# Self-similar group actions and their associated monoids

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This is joint work-in-progress with Alistair Wallis.

# Outline of talk

1. Old results: setting the scene.

2. New results.

3. Further work.

# 1. Old results: setting the scene

## Introduction: the background story

The Thompson groups $G_{n,r}$ are constructed from free monoids and their free actions.

My original motivation for studying self-similar group actions came from trying to generalize free monoids to obtain generalizations of the Thompson groups.

The monoids that arose were left Rees monoids, described in this talk.

It turned out that they could be constructed from self-similar group actions.

In addition, I discovered that they had been introduced in 1972 in Perrot's thesis.

Thus self-similar group actions were first discovered within semigroup theory.

## What is a self-similar group action?

Let $X$ be a finite alphabet. Let $X^*$ be the free monoid on $X$; we shall think of this as being ordered by the *prefix ordering*.

Let $X^\omega$ be the set of all right-infinite strings over $X$. We may regard $X^\omega$ as the set of infinite paths from the root of the $|X|$-ary tree determined by $X^*$.

Let $\theta$ be an automorphism of $X^\omega$.

We show how the self-similarity structure of $X^\omega$ leads to a recursive method for computing $\theta$.

Let $w' = xw$ where $x$ is a letter and $w$ is infinite. By assumption, any string beginning with $x$ will be mapped to a string beginning with the same letter. We denote this letter by $\theta(x)$.

We have that $w' \in xX^\omega$ and $\theta(w') \in \theta(x)X^\omega$.

It follows that $\theta$ restricts to an isomorphism between $xX^\omega$ and $\theta(x)X^\omega$. Let's call this isomorphism $\theta'$.

There are are self-similarities $\lambda_x\colon X^\omega \to xX^\omega$ and $\lambda_{\theta(x)}\colon X^\omega \to \theta(x)X^\omega$.

It follows that

$$\theta_x = \lambda_{\theta(x)}^{-1}\theta'\lambda_x$$

is another automorphism and, crucially,

$$\theta(xw) = \theta(x)\theta_x(w).$$

A *faithful* action of $G$ on $X^*$ is said to be *self-similar* if for each $x \in X$ and $w \in X^*$ we have that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w)$$

where $g_x \in G$ is uniquely determined by $g$ and $x$.

**Remarks**

1. Such actions can also be interpreted in terms of automata-theory.

2. The self-similarity properties of $X^\omega$ are the subject of Rees (1948), and led to the concept of *uniform posets* which form the basis of the theory of 0-bisimple inverse monoids to be found in McAlister (1974) and which ultimately lie behind Perrot's work.

## Why are such groups interesting?

See Nekrashevych's book (2005). But, for example, the Grigorchuk group is self-similar; this group is an infinite, finitely generated torsion group of intermediate growth.

## Generalizations

The definition of a self-similar group action requires the action to be *faithful*. We shall now construct a definition that does not require this assumption.

We shall arrive at this definition from a different direction.

## Zappa-Szép products

Let $S$ be a monoid and let $A$ and $B$ be submonoids. Assume that $S = AB$ and that every element of $S$ can be written *uniquely* as an element of $A$ mutiplied by an element of $B$. We say that $S$ can be *uniquely factorized*.

It follows that

$$ba = a'b'$$

where $a'$ and $b'$ are uniquely determined by $a$ and $b$. To signal this, we write

$$a' = b \cdot a \text{ and } b' = b|_a.$$

We therefore have two maps

$$B \times A \to A \text{ where } (b, a) \mapsto b \cdot a$$

and

$$B \times A \to B \text{ where } (b, a) \mapsto b|_a.$$

To work out the properties satisfied by these maps we use the associative and identity laws in varous cases. Here is an important example. We have that

$$b(a_1a_2) = (ba_1)a_2.$$

But

$$b(a_1a_2) = [b \cdot (a_1a_2)]b|_{b_{a_1a_2}}$$

and

$$(ba_1)a_2 = (b \cdot a_1)(b|_{a_1} \cdot a_2)(b|_{a_1})|_{b_2}.$$

We now use the uniqueness of the factorization to deduce that

$$b \cdot (a_1a_2) = (b \cdot a_1)(b|_{a_1} \cdot a_2)$$

and

$$b|_{a_1a_2} = (b|_{a_1})|_{a_2}.$$

The first equation provides the link with self-similar group actions.

Eight properties arise by continuing in this way.

(SS1) $1 \cdot a = a$.

(SS2) $(b_1 b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$.

