(1_T)\Big) = \big((uv')\psi_\ell\big)(tv'')$. The remaining equality is a particular case of the other one. $\square$

In order to simplify notation, define maps

$$\begin{array}{rccc} \eta_\ell\colon & T * Y & \longrightarrow & \mathcal{N}_\ell \\ & u & \longmapsto & (u\psi_\ell)(1_T) \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \eta_r\colon & T * Y & \longrightarrow & \mathcal{N}_r \\ & u & \longmapsto & (1_T)(u\psi_r) \end{array}$$

Thus $u\eta_\ell$ ($u\eta_r$) are, respectively, the unique elements of $\mathcal{N}_\ell$ ($\mathcal{N}_r$) in the $\sim_\ell$ ($\sim_r$)-class of u. We observe that $\ker\psi_\ell \subseteq \ker\eta_\ell$, whence, in view of Lemma 4.6, we have $\sim_\ell \subseteq \ker\psi_\ell \subseteq \ker\eta_\ell$. This assures that, given $v \in T * Y$ and $u, w \in (T * Y)^1$, since $v \sim_\ell v\eta_\ell$, then

$$(uv^+vw)\eta_\ell = (uvw)\eta_\ell = \big(u(v\eta_\ell)w\big)\eta_\ell.$$

Dually, we have

$$(uvv^*w)\eta_r = (uvw)\eta_r = \big(u(v\eta_r)w\big)\eta_r.$$

The next result strengthens Lemma 4.6 though we will not use it in the sequel.

Proposition 4.8. *We have* $\sim_\ell = \ker\psi_\ell = \ker\eta_\ell$.

Proof. By Lemma 4.6 and the observation above, we only have to prove that $\ker\eta_\ell \subseteq \sim_\ell$. If $u, u' \in T * Y$ are such that $u\eta_\ell = u'\eta_\ell$, then $u \sim_\ell u\eta_\ell = u'\eta_\ell \sim_\ell u'$, whence $u \sim_\ell u'$, as desired. □

For any $u \in T*Y$, the element $u\eta_\ell\eta_r\eta_\ell$ is $\sim$-equivalent to u and we will see that $u\eta_\ell\eta_r\eta_\ell$ is normal.

Lemma 4.9. *Suppose that the left and right actions of T on Y satisfy (CC1) (resp. (CC2)). For any $u, v \in T * Y$, if $u \sim v$, then $u^+ = v^+$ (resp. $u^* = v^*$).*

Proof. Suppose that the left and right actions of T on Y satisfy (CC1). Let us show that for any $u, v \in T*Y$, if $u \sim_\ell v$ or $u \sim_r v$, then $u^+ = v^+$. If $u \sim_\ell v$, then, by Proposition 4.8, $u\eta_\ell = v\eta_\ell$, whence Lemma 4.4(a) implies that $u^+ = v^+$. If $u \sim_r v$, then, by the dual of Proposition 4.8, $u\eta_r = v\eta_r$, whence the dual of Lemma 4.4(b) gives $u^+ = v^+$. Since $\sim$ is the semigroup congruence on $T * Y$ generated by $H_\ell \cup H_r$, we have that $\sim$ is the equivalence relation generated by $\sim_\ell \cup \sim_r$, and hence if $u \sim v$, then $u^+ = v^+$. The other part is dual. □

Following the notation for $\mathcal{P}_\ell(T, Y)$ of [9], given in Section 3, we define

$$\mathcal{P} = \mathcal{P}(T * Y) = (T * Y)/{\sim}$$

and let $\nu\colon T * Y \to \mathcal{P}$ be the natural morphism associated with $\sim$. With the assumption that the left and right actions of T on Y satisfy (CC1), Lemma 4.9 allows us to define a unary operation $^+$ on $\mathcal{P}$ by $[u]_\sim^+ = [u^+]_\sim$ for any $u \in T * Y$. Dually, if (CC2) is satisfied, we can define a unary operation * by $[u]_\sim^* = [u^*]_\sim$.

Set

$$T' = \big\{[t]_\sim \colon t \in T\big\} \quad\text{and}\quad Y' = \big\{[e]_\sim \colon e \in Y\big\}.$$

Lemma 4.10. *If the left and right actions of T on Y satisfy (CC1) (resp. (CC2)), then the semigroup $\mathcal{P}$ is a left Ehresmann (resp. right Ehresmann) monoid with distinguished semilattice Y'.*

Proof. Since T and Y are monoids and $1_T \sim 1_Y$, the semigroup $\mathcal{P}$ is in fact a monoid. Clearly, Y' is a semilattice in $\mathcal{P}$. Assume that the left and right actions of T on Y satisfy (CC1). Let $u \in T * Y$. The fact that $u \sim u^+u$ gives immediately $[u]_\sim = [u]_\sim^+[u]_\sim$. Let $e \in Y$ such that $[u]_\sim = [e]_\sim[u]_\sim$. Then $u \sim eu$, whence, by Lemmas 4.9 and 4.1, $u^+ = (eu)^+ = eu^+$, and therefore $[u]_\sim^+ = [e]_\sim[u]_\sim^+$. Hence $[u]_\sim \widetilde{\mathcal{R}}_{Y'} [u]_\sim^+$.

