# Varieties of Restriction Semigroups

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# Inverse Semigroups

### Definition

An element $a' \in S$ is an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. If each element of $S$ has exactly one inverse in $S$, then $S$ is an *inverse semigroup*.

### Definition

For $a, b \in S$,

$$a \,\mathcal{R}\, b \Leftrightarrow a = bt \text{ and } b = as \text{ for some } s, t \in S$$

and

$$\begin{aligned} a \,\sigma\, b &\Leftrightarrow ea = eb \text{ for some } e \in E(S) \\ &\Leftrightarrow af = bf \text{ for some } f \in E(S). \end{aligned}$$

# E-unitary and Proper Inverse Semigroups

## Definition

An inverse semigroup is *proper* if and only if $\mathcal{R} \cap \sigma = \iota$, i.e.

$$a \, \mathcal{R} \, b \text{ and } a \, \sigma \, b \Leftrightarrow a = b.$$

## Definition

An inverse semigroup $S$ is *E-unitary* if for all $a \in S$ and all $e \in E(S)$, if $ae \in E(S)$, then $a \in E(S)$.

## Proposition

*Let $S$ be an inverse semigroup. Then the following are equivalent:*
*i) $S$ is E-unitary;*
*ii) $S$ is proper;*
*iii) $\mathcal{L} \cap \sigma = \iota$.*

# Proper Covers

**Definition**

Let $S$ be an inverse semigroup. A *proper cover* of $S$ is a proper inverse semigroup $U$ together with an onto morphism

$$\psi : U \to S$$

where $\psi$ is idempotent separating.

**McAlister's Covering Theorem**

*Every inverse semigroup has a proper cover.*

# Restriction Semigroups

### Definition

Suppose $S$ is a semigroup and $E$ a set of idempotents of $S$. Let $a, b \in S$. Then $a \widetilde{\mathcal{R}}_E b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b.$$

### Definition

A semigroup $S$ is *left restriction* (formerly known as *weakly left E-ample*) if the following hold:

1) $E$ is a subsemilattice of $S$;
2) Every element $a \in S$ is $\widetilde{\mathcal{R}}_E$-related to an idempotent in $E$ (idempotent denoted by $a^+$);
3) $\widetilde{\mathcal{R}}_E$ is a left congruence;
4) For all $a \in S$ and $e \in E$,

$$ae = (ae)^+ a \text{ (the } \textit{left ample condition}\text{)}.$$

# Proper Restriction Semigroups

Let $S$ be a left restriction semigroup with distinguished semilattice $E$. Then for $a, b \in S$,

$$a \, \sigma_S \, b \Leftrightarrow ea = eb \text{ for some } e \in E.$$

**Definition**

A left restriction semigroup is *proper* if and only if $\widetilde{\mathcal{R}}_E \cap \sigma_S = \iota$.

A right restriction semigroup is *proper* if and only if $\widetilde{\mathcal{L}}_E \cap \sigma_S = \iota$.

**Definition**

A *proper cover* of $S$ is a proper left restriction semigroup $U$ together with an onto morphism $\psi : U \to S$, which is idempotent-separating on $E$.

# Varieties

A *variety* is a non-empty class of algebras of a certain type which is closed under subalgebras, homomorphic images and direct products.

A variety $\mathscr{V}$ of restriction semigroups has *proper covers* if, for every $S \in \mathscr{V}$, there is a proper cover of $S$ in $\mathscr{V}$.

### Theorem

*Let $\mathscr{V}$ be a variety of restriction semigroups. Then the following are equivalent:*

(i) *$\mathscr{V}$ has proper covers;*

(ii) *the free objects in $\mathscr{V}$ are proper;*

(iii) *$\mathscr{V}$ is generated by its proper members.*

# Varieties

## Definition

A left restriction semigroup has a *proper cover over* $\mathcal{U}$, where $\mathcal{U}$ is a variety of monoids, if it has a proper cover $R$ such that $R/\sigma \in \mathcal{U}$. We put

$$\hat{\mathcal{U}} = \{N \in \mathcal{LR} : N \text{ has a proper cover over } \mathcal{U}\}.$$

## Theorem

*The class of left restriction semigroups having a cover over $\mathcal{U}$, where $\mathcal{U}$ is a variety of monoids, is a variety of left restriction semigroups and is determined by*

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \textit{ is a law in } \mathcal{U}\}.$$

# Subhomomorphisms

**Definition (Petrich/Reilly)**

*Let $S$ and $T$ be inverse semigroups. Then a mapping $\varphi : S \to 2^T$ is an inverse subhomomorphism of $S$ into $T$, if for all $s, t \in S$,*

(i) $s\varphi \neq \emptyset$;

(ii) $(s\varphi)(t\varphi) \subseteq (st)\varphi$;

(iii) $s'\varphi = (s\varphi)'$,

*where for any subset $A$ of $T$, $A' = \{a' : a \in A\}$.*

# Subhomomorphisms

**Definition**

Let $S$ and $T$ be left restriction semigroups. Then a mapping $\varphi : S \to 2^T$ is a *left subhomomorphism* of $S$ into $T$, if for all $s, t \in S$,

(i) $s\varphi \neq \emptyset$;

(ii) $(s\varphi)(t\varphi) \subseteq (st)\varphi$;

(iii) $(s\varphi)^+ \subseteq s^+\varphi$,

where for any subset $A$ of $T$, $A^+ = \{a^+ : a \in A\}$.

