Completely regular semigroups and
(Completely) \((E, \overset{\sim}{\mathcal{H}}_E)\)-abundant
semigroups (a.k.a. \(U\)-superabundant
semigroups):

Similarities and Contrasts

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The setting

- Green’s relations $\rightarrow$ regular semigroup, simple semigroups, completely regular semigroups, inverse semigroups ...
- Generalizations to extended Green’s relations $\mathcal{L}^*, \mathcal{L}, \mathcal{L}_E, \mathcal{L}^{(l)}$ ...
  (Fountain, Lawson, Shum, Pastijn...)
- **Objective**: study the analogs to completely regular (completely simple, Clifford) semigroups for relations $\mathcal{K}_E$.
  - Emphazise on the similarities and differences.
  - Description as unary semigroups.
  - Application to regular semigroups.
Terminology varies: abundant, semiabundant, weakly left abundant, left semiabundant, superabundant, U-semiabundant, weakly U-superabundant with $C$, weakly left ample, left $E$-ample, ...

Shum et al. proposed:

**Definition**

*S* is $(A, \sigma)$-abundant if each $\sigma$-class intersects $A$. 
Green’s extended relations

Extended Green’s relations $\tilde{\mathcal{L}}_E, \tilde{\mathcal{R}}_E$ are based on (right, left) identities (El-Qallali’80, Lawson’90)

$$a \tilde{\mathcal{L}}_E b \iff \{(\forall e \in E) \ be = b \iff ae = a\};$$

$$a \tilde{\mathcal{R}}_E b \iff \{(\forall e \in E) \ eb = b \iff ea = a\}.$$

In general, $\tilde{\mathcal{L}}_E$ is not a right congruence, $\tilde{\mathcal{R}}_E$ is not a left congruence and the relations do not commute.

- $\tilde{\mathcal{H}}_E = \tilde{\mathcal{L}}_E \wedge \tilde{\mathcal{R}}_E$;
- $\tilde{\mathcal{D}}_E = \tilde{\mathcal{L}}_E \vee \tilde{\mathcal{R}}_E$;
- $\tilde{\mathcal{J}}_E$ defined be equality of ideals.
The semigroups of this talk

- $S$ is $(E, \tilde{H}_E)$-abundant if $(\forall a \in S, \exists e \in E) \ a \tilde{H}_E e$.
- $S$ is completely $(E, \tilde{H}_E)$-abundant if it is $(E, \tilde{H}_E)$-abundant and $\tilde{\mathcal{L}}_E, \tilde{\mathcal{R}}_E$ are right and left congruences.
- $S$ is completely $E$-simple if it is $(E, \tilde{H}_E)$-abundant and $\tilde{\mathcal{D}}_E$-simple.
- $S$ is an $E$-Clifford restriction semigroup if it is completely $(E, \tilde{H}_E)$-abundant with $E$ central idempotents.

Other names exist (weakly $U$-superabundant, $U$-superabundant or weakly $U$-superabundant with $C$, completely $\tilde{\mathcal{J}}_U$-simple).
Outline of the talk

1. Study as plain semigroups.
2. Study as unary semigroups.
3. Clifford and $E$-Clifford restriction semigroups.
Study as plain semigroups
Lemma

Let $S$ be a $(E, \tilde{\mathcal{H}}_E)$-abundant semigroup and $e, f \in E$. Then:

1. $e \tilde{D}_E f \iff e D f$;
2. $\tilde{D}_E = \tilde{L}_E \circ \tilde{R}_E = \tilde{R}_E \circ \tilde{L}_E$.

Proposition

Let $S$ be a $(E, \tilde{\mathcal{H}}_E)$-abundant semigroup, and let $e \in E$. Then

1. $\bigcup_{f \in E, f < e} f S f$ is an ideal of $e S e$;
2. $\tilde{\mathcal{H}}_E(e) = e S e \setminus (\bigcup_{f \in E, f < e} f S f)$.
Let $S$ be a semigroup. Then $S$ $(E, \tilde{\mathcal{H}}_E)$-abundant does not imply $S$ is a disjoint union of monoids.

Consider $S = \{0, a, 1_a, b, 1_b\}$ with multiplication table

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Pose $E = \{0, 1_a, 1_b\}$. Then $S$ is $(E, \tilde{\mathcal{H}}_E)$-abundant, but not a disjoint union of monoids (in particular, $\tilde{\mathcal{H}}_E(1_a)$ is not a monoid).
Theorem

Let $S$ be a $(E, \tilde{\mathcal{H}}_E)$-abundant semigroup. Then the following statements are equivalent:

1. $\tilde{\mathcal{L}}_E$ and $\tilde{\mathcal{R}}_E$ are right and left congruences;
2. $\tilde{\mathcal{D}}_E$ is a semilattice congruence;
3. $\tilde{\mathcal{D}}_E$ is a congruence.