Let us see that $\widetilde{\mathcal{R}}_{Y'}$ is a left congruence. Let $u, v, w \in T * Y$ be such that $[u]_\sim \widetilde{\mathcal{R}}_{Y'} [v]_\sim$. Then $[u]_\sim^+ = [v]_\sim^+$, and therefore $u^+ = v^+$ by Lemma 4.9. It follows, in view of Lemma 4.1, that $(wu)^+ = w \cdot u^+ = w \cdot v^+ = (wv)^+$, so $\big([w]_\sim [u]_\sim\big)^+ = \big([w]_\sim [v]_\sim\big)^+$, which is equivalent to $[w]_\sim [u]_\sim \widetilde{\mathcal{R}}_{Y'} [w]_\sim [v]_\sim$. Hence $\mathcal{P}$ is a left Ehresmann monoid.

Dually $\mathcal{P}$ is a right Ehresmann monoid under the Condition (CC2). □

Remark 4.11. Lemma 4.10 can also be obtained using the following facts: $(T * Y)/{\sim_\ell}$ is a left Ehresmann monoid [9]; and under Condition (CC1) the semigroup $(T * Y)/{\sim}$ is a $(2, 1, 0)$-algebra, which is a $(2, 1, 0)$-quotient of the left Ehresmann monoid $(T * Y)/{\sim_\ell}$. The dual result follows similarly.

Define $\tau \colon T * Y \to T$ as being the unique morphism from $T * Y$ to T that extends the morphisms $\mathrm{id}_T \colon T \to T$ and $Y \to T$, $e \mapsto 1_T$.

Lemma 4.12. *We have $\sim \subseteq \ker \tau$, and, for any $u \in T * Y$,*

(a) $\big((u\psi_\ell)(v)\big)\tau = (uv)\tau$, for all $v \in \mathcal{N}_\ell$.

(b) $\big((v)(u\psi_r)\big)\tau = (vu)\tau$, for all $v \in \mathcal{N}_r$.

Proof. Since $u^+\tau = 1_T = u^*\tau$ for all $u \in T * Y$, it is clear that $H_\ell \cup H_r \subseteq \ker \tau$, whence $\sim \subseteq \ker \tau$.

(a) Let $u \in T * Y$ and $v \in \mathcal{N}_\ell$. By Lemma 4.7, $(u\psi_\ell)(v) = \big((uv)\psi_\ell\big)(1_T) = (uv)\eta_\ell \sim_\ell uv$. Therefore $\big((u\psi_\ell)(v)\big)\tau = (uv)\tau$.

(b) It is dual to (a). □

Lemma 4.13. *The morphism $\nu \colon T * Y \to \mathcal{P}$ is injective on T. If the left and right actions of T on Y satisfy (CC1) or (CC2), then ν is also injective on Y.*

Proof. The injectivity of ν on T comes from the inclusion $\sim \subseteq \ker \tau$ in Lemma 4.12. The other part follows from Lemma 4.9. □

Corollary 4.14. *If T is unipotent and has no units other than 1_T, then the set of idempotents of $\mathcal{P}$ is $Y' = \big\{[e]_\sim \colon e \in Y\big\}$.*

Proof. Let f be an idempotent of $\mathcal{P}$. Then $f = [u]_\sim$, where $u \in T * Y$ is such that $u = t_0 e_1 t_1 \dots e_n t_n$, with $t_0, t_1, \dots, t_n \in T$ and $e_1, \dots, e_n \in Y$. From Lemma 4.12, $\sim \subseteq \ker \tau$, whence $t_0 t_1 \dots t_n = (t_0 t_1 \dots t_n)^2$. Since T is unipotent, $t_0 t_1 \dots t_n = 1_T$, and, as T has trivial group of units, $t_0 = t_1 = \dots = t_n = 1_T$. Hence $f = [u]_\sim = [t_0 e_1 t_1 \dots e_n t_n]_\sim = [e_1 \dots e_n]_\sim \in Y'$, as required. □

Lemma 4.13 shows that T' is isomorphic to T and, under Condition (CC1) or Condition (CC2), Y' is isomorphic to Y, both via the natural morphism ν. Moreover, $\mathcal{P} = \langle T' \cup Y' \rangle_{(2)}$ since $T * Y = \langle T \cup Y \rangle_{(2)}$.