A left or right subhomomorphism is said to be *surjective* if $S\varphi = T$, where $S\varphi = \cup\{s\varphi : s \in S\}$.

# Subhomomorphisms

### Proposition

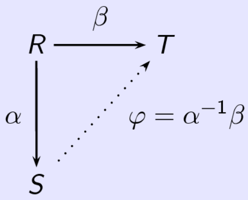
*Let $\varphi$ be a left subhomomorphism of $S$ into $T$, where $S$ and $T$ are left restriction semigroups. Then $S\varphi$ is a left restriction semigroup with respect to the distinguished semilattice*

$$E_{S\varphi} = \cup\{(s\varphi)^+ : s \in S\}.$$

# Subhomomorphisms

**Theorem**

*Let $R, S$ and $T$ be left restriction semigroups. Let $\alpha : R \to S$ be an epimorphism and $\beta : R \to T$ a morphism. Then $\varphi = \alpha^{-1}\beta$ is a left subhomomorphism of $S$ into $T$ and every such left subhomomorphism is obtained in this way.*



# Subhomomorphisms

## Proposition

*Let $S$ and $T$ be left restriction semigroups and let $\varphi$ be a (surjective) left subhomomorphism of $S$ into $T$. Then*

$$\Pi(S, T, \varphi) = \{(s, t) \in S \times T : t \in s\varphi\}$$

*is a left restriction semigroup (which is a subdirect product of $S$ and $T$).*

*Conversely, suppose that $V$ is a left restriction semigroup which is a subdirect product of $S$ and $T$. Then $\varphi$, defined by*

$$s\varphi = \{t \in T : (s, t) \in V\}$$

*is a surjective left subhomomorphism of $S$ into $T$. Furthermore, $V = \Pi(S, T, \varphi)$.*

# Subhomomorphisms

**Proposition**

*Let $\varphi$ be a left subhomomorphism of $S$ into $M$, where $S$ is a left restriction semigroup and $M$ a monoid. Then $\Pi(S, M, \varphi)$ is E-unitary if and only if $\varphi$ satisfies*

$$1 \in s\varphi, es \in E_S \Rightarrow s \in E_S, \tag{S3}$$

*for $s \in S$ and $e \in E_S$.*

# Subhomomorphisms

### Proposition

*Let $\varphi$ be a left subhomomorphism of $S$ into $M$, where $S$ is a left restriction semigroup and $M$ a monoid. Then $\Pi(S, M, \varphi)$ is proper if and only if $\varphi$ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b, \tag{S1}$$

*for $a, b \in S$.*

# Subhomomorphisms

### Proposition

*Let $\varphi$ be a left subhomomorphism of $S$ into $T$, where $S$ and $T$ are left restriction semigroups. Then the following are equivalent for $a, b \in S$:*

(i) *(S4)* $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+ b = b^+ a$;

(ii) *(S6)* $a\varphi \cap b\varphi \neq \emptyset, a^+ = b^+ \Rightarrow a = b$;

(iii) *Conditions (S1) and (S9).*

$$a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b. \tag{S1}$$

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a\,\sigma_S\,b. \tag{S9}$$

# Subhomomorphisms

### Proposition

*Let $\varphi$ be a left subhomomorphism of $S$ into $T$, where $S$ and $T$ are left restriction semigroups. Then*

(S4) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a$, *for* $a, b \in S$,

*implies*

(S9) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a\,\sigma_S\,b$, *for* $a, b \in S$.

**Proposition**

*Let $\varphi$ be an inverse subhomomorphism of $S$ into $G$, where $S$ is an inverse semigroup and $G$ a group. Then*

(S8) $1 \in s\varphi \Rightarrow s \in E(S)$, *for* $s \in S$,

*implies*

(S9) $s\varphi \cap t\varphi \neq \emptyset \Rightarrow s \, \sigma_S \, t$, *for* $s, t \in S$.