In this case, $\tilde{\mathcal{J}}_E = \tilde{\mathcal{D}}_E$, and each $\tilde{\mathcal{H}}$-class is a monoid.

In particular, $S$ is $(E, \tilde{\mathcal{H}}_E)$-abundant and $\tilde{\mathcal{D}}_E$-simple (completely $E$-simple) iff it is completely $(E, \tilde{\mathcal{H}}_E)$-abundant and $\tilde{\mathcal{J}}_E$-simple.
Theorem

Let $S$ be a completely $(E, \overline{\mathcal{H}}_E)$-abundant semigroup. Then

1. $S$ is a disjoint union of monoids;
2. $S$ is a semilattice of completely $E$–simple semigroups (see also Ren’2010).

Converse is false.

Consider $S = \{0_a, 1_a\} \cup \{0_b, 1_b\}$ with multiplication table

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Pose $E = \{1_a, 1_b\}$. Then $S$ is not $(E, \overline{\mathcal{H}}_E)$-abundant.
**Proposition**

Let $S = \bigcup_{e \in E} M_e$ be a disjoint union of monoids such that

$$(\forall a \in S, \forall e, f \in E) \; ae \in M_f \Rightarrow fe = f \; \text{and} \; ea \in M_f \Rightarrow ef = f$$

Then $S$ is $(E, \tilde{\mathcal{H}}_E)$-abundant.

Conversely, any $(E, \tilde{\mathcal{H}}_E)$-abundant semigroup such that each $\tilde{\mathcal{H}}_E$-class is a monoid is a union of monoids with this property.

$S$ is not completely $(E, \tilde{\mathcal{H}}_E)$-abundant in general.

Let $S = \{f, e, d, a, a^2, \ldots\}$ such that $E = \{d, e, f\} = E(S)$ with $f \leq e \leq d$ and relations $ad = da = a$, $ae = ea = f$. It satisfies the assumptions of the proposition but $a \in \tilde{\mathcal{L}}_E(d)$ whereas $f = ae \not\in \tilde{\mathcal{L}}_E(de = e)$.
Theorem

$S$ is completely $(E, \tilde{\mathcal{H}}_E)$-abundant if and only if it is a semilattice $Y$ of $(E_\alpha, \tilde{\mathcal{H}}_{E_\alpha})$-abundant, $\tilde{\mathcal{D}}_{E_\alpha}$-simple semigroups such that:

$(\forall a \in S_\alpha, e \in E_\beta)$

\[ f \in E_{\alpha \beta} \cap \tilde{\mathcal{H}}_{E_{\alpha \beta}} (ae) \Rightarrow fe = f \]

and

\[ f \in E_{\beta \alpha} \cap \tilde{\mathcal{H}}_{E_{\beta \alpha}} (ea) \Rightarrow ef = f \]

The additional assumption is automatically satisfied for relation $\mathcal{H}$.
Theorem

The following statements are equivalent:

1. $S$ is $(E, \tilde{\mathcal{H}}_E)$-abundant and $\tilde{D}_E$-simple;
2. $S$ is completely $(E, \tilde{\mathcal{H}}_E)$-abundant and $\tilde{J}_E$-simple;
3. $S$ is $(E, \tilde{\mathcal{H}}_E)$-abundant and the idempotents of $E$ are primitive (within $E$).

In particular,

\[ E = \{ e \in E(S) | (\forall f \in E(S)) \, ef = fe = e \Rightarrow e = f \} = \text{Max} \]

set of maximal idempotents of $S$. 

Completely $E$-simple semigroups
Completely \(E\)-simple semigroups

**Proposition**

\(S\) is completely \(E\)-simple iff it is the disjoint union of its local submonoids \(eSe, e \in E\) and satisfies: \(e, f \in E, efe = fe \Rightarrow fe \in E\) and \(e, f \in E, efe = ef \Rightarrow ef \in E\).

**Example:** \( \mathcal{C} = \text{FinSet}_n \)

For \(a \in \text{Obj}(\mathcal{C})\) choose \(a \mapsto \{n\} = \{0, 1, \ldots, n-1\}\).