Let $\widehat{\sigma}_Y$ be the congruence on $T * Y$ generated by $(Y \times Y) \cup \{(1_T, 1_Y)\}$ and let $\tau' \colon \mathcal{P} \to T'$, $[u]_\sim \mapsto [u\tau]_\sim$; this map is well defined, since $\sim \subseteq \ker \tau$ by Lemma 4.12. Clearly τ' is a surjective monoid morphism. Moreover, regarding T as a reduced left (right, two-sided) Ehresmann monoid, τ' preserves $^+$ and * whenever these are defined on $\mathcal{P}$.

Lemma 4.15. *We have* $\widehat{\sigma}_Y = \ker\tau$ *and* $\sigma = \ker\tau' = \widehat{\sigma}_Y/\sim$, *and consequently* $T' \simeq (T*Y)/\widehat{\sigma}_Y \simeq \mathcal{P}/\sigma$.

Proof. We have $\widehat{\sigma}_Y \subseteq \ker\tau$ since $\ker\tau$ is a congruence on $T*Y$ containing $(Y\times Y)\cup\{(1_T,1_Y)\}$. To prove the opposite inclusion, it suffices to show that $u\ \widehat{\sigma}_Y\ u\tau$ for any $u\in T*Y$. For that we need consider only the elements $u\in T(T*Y)T$, since $u\ \widehat{\sigma}_Y\ 1_Tu1_T$ and $u\tau = (1_Tu1_T)\tau$. Thus, let $u = t_0e_1t_1\ldots e_nt_n$, where $n\in\mathbb{N}_0$, $t_0,\ldots,t_n\in T$ and $e_1,\ldots,e_n\in Y$. Then

$$u = t_0e_1t_1\ldots e_nt_n\ \widehat{\sigma}_Y\ t_01_Tt_1\ldots 1_Tt_n = t_0t_1\ldots t_n = u\tau.$$

Hence $\widehat{\sigma}_Y = \ker\tau$.

As $\ker\tau'$ is a congruence on $\mathcal{P}$ containing $Y'\times Y'$, we have $\sigma\subseteq\ker\tau'$. Similarly to the above, $[u]_\sim\ \sigma\ [u\tau]_\sim$ for every $u\in T*Y$, which gives the inclusion $\ker\tau'\subseteq\sigma$.

The third equality follows easily from Lemma 4.13 and the first equality: for every $u,v\in T*Y$,

$$\begin{aligned}([u]_\sim,[v]_\sim)\in\ker\tau' &\iff [u\tau]_\sim = [v\tau]_\sim\\ &\iff u\tau = v\tau\\ &\iff u\ \widehat{\sigma}_Y\ v.\end{aligned}$$

□

Corollary 4.16. *If the left and right actions of T on Y satisfy (CC1) (resp. (CC2)), then the semigroup $\mathcal{P}$ is a T'-generated strongly T'-proper left Ehresmann (resp. right Ehresmann) monoid.*

Proof. This is immediate from Lemmas 4.13 and 4.15. □

Lemma 4.17. *If the left action (resp. right action) of T on Y is by morphisms and the left and right actions of T on Y satisfy (CC1) (resp. (CC2)), then in $\mathcal{P}$ the actions of T' on Y' are also by morphisms and so $\mathcal{P}$ is left (right) hedged.*

Proof. Suppose that the left action of T on Y is by morphisms and (CC1) holds. Let $t\in T$ and $e,f\in Y$. We have $t\cdot(ef) = (t\cdot e)(t\cdot f)$, which means $(tef)^+ = (te)^+(tf)^+$. Then

$$\begin{aligned}[t]_\sim\cdot\big([e]_\sim[f]_\sim\big) &= [t]_\sim\cdot[ef]_\sim = \big([t]_\sim[ef]_\sim\big)^+ = [tef]_\sim^+\\ &= [(tef)^+]_\sim = [(te)^+(tf)^+]_\sim = [(te)^+]_\sim[(tf)^+]_\sim\\ &= [te]_\sim^+[tf]_\sim^+ = \big([t]_\sim[e]_\sim\big)^+\big([t]_\sim[f]_\sim\big)^+\\ &= \big([t]_\sim\cdot[e]_\sim\big)\big([t]_\sim\cdot[f]_\sim\big).\end{aligned}$$

The remaining part is dual. □

At this point we summarize the main achievements, obtained so far, about the quotient monoid $\mathcal{P}$.