# Subhomomorphisms

Let $\alpha : R \to S$ and $\beta : R \to T$ are morphisms between restriction semigroups $R$, $S$ and $T$. Let

$$\text{Ker } \alpha = \{(a, b) \in R \times R : a\alpha = b\alpha\}$$

and

$$\ker \alpha = \{a \in R : a\alpha \in E_S\}.$$

## Proposition

*Let $R$, $S$ and $T$ be left restriction semigroups. Let $\alpha : R \to S$ and $\beta : R \to T$ be morphisms. Then*

$$\text{Ker } \beta \subseteq \text{Ker } \alpha \text{ implies } \ker \beta \subseteq \ker \alpha.$$

# Subhomomorphisms

## Proposition

*Let $R$, $S$ and $T$ be left restriction semigroups. Let $\alpha : R \to S$ be an epimorphism and $\beta : R \to T$ a morphism. Then the left subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a, \qquad \text{(S4)}$$

*for $a, b \in S$, if and only if*

$$s\beta = t\beta \Rightarrow (s^+t)\alpha = (t^+s)\alpha, \qquad \text{(*)}$$

*for $s, t \in R$.*

# Subhomomorphisms

## Proposition

*Let $R$ and $S$ be left restriction semigroups and let $T$ be a monoid. Let $\alpha : R \to S$ be an epimorphism and $\beta : R \to T$ a morphism. Then the left subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a = b,$$

*for $a, b \in S$, if and only if*

$$Ker\ \beta \subseteq Ker\ \alpha.$$

*for $s, t \in R$.*

# Subhomomorphisms

### Proposition

*Let $R$ and $S$ be left restriction semigroups and $T$ a monoid. Let $\alpha : R \to S$ be an epimorphism and $\beta : R \to T$ a morphism. Then the left subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$1 \in s\varphi \Rightarrow s \in E_S, \tag{S8}$$

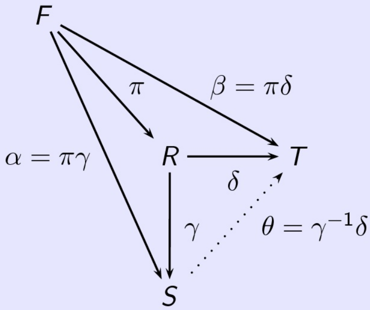
*for $s \in S$, if and only if*

$$\ker \beta \subseteq \ker \alpha.$$

# Subhomomorphisms

## Proposition

*Let $\theta$ be a left subhomomorphism of $S$ into $T$, where $S$ and $T$ are left restriction semigroups. Then there exist a free left restriction semigroup $F$, an epimorphism $\alpha : F \to S$, and a morphism $\beta : F \to T$ such that $\theta = \alpha^{-1}\beta$.*



# Proper Covers

### Proposition

*Let $R$ be a left restriction semigroup and $M$ a monoid. Let $\phi$ be a surjective left subhomomorphism of $R$ into $M$ such that*

$$a\phi \cap b\phi \neq \emptyset \Rightarrow a^+b = b^+a, \tag{S4}$$

*for $a, b \in S$. Then*

$$\Pi(R, M, \phi) = \{(r, m) \in R \times M : m \in r\phi\}$$

*is a proper cover of $R$ over $M$.*

# Proper Covers

Conversely, let $P$ be a proper cover of $R$ over $M$ with the induced morphism $\psi : P \to R \times M$. Then $\phi$, defined by

$$s\phi = \{g \in M : (s, g) \in P\psi\},$$

for $s \in R$, is a surjective left subhomomorphism of $R$ into $M$ such that Condition (S4) holds and

$$P \cong \Pi(R, M, \phi).$$

# Proper Covers

## Proposition

*Let $R$ be a left restriction semigroup and $M$ a monoid. Let $\phi$ be a surjective left subhomomorphism of $R$ into $M$ such that*

$$1 \in s\phi \Rightarrow s \in E_R \tag{S8}$$

*and*

$$s\phi \cap t\phi \neq \emptyset \Rightarrow s \, \sigma_R \, t, \tag{S9}$$

*for $s, t \in R$. Then*

$$\Pi(R, M, \phi) = \{(r, m) \in R \times M : m \in r\phi\}$$

*is an E-unitary cover of $R$ over $M$.*

# Proper Covers

Conversely, let $P$ be an E-unitary cover of $R$ over $M$ with the induced morphism $\psi : P \to R \times M$. Then $\phi$ defined by

$$s\phi = \{g \in M : (s, g) \in P\psi\},$$

for $s \in R$, is a surjective left subhomomorphism of $R$ into $M$ such that Conditions (S8) and (S9) hold.

# Proper Covers

**Theorem**

*Let $S$ be a left restriction semigroup and $\mathcal{U}$ a variety of monoids. Then the following are equivalent:*

(1) *$S$ has proper covers over $\mathcal{U}$;*

(2) *if $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{U}$, then $\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ is a law in $S$.*

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