\((S = \text{Mor}(\mathcal{C}), \odot)\) with product

\((a \to b) \odot (c \to d) = a \to b \to \{n\} \to c \to d\)

is completely \(E\)-simple, with \(E = \{a \to \{n\} \to b | a, b \in \text{Obj}(\mathcal{C})\}\).

For \(e = a \to \{n\} \to b\),

\(eSe = \text{Mor}(a, b) = \tilde{\mathcal{H}}_E(e)\).
Theorem

Let \( \mathcal{M}(I, M, \Lambda, P) \) be a Rees matrix semigroup over a monoid with sandwich matrix with values in the group of units. Then \( \mathcal{M}(I, M, \Lambda, P) \) is completely \( E \)-simple.

Conversely, any completely \( E \)-simple semigroup is isomorphic to a Rees matrix semigroup over a monoid with sandwich matrix with values in the group of units.

Corollary

\( S \) is completely \( E \)-simple iff \( G_E = \dot{\bigcup}_{e \in E} G_e \) is a (completely simple) subsemigroup of \( S \) and \( S = \dot{\bigcup}_{e \in E} eSe \).
example: \( \mathcal{C} = \text{FinSet}_n \)

Let \((S = \text{Mor}(\mathcal{C}), \odot)\) as before. Then \(S \sim \mathcal{M}(\text{obj}(\mathcal{C}), \text{Mor}([n], [n]), \text{obj}(\mathcal{C}), (1))\).

example: \(N = \langle a \rangle\) free monogenic semigroup, \(B\) nowhere commutative band. Pose \(S = N \mathbin{\dot{\cup}} B\) with product \(a^n b = ba^n = a^n, b \in B, n \geq 0\). Then for any \(e \in B\), \(S = (N \cup e) \cup \bigcup_{f \in B \setminus \{e\}} \{f\}\) union of disjoint monoids with set of identities \(E = B\).

\(G_E = B\) completely simple but \(S\) is not \((B, \tilde{\mathcal{H}}_B)\)-abundant.

Assume \(a\) is \((B, \tilde{\mathcal{H}}_B)\)-related to \(a_0 \in B\). As \(fa = a = af\) then \(fa_0 = a_0 f\) for any \(f\), absurd.
Theorem (Hickey’10)

Let $S$ be regular $J \subseteq S$ completely simple. $S = \dot{\cup}_{e \in E(J)} eSe$, if and only if $S \sim \mathcal{M}(I, T, \Lambda, P)$, $T$ regular monoid and $P_{\lambda,i} \in T^{-1}$. In this case, $J \subseteq \text{RP}(S)$ and $E(J) = E(\text{RP}(S))$ where

$$\text{RP}(S) = \text{Regularity Preserving elements of } S.$$ 

Corollary

Let $S$ be a semigroup. Then the following statements are equivalent:

1. $S$ is a regular completely $E$-simple semigroup;
2. $S$ is regular and $S = \dot{\cup}_{e \in E(J)} eSe$ for a completely simple subsemigroup $J$ of $S$;
3. $S$ is regular and $S = \dot{\cup}_{e \in E(\text{RP}(S))} eSe$.

In this case, $J \subseteq \text{RP}(S) = \dot{\cup}_{e \in E} G_e$ and $E = E(J) = E(\text{RP}(S))$. 
Petrich (’87) gives a construction of a completely regular semigroups from a given semilattice $Y$ of Rees matrix semigroups. The same construction works in the setting of completely $(E, \tilde{\mathcal{H}}_E)$-abundant semigroups (see also Yuan’14).

Extra ingredient needed: the structure maps $(\beta \leq \alpha)$

$$\phi_{\alpha, \beta} : S_\alpha = \mathcal{M}(I_\alpha, M_\alpha, \Lambda_\alpha, P_\alpha) \to M_\beta$$

must map $\mathcal{M}(I_\alpha, M_\alpha^{-1}, \Lambda_\alpha, P_\alpha)$ to $M_\beta^{-1}$. 
Study as unary semigroups
We define a unary operation on \((E, \tilde{\mathcal{H}}_E)\)-abundant semigroups by:

\[ (\forall x \in S) \, x^+ \text{ is the unique element in } E \cap \tilde{\mathcal{H}}_E(x) \]

Conversely, for \((S, \cdot^+)\) unary semigroup we pose

\[ E = S^+ = \{x^+, x \in S\} \]

and

\[ x \sigma^+ y \iff x^+ = y^+. \]
Let \((S, ., ^+)\) be a unary semigroup. We consider the following identities on \((S, ., ^+)\).