Theorem 4.18. *Let T be a monoid acting on both sides on a semilattice Y by order preserving maps. Suppose that the left and right actions of T on Y satisfy (CC1) and (CC2). Then, the quotient $\mathcal{P} = (T * Y)/\!\sim$ is an Ehresmann monoid with semilattice of projections*

$$Y' = \big\{[e]_\sim \colon e \in Y\big\}$$

and $[u]^+_\sim = [u^+]_\sim$ and $[u]^*_\sim = [u^*]_\sim$, for any $u \in T * Y$, and $1_\mathcal{P} = [1_T]_\sim = [1_Y]_\sim$.

Moreover, Y' is isomorphic to Y and the submonoid $T' = \big\{[t]_\sim \colon t \in T\big\}$ of $\mathcal{P}$ is isomorphic to T under the natural morphism $\nu\colon T * Y \to \mathcal{P}$, $u \mapsto [u]_\sim$. Further, $\mathcal{P}$ is T'-generated and:

(a) $\mathcal{P}/\sigma_{Y'} \simeq T'$;

(b) $\mathcal{P}$ is strongly T'-proper;

(c) if the left and the right actions of T on Y are by morphisms, then $\mathcal{P}$ is hedged.

We may relate one sided normal factorizations in $\mathcal{P}$ to those in $\mathcal{P}_\ell$, and dually, to those in $\mathcal{P}_r = \mathcal{P}_r(T, Y) = T * Y/\sim_r$, as follows.

Proposition 4.19. *Let $t_0, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in Y$. If the left and right actions of T on Y satisfy (CC1), then for any $i \in \{1, \ldots, n\}$,*

$$[e_i]_\sim < \big([t]_\sim[e_{i+1}]_\sim \ldots [e_n]_\sim[t_n]_\sim\big)^+$$

in $\mathcal{P}$ if and only if

$$e_i < \big(t_i e_{i+1} \ldots e_n t_n\big)^+$$

if and only if

$$[e_i]_{\sim_\ell} < \big([t]_{\sim_\ell}[e_{i+1}]_{\sim_\ell} \ldots [e_n]_{\sim_\ell}[t_n]_{\sim_\ell}\big)^+$$

in $\mathcal{P}_\ell$. Consequently, the factorization

$$[t_0]_\sim\, [e_1]_\sim\, [t_1]_\sim \ldots [e_n]_\sim\, [t_n]_\sim$$

is left T'-normal in $\mathcal{P}$ if and only if

$$t_0 e_1 t_1 \ldots e_n t_n$$

is left normal if and only if

$$[t_0]_{\sim_\ell}\, [e_1]_{\sim_\ell}\, [t_1]_{\sim_\ell} \ldots [e_n]_{\sim_\ell}\, [t_n]_{\sim_\ell}$$

is left T'_ℓ-normal in $\mathcal{P}_\ell$.

Proof. As the identity of $\mathcal{P}$ is $[1_T]_\sim$, Lemma 4.13 assures that $[t]_\sim \neq 1_\mathcal{P}$ if and only if $t \neq 1_T$, for any $t \in T$. Under Condition (CC1) or (CC2), Lemma 4.13 also assures that

$[e]_\sim < [f]_\sim$ if and only if $e < f$, for any $e, f \in Y$. Therefore, if (CC1) holds, then, for any $i \in \{1, \dots, n\}$,

$$\begin{aligned}
[e_i]_\sim < \big([t_i]_\sim\,[e_{i+1}]_\sim\,[t_{i+1}]_\sim \dots [e_n]_\sim\,[t_n]_\sim\big)^+ &\iff [e_i]_\sim < [t_i e_{i+1} t_{i+1} \dots e_n t_n]^+_\sim \\
&\iff [e_i]_\sim < [(t_i e_{i+1} t_{i+1} \dots e_n t_n)^+]_\sim \\
&\iff e_i < (t_i e_{i+1} t_{i+1} \dots e_n t_n)^+.
\end{aligned}$$

The corresponding statements with $\sim$ replaced by $\sim_\ell$ follow from (the proof) of [9, Lemma 2.10]. □

Proposition 4.20. *For any $u \in T * Y$, if $u\eta_\ell\eta_r\eta_\ell = t_0 e_1 \dots e_n t_n$, where $t_0, \dots, t_n \in T$ and $e_1, \dots, e_n \in Y$. Then*

$$[u]_\sim = [t_0]_\sim [e_1]_\sim \dots [e_n]_\sim [t_n]_\sim$$

is in T'-normal form.

Proof. Let $u\eta_\ell = s_0 f_1 \dots f_m s_m$, where $s_0, \dots, s_m \in T$ and $f_1, \dots, f_m \in Y$. By [9, Lemmas 2.5 and 2.6],

$$[u]_{\sim_\ell} = [u\eta_\ell]_{\sim_\ell} = [s_0]_{\sim_\ell}[f_1]_{\sim_\ell} \dots [f_m]_{\sim_\ell}[s_m]_{\sim_\ell}$$

where the right hand side is in left T'_ℓ-normal form in $\mathcal{P}_\ell$. Since $\mathcal{P}_\ell$ has uniqueness of left T'_ℓ-normal forms, we must have that $[s_0]_{\sim_\ell}[f_1]_{\sim_\ell} \dots [f_m]_{\sim_\ell}[s_m]_{\sim_\ell}$ is the element of $\mathcal{P}_\ell$ obtained by applying ($\mathrm{P_L}$) to the original factorisation of $[u]_\ell$ obtained from the expression for $u \in T * Y$. In view of Proposition 4.19, the same process can be simulated in $\mathcal{P}$, reducing $[u]_\sim$ to the left T'-normal form $[s_0]_\sim[f_1]_\sim \dots [f_m]_\sim[s_m]_\sim$.

Repeating this process twice, first with η_r and then again with η_ℓ (i.e. applying ($\mathrm{P_R}$) to $[s_0]_\sim[f_1]_\sim \dots [f_m]_\sim[s_m]_\sim$ and then ($\mathrm{P_L}$) to the resulting factorisation), yields the result. □

Contrary to what happens with the congruence $\sim_\ell$ [9, Section 2], we will see that, in general, a *$\sim$-class may have more than one normal element.*

In the case T and Y are not necessarily disjoint, we also denote by $T * Y$ the semigroup $\overline{T} * \overline{Y}$ and by $\mathcal{P}(T, Y)$ the monoid $\mathcal{P}(\overline{T}, \overline{Y})$, where $\overline{T}$ and $\overline{Y}$ are disjoint fixed copies of T and Y, respectively. If there is no ambiguity, we identify (the elements of) $\overline{T}$ and (the elements of) $\overline{Y}$ with (those of) T and Y, respectively.

Proposition 4.21. *Let T be a monoid that acts on a semilattice Y on both sides by order preserving maps. Let U be a monoid and let $\alpha\colon U \to T$ be a monoid morphism. Then U acts on the left and on the right of Y via order preserving maps by, respectively, $u \cdot e = (u\alpha) \cdot e$ and $e \circ u = e \circ (u\alpha)$ for all $u \in U$ and $e \in Y$.*

Moreover, if the actions of T on Y satisfy (CC1) and (CC2), then the actions of U on Y also satisfy (CC1) and (CC2). In this case, there exists a (2,1,1,0)-morphism $\theta\colon \mathcal{P}(U, Y) \to \mathcal{P}(T, Y)$ such that $\big([u]_{\sim'}\big)\theta = [u\alpha]_\sim$ and $\big([e]_{\sim'}\big)\theta = [e]_\sim$, for any $u \in U$ and $e \in Y$, where $\sim'$ is the semigroup congruence on $U * Y$ such that $\mathcal{P}(U, Y) = (U * Y)/{\sim'}$. The morphism θ is surjective if α is surjective. Moreover, θ is an isomorphism if α is bijective.

Proof. It is straightforward to show that U acts on the left and on the right of Y via order preserving maps as described in the statement, and that those actions satisfy (CC1) and (CC2) if the actions of T on Y satisfy (CC1) and (CC2). Let $\beta\colon U * Y \to T * Y$ be the morphism such that $u\beta = u\alpha$ and $e\beta = e$, for any $u \in U$ and $e \in Y$. Then, for any $x \in U * Y$,

$$x^+ = x \cdot 1_Y = (x\beta) \cdot 1_Y = (x\beta)^+$$

and

$$x^* = 1_Y \circ x = 1_Y \circ (x\beta) = (x\beta)^*.$$

Therefore, for any $x \in U*Y$, we have $(x^+x)\beta = (x^+\beta)(x\beta) = (x\beta)^+(x\beta) \sim x\beta$, and, dually, $(xx^*)\beta \sim x\beta$. Moreover, $(1_U)\beta = 1_T \sim 1_Y = (1_Y)\beta$. Then $\sim'$ is contained in the kernel of the semigroup morphism $U*Y \to \mathcal{P}(T,Y)$, $x \mapsto [x\beta]_\sim$. The resulting semigroup morphism $\theta\colon \mathcal{P}(U,Y) \to \mathcal{P}(T,Y)$ is easily seen to the the $(2,1,1,0)$-morphism of the statement.