\[
\begin{align*}
x^+ x &= x & (1) \\
xx^+ &= x & (2) \\
(xy^+)^+ y^+ &= (xy^+)^+ & (3) \\
y^+(y^+ x)^+ &= (y^+ x)^+ & (4) \\
(x^+ y)(xy)^+ &= x^+ y & (5) \\
(yx)^+(yx^+) &= yx^+ & (6) \\
(xy)^+ &= (x^+ y^+)^+ & (7) \\
(xy)^+ x^+ &= x^+ & (8) \\
x^+(yx)^+ &= x^+ & (9) \\
x^+(xy)^+ y^+ &= (xy)^+ & (10)
\end{align*}
\]
Theorem

1. \( S^+A = \mathcal{V}(1, 2, 3, 4) \) is the variety of unary \((S^+, \tilde{\mathcal{H}}_{S^+})\)-abundant semigroups;

2. \( \mathcal{C}S^+A = \mathcal{V}(1, 2, 3, 4, 5, 6) \) is the subvariety of unary completely \((S^+, \tilde{\mathcal{H}}_{S^+})\)-abundant semigroups;

3. \( S^+G = \mathcal{V}(1, 2, 3, 4, 7) \) is the subvariety of unary completely \((S^+, \tilde{\mathcal{H}}_{S^+})\)-abundant, \(\tilde{\mathcal{H}}_{S^+}\)-congruent semigroups \((S^+\)-cryptogroups);

4. \( \mathcal{C}S^+S = \mathcal{V}(1, 2, 8, 9, 10) \) is the subvariety of unary completely \(S^+\)-simple semigroups.

Moreover, \( \mathcal{C}S^+S \subseteq \mathcal{C}S^+G \subseteq \mathcal{C}S^+A \subseteq S^+A \).

If \((S, ., +)\) belongs to any of these families, then \(\sigma^+ = \tilde{\mathcal{H}}_{S^+}\).
Clifford and $E$-Clifford restriction semigroups
A unary semigroup \((S, ., ^+)\) is a left restriction semigroup if

\[
\begin{align*}
x^+ x &= x \\
x^+ y^+ &= y^+ x^+ \quad (S) \\
(x^+ y)^+ &= x^+ y^+ \quad (LC) \\
xy^+ &= (xy)^+ x \quad (LA)
\end{align*}
\]

In this case, \(E = S^+ = \{x^+, x \in S\}\) is a semilattice and the unary operation is the identity on \(E\).
Let $S$ be a semigroup and $E \subseteq E(S)$ be a semilattice. Then $S$ is weakly left $E$-ample if:

1. Every $\tilde{R}_E$-class $\tilde{R}_E(a)$ contains a (necessarily unique) idempotent $a^+$;
2. The relation $\tilde{R}_E$ is a left congruence;
3. The left ample condition $(\forall a \in S, \forall e \in E) ae = (ae)^+ a$ is satisfied.

Weakly left $E$-ample semigroups are precisely left restriction semigroups.
Clifford and $E$-Clifford restriction semigroup

**Definition**

A Clifford restriction semigroup $(S, ., ^+)$ is a unary semigroup that satisfies the following identities:

\[
x^+x = x \\
x^+y = yx^+ \\
(xy)^{++} = x^+y^+
\]

**Definition**

$S$ is a $E$-Clifford restriction semigroup if it is completely $(E, \tilde{\mathcal{H}}_E)$-abundant with $E$ central idempotents.

**Theorem**

Clifford restriction semigroup $\iff$ $E$-Clifford restriction semigroup.
Theorem

Let \((S, ., +)\) be a unary semigroup. Then the following statements are equivalent:

1. \(S\) is a left restriction semigroup with \((xy)^+ = x^+y^+\);
2. \(S\) is a left restriction semigroup with \(S^+ = \{x^+, x \in S\}\) semilattice of central idempotents;
3. \(S\) is a Clifford restriction semigroup;
4. \((S, ., +, ^+)\) is a restriction semigroup.
Theorem

The following statements are equivalent:

1. $S$ is a $E$-Clifford restriction semigroup;
2. $S$ is completely $(E, \tilde{\mathcal{H}}_E)$-abundant and idempotents of $E$ commute;
3. $S$ is completely $(E, \tilde{\mathcal{H}}_E)$-abundant and $\tilde{\mathcal{H}}_E = \tilde{\mathcal{D}}_E$;
4. $S$ is $(E, \tilde{\mathcal{H}}_E)$-abundant and $\tilde{\mathcal{H}}_E = \tilde{\mathcal{D}}_E$ is a congruence;
5. $S$ is a semilattice $Y$ of monoids $\{F_\alpha, \alpha \in Y\}$, with $1_\alpha 1_\beta = 1_{\alpha\beta} (\forall \alpha, \beta \in Y)$;
6. $S$ is a strong semilattice $A$ of monoids $\{F_\alpha, \alpha \in Y\}$.