Clearly, θ is onto if α is onto. Assume that α is bijective. Then the actions of T on Y can be recovered from those of U on Y by $t \cdot e = (t\alpha^{-1}) \cdot e$ and $e \circ t = e \circ (t\alpha^{-1})$, for any $t \in T$ and $e \in Y$. Moreover, β is bijective and $\beta^{-1}\colon T*Y \to U*Y$ is such that $t\beta^{-1} = t\alpha^{-1}$ and $e\beta^{-1} = e$, for any $t \in T$ and $e \in Y$. By the first part, there exists a (2,1,1,0)-morphism $\gamma\colon \mathcal{P}(T,Y) \to \mathcal{P}(U,Y)$ such that $\big([t]_\sim\big)\gamma = [t\alpha^{-1}]_{\sim'}$ and $\big([e]_\sim\big)\gamma = [e]_\sim$, for any $t \in T$ and $e \in Y$. It follows now that $\theta\gamma = \mathrm{id}_{\mathcal{P}(U,Y)}$ and $\gamma\theta = \mathrm{id}_{\mathcal{P}(T,Y)}$. In particular, θ is an isomorphism. □

5 Covers

Let M be a monoid with submonoid T and let E be a semilattice in M. For convenience, in this section we represent an element of $T * E$ with parentheses and commas. Given $u = (u_1, \ldots, u_n) \in T * E$, we denote by $\overline{u}$ the product in M of $u_1, \ldots, u_n$, that is $\overline{u} = u_1 \ldots u_n$. Notice that $\overline{u}$ is the image of u under the semigroup morphism from $T*E$ to M determined by the inclusion mapping $t \mapsto t$ for any $t \in T$ and $e \mapsto e$ for any $e \in E$.

If M is an Ehresmann monoid with distinguished semilattice E, then the left and the right actions of T on E given by Lemma 2.2 satisfy (CC1) and (CC2), by Remark 2.4. Hence we may consider the Ehresmann monoid $\mathcal{P}(T,E)$. We observe that, in this case, the left and right actions of $T * E$ on E of Section 4 are defined, respectively, by $u \cdot e = (\overline{u}\, e)^+$ and $e \circ u = (e\, \overline{u})^*$, for any $u \in T * E$ and any $e \in E$. In particular, $u^+ = \overline{u}^+$ and $u^* = \overline{u}^*$ for any $u \in T * E$.

Proposition 5.1. *Let M be an Ehresmann monoid with distinguished semilattice E and let T be a submonoid of M. Let $t_0, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in E$. Then the factorization*

$$[t_0]_\sim\, [e_1]_\sim\, [t_1]_\sim \ldots [e_n]_\sim\, [t_n]_\sim$$

in $\mathcal{P}(T,E)$ is left (resp. right) T'-normal, where $T' = \big\{[t]_\sim \colon t \in T\big\}$, if and only if the factorization

$$t_0 e_1 t_1 \ldots e_n t_n$$

in M is left (resp. right) T-normal.

Proof. It is clear that the element $(t_0, e_1, t_1, \ldots, e_n, t_n)$ of $T * E$ is left (respectively right) normal if and only if the factorization $t_0 e_1 t_1 \ldots e_n t_n$ of $\overline{(t_0, e_1, t_1, \ldots, e_n, t_n)}$ in M is left (resp. right) T-normal. The result now follows from Proposition 4.19. □

Theorem 5.2. *Let M be an Ehresmann monoid with distinguished semilattice E and let T be a submonoid of M. Then $\pi\colon \mathcal{P}(T,E) \to M$ defined by $([u]_\sim)\pi = \overline{u}$, for all $u \in T * E$, is a (2,1,1,0)-morphism which induces a bijection from $T' = \{[t]_\sim \colon t \in T\}$ to T and a bijection from $E' = \{[e]_\sim \colon e \in E\}$ to E.*

Proof. Let us see that π is a map, for which is enough to see that $\overline{u} = \overline{v}$, for all $(u, v) \in H_\ell \cup H_r$. It is obvious that $\overline{1_T} = 1_M = \overline{1_E}$. For any $u \in T * E$, we have $\overline{u^+ u} = \overline{u^+}\,\overline{u} = u^+ \overline{u} = \overline{u}^+\,\overline{u} = \overline{u}$, and dually we obtain $\overline{uu^*} = \overline{u}$. Hence π is a map.

It is clear that π is a monoid morphism. Let $u \in T * E$. We have $([u]_\sim^+)\pi = ([u^+]_\sim)\pi = u^+ = \overline{u}^+ = \big(([u]_\sim)\pi\big)^+$, and dually we obtain $([u]_\sim^*)\pi = \big(([u]_\sim)\pi\big)^*$. Hence π is (2,1,1,0)-morphism. The remaining part is immediate from Theorem 4.18. □

From Proposition 5.1 and Theorem 5.2 we obtain immediately the following.

Corollary 5.3. *Let M be an Ehresmann monoid with distinguished semilattice E and let T be a submonoid of M.*

If M has uniqueness of T-normal factorizations, then:

(a) $\mathcal{P}(T,E)$ has uniqueness of T'-normal factorizations;

(b) the morphism $\pi\colon \mathcal{P}(T,E) \to M$ of Theorem 5.2 is injective.

We say that an Ehresmann monoid N with distinguished semilattice F is a *cover* of an Ehresmann monoid M with distinguished semilattice E if there exists a (2,1,1,0)-morphism from N onto M injective on F. We also say that such a morphism is a *cover* of M.

Given a set X, we denote by X^* the free monoid on X. In Theorem 4.2 of [2] the authors show that any Ehresmann monoid $M = \langle X \rangle_{(2,1,1,0)}$ has a strongly X^*-proper cover[(b)]. An alternative proof of this result was privately communicated to the authors by Peter Jones [14]. Now we improve on that result and show that M has a cover of the form $\mathcal{P}(X^*, E)$, where E is the distinguished semilattice of M.

Theorem 5.4. *Let M be an Ehresmann monoid with distinguished semilattice E such that $M = \langle X \rangle_{(2,1,1,0)}$ for some $X \subseteq M$. Then*

(a) M has a strongly T-proper cover $\mathcal{P}(T,E)$, where $T = \langle X \rangle_{(2,0)}$.

(b) M has a strongly X^*-proper cover $\mathcal{P}(X^*, E)$.

Proof. (a) By Lemma 2.6, we have $M = \langle E \cup T \rangle_{(2)}$, where $T = \langle X \rangle_{(2,0)}$, and therefore the result follows from Theorem 5.2.

(b) Let $T = \langle X \rangle_{(2,0)}$ and let $\alpha\colon X^* \to T$ be the monoid epimorphism such that $x\alpha = x$ for every $x \in X$. Then, by Proposition 4.21, there exists a surjective (2,1,1,0)-morphism from $\mathcal{P}(X^*, E)$ to $\mathcal{P}(T, E)$, which is injective on the distinguished semilattice by Lemma 4.13. It follows, by (a), that $\mathcal{P}(X^*, E)$ is a X^*-proper cover of M. □

[(b)] 'Strongly' is not stated in the theorem, but follows from its proof

6 The free Ehresmann monoid

Since the class of all Ehresmann monoids is a variety of algebras, free objects exist. Let us fix a set X and denote by F_X the free Ehresmann monoid on X. Let $\iota\colon X \to F_X$ be the underlying map. In [16] Kambites gave an elegant description of F_X (which coincides with the free left adequate monoid on X) using birooted X-labelled trees. Our aim is to show that $F_X \cong \mathcal{P}(X^*, E_X)$ for the semilattice E_X of projections of F_X. In [2, Section 5] it was shown that the submonoid $T_X = \langle X\iota\rangle_{(2,0)}$ of F_X is isomorphic to the free monoid X^* via the monoid morphism $\psi\colon X^* \to F_X$ that extends ι, and that F_X is a (two-sided) T_X-proper Ehresmann monoid (by Theorem 5.2 of [2] and its dual for right Ehresmann monoids). By Theorem 4.18(b), this last result can now be obtained as a corollary of the following theorem.

Theorem 6.1. *Let X be a set. Then $F_X \simeq \mathcal{P}(T_X, E_X)$.*

Proof. We may suppose, to simplify, that $X \subseteq F_X$ and that ι is the inclusion map, since ι is one-to-one. Consider the morphism $\pi\colon \mathcal{P}(T_X, E_X) \to F_X$, $[u]_\sim \mapsto \overline{u}$, given by Theorem 5.2. Since F_X is the free Ehresmann monoid on X and $\mathcal{P}(T_X, E_X)$ is an Ehresmann monoid, there exists a (unique) morphism $\varphi\colon F_X \to \mathcal{P}(T_X, E_X)$ such that $x\varphi = [x]_\sim$ for all $x \in X$ (notice that we identify the elements of $T_X \cup E_X$, and in particular those of X, as a subset of F_X with the elements of $T_X \cup E_X$ as subset of $T_X * E_X$). Then $x\varphi\pi = ([x]_\sim)\pi = \overline{x} = x$ for all $x \in X$, which implies that $\varphi\pi = \mathrm{id}_{F_X}$. Then φ is injective. Moreover, by Lemma 2.6, $F_X = \langle T_X \cup E_X\rangle_{(2)}$.