Also, $S$ is a subdirect product of monoids but the converse does not hold.
Proposition

Let \( S \) be a \((E, \tilde{\mathcal{H}}_E)\)-abundant semigroup with \( E \) set of central idempotents of \( S \). Then \( S \) is a subdirect product of the factors

\[
\tilde{\mathcal{H}}^0_E(e) = eSe / \left( \bigcup_{f \in E, f < e} fSf \right), \quad e \in E
\]

Let \( M = \{0, n, 1\} \), \( n^2 = 0 \). The direct product \( P = \{0\} \times M \times M \) is a commutative monoid and

\[
S = \{(0, 0, 0); (0, n, 0); (0, 1, 0); (0, 0, n); (0, 0, 1)\}
\]

is a subdirect product of \( \{0\} \times M \times M \). \( S \) is \((E(S), \tilde{\mathcal{H}})\)-abundant, but not completely \((E(S), \tilde{\mathcal{H}})\)-abundant.
Proper Clifford restriction semigroup

$E$-Clifford restriction semigroup is proper if $\overline{\mathcal{H}}_E \cap \sigma = \iota$, where

$\sigma = \{(a, b) \in S^2 | \exists e \in E, ea = eb\}.$

Let $E$ be a semilattice, $M$ a monoid, $\text{OrdI}(E)$ the set of order ideals of $E$. Let

$$I : (M, \leq_J) \rightarrow (\text{OrdI}(E), \subseteq)$$

be a non-decreasing function (with $I(1) = E$). Then

$$\mathcal{M}(M, E, I) = \{(e, m) \in E \times M, e \in I(m)\}$$

with $(e, m)(f, n) = (ef, mn)$ and $(e, m)^+ = (e, 1)$ is a proper Clifford restriction semigroup.

**Theorem**

$S$ is a proper $E$-Clifford restriction semigroup if and only if it is isomorphic to a semigroup $\mathcal{M}(M, E, I)$. 
Application: \( T \)-regular semigroups
Motivation

- Monoid $M$ is a factorisable monoid (unit regular monoid) if
  \[(\forall a \in S) \ a \in aM^{-1}a \quad (1)\]

- $M$ inverse monoid is factorisable iff
  \[(\forall a \in S, \exists x \in M^{-1}) \ a \omega x \quad (2)\]

- **Question**: How to move from monoids to semigroups? Can we get structure theorems?

- **Answer**: To move from monoids to semigroups, replace $M^{-1}$ by (some, all) maximal subgroups in (1) or (2).
  - If $a \omega x$ with $x \in G_e = \mathcal{H}(e)$, then $ex = xe = x$. 
**Definition**

Let $S$ be a regular semigroup, $T$ a subset of $S$. $a \in S$ is $T$-regular (resp. $T$-dominated) if it admits an associate (resp. a majorant for the natural partial order) $x \in T$. $S$ is $T$-regular (resp. $T$-dominated) if each element is $T$-regular (resp. $T$-dominated).

**Lemma**

Let $a \in S$, $x \in G_e$, $e \in E(S)$. Then

\[
 a \omega x \iff ax \# a = a, \ a \leq_{\mathcal{H}} x. \]
Structure Theorems

For $E \subseteq E(S)$, $G_E = \bigcup_{e \in E} G_e = \bigcup_{e \in E} \mathcal{K}(e)$.

**Theorem**

Let $S$ be a semigroup. Then the following statements are equivalent:

1. $S$ is a completely $E$-simple, $G_E$-dominated semigroup;
2. $S$ is a completely $E$-simple, $G_E$-regular semigroup;
3. $S$ is isomorphic to a Rees matrix semigroup $\mathcal{M}(I, M, \Lambda, P)$ over a unit-regular monoid $M$ with sandwich matrix with values in the group of units;
4. There exists a completely simple subsemigroup $J$ of $S$, $S$ is $J$-dominated and the local submonoids $eSe$, $e \in J$ are disjoint.

Extends to completely $(E, \tilde{\mathcal{K}}_E)$-abundant semigroups and $E$-Clifford restriction semigroups.