The fact that $T_X = \langle X\rangle_{(2,0)}$ implies that $t\varphi = [t]_\sim$ for any $t \in T_X$. Let $e \in E_X$. We aim to see that $e\varphi = [e]_\sim$. Since $T_X \simeq X^*$, Corollary 4.14 guaranties that the set of idempotents of $\mathcal{P}(T_X, E_X)$ is $\big\{[f]_\sim\colon f \in E_X\big\}$. It follows that $e\varphi = [f]_\sim$, for some $f \in E_X$. Then $e = e\varphi\pi = \big([f]_\sim\big)\pi = \overline{f} = f$, and hence $e\varphi = [e]_\sim$, as desired.

Let $T'_X = \big\{[t]_\sim\colon t \in T_X\big\}$ and $E'_X = \big\{[e]_\sim\colon e \in E_X\big\}$. Now,

$$(F_X)\varphi = \big(\langle T_X \cup E_X\rangle_{(2)}\big)\varphi = \langle (T_X)\varphi \cup (E_X)\varphi\rangle_{(2)} = \langle T'_X \cup E'_X\rangle_{(2)} = \mathcal{P}(T_X, E_X).$$

Hence φ is surjective. Then φ is an isomorphism (whose inverse is π). $\square$

We now aim to show that F_X does not have uniqueness of T_X-normal factorizations. We start with the following general result.

Proposition 6.2. *Let M be a T-generated Ehresmann monoid.*

(a) If M has uniqueness of left T-normal factorizations, then $t^* = 1$ for any $t \in T$.

(b) If M has uniqueness of right T-normal factorizations, then $t^+ = 1$ for any $t \in T$.

(c) If M has uniqueness of (two-sided) T-normal factorizations, then for each $s, t \in T$, we have $s^* \leqslant t^+$ or $t^+ \leqslant s^*$. Consequently, for each $t \in T$, we also have $t = t^*t$ or $t = tt^+$.

Proof. (a) Suppose that M has uniqueness of left T-normal factorizations. Let $t \in T$. Since $t = tt^*1$, we necessarily have $t^* = 1$.

(b) Dual to (a).

(c) Suppose that M has uniqueness of T-normal factorizations. Let $s, t \in T$. Then $st \in T$ and $st = ss^*t^+t$, with $s^*t^+ \in E$, $s^*t^+ \leqslant s^*$ and $s^*t^+ \leqslant t^+$. It follows that $s^*t^+ = s^*$ or $s^*t^+ = t^+$, that is, $s^* \leqslant t^+$ or $t^+ \leqslant s^*$. When $s = t$ we obtain $t^*t^+ = t^*$ or $t^*t^+ = t^+$, which implies that $tt^+ = t$ or $t^*t = t$. □

Proposition 6.3. *The free Ehresmann monoid F_X does not have uniqueness of T_X-normal factorizations.*

Proof. Assume that F_X has uniqueness of T_X-normal factorizations. Let M be an Ehresmann monoid, with distinguished semilattice E, and let $s \in M$. Consider a morphism $\varphi\colon F_X \to M$ such that, for some $x \in X$, $x\iota\varphi = s$ (such an x and φ can certainly be found by the freeness of F_X). By Proposition 6.2(c), $(x\iota)^* \leqslant (x\iota)^+$ or $(x\iota)^+ \leqslant (x\iota)^*$, and, therefore, $s^* \leqslant s^+$ or $s^+ \leqslant s^*$. However, not all Ehresmann monoids satisfy this property, as it is the case of the inverse monoid of all partial injective transformations on a set with more than one element, where $\alpha^+ = \alpha\alpha^{-1} = \mathrm{id}_{\mathrm{dom}\alpha}$ and $\alpha^* = \alpha^{-1}\alpha = \mathrm{id}_{\mathrm{im}\alpha}$. Hence F_X does not have uniqueness of T_X-normal factorizations. □

We saw in the proof of Theorem 6.1 that there exists an isomorphism $\varphi\colon F_X \to \mathcal{P}(T_X, E_X)$ such that $t\varphi = [t]_\sim$ for any $t \in T_X$ and $e\varphi = [e]_\sim$ for any $e \in E_X$. Then, by Proposition 6.3, $\mathcal{P}(T_X, E_X)$ does not have uniqueness of T'_X-normal factorizations, where $T'_X = \big\{[t]_\sim\colon t \in T_X\big\}$. Proposition 4.19 allows us to conclude now that there are $\sim$-classes of $T_X * E_X$ with more than one normal element. This justifies last part of the comment following Proposition 4.20.

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