(SS3) $b \cdot 1 = 1$.

(SS4) $b \cdot (a_1 a_2) = (b \cdot a_1)(b|_{a_1} \cdot a_2)$.

(SS5) $b|_1 = b$.

(SS6) $b|_{a_1 a_2} = (b|_{a_1})|_{a_2}$.

(SS7) $1|_a = 1$.

(SS8) $(b_1 b_2)|_a = b_1|_{b_2 \cdot a} b_2|_a$.

Conversely, if $A$ and $B$ are monoids with a pair of actions satisfying the above axioms we may define a binary operation on $A \times B$ by

$$(a_1, b_1)(a_2, b_2) = (a_1(b_1 \cdot a_2), (b_1|_{a_2})b_2.)$$

In this way, we obtain a monoid $A \bowtie B$ called the *Zappa-Szép* product of $A$ and $B$. It may be uniquely factorized by isomorphic copies of $A$ and $B$.

## Self-similar group actions

We suppose that the monoids above are $A = X^*$, the free monoid on $X$, and $B = G$, a group. Then we say that there is a *self-similar group action of $G$ on $X^*$* if the axioms (SS1)—(SS8) are satisfied.

**Remark** If the action is faithful then we obtain the classical definition of a self-similar group action.

What about the monoid $X^* \bowtie G$?

**Theorem** *The monoid* $X^* \bowtie G$ *satisfies the following three axioms:*

(LR1) $S$ *is a left cancellative monoid.*

(LR2) *Incomparable principal right ideals are disjoint; that is, the monoid is right rigid.*

(LR3) *Each principal right ideal is properly contained in only a finite number of principal right ideals.*

*Every monoid satisfying these axioms, which is not a group, is isomorphic to a Zappa-Szép product of a free monoid by a group.*

A monoid, not a group, satisfying the above three axioms, is called a *left Rees monoid*. If it is also right cancellative we say that it is a *Rees monoid.*

**Remark** Left Rees monoids may be used to construct unambiguous, 0-bisimple inverse monoids in which there are only a finite number of idempotents above every non-zero idempotent. We call these *Perrot monoids*. The self-similar group action is faithful iff the Perrot monoid is *fundamental* iff the Perrot monoid is congruence-free.

# 2. New results

## Some definitions

Let $S$ be a monoid. An *atom* in $S$ is an element $a$ such that if $a = bc$ then either $b$ or $c$ is invertible.

A *length function* on a monoid $S$ is a function $\lambda\colon S \to \mathbb{N}$ such that $\lambda(ab) = \lambda(a) + \lambda(b)$; $\lambda^{-1}(0)$ is the group of units of $S$; $\lambda^{-1}(1)$ is the set of atoms of $S$.

The monoid $S$ is said to be *equidivisible* if if for all $a, b, c, d \in S$ we have that $ab = cd$ implies $a = cu$ and $ub = d$ for some $u \in S$ or $av = c$ and $b = vd$ for some $v \in S$.

The following theorem is well-known and we include it to set the scene.

**Theorem** *A monoid is free if and only if it is an equidivisible monoid with a length function having a trivial group of units.*

## Irreducible left Rees monoids

A left Rees monoid is said to be *irreducible* if there is a maximum proper principal ideal. The proof of the following uses a result due to N. Bourbaki.

**Theorem** *Let $S_1, \ldots, S_m$ be any set of irreducible left Rees monoids where $G = G(S_i)$ for all $i$. Suppose in addition that $S_i \cap S_j = G$ whenever $i \neq j$. Then the free product with amalgamation $S_1 *_G S_2 *_G \ldots *_G S_m$ is a left Rees monoid with group of units $G$ and irreducible components isomorphic to the $S_i$.*

*Every left Rees monoid is either irreducible or a free product with amalgamation of irreducible left Rees monoids having the same groups of units.*

Thus irreducible left Rees monoids are the building blocks of all left Rees monoids.

My goal is to show you two methods for building all irreducible left Rees monoids.

## Partial endomorphisms

The following construction is fundamental to our work.

Let $S$ be a left cancellative monoid. Let $s \in S$. Define

$G_s^+ = \{g \in G\colon gs \in sG\}$ and $G_s^- = \{g \in G\colon sg \in Gs\}$.

By left cancellation, if $gs = sg' = sg''$ then $g' = g''$. We may therefore write

$$gs = s\phi_s(g).$$

In fact,

$$\phi_s\colon G_s^+ \to G_s^-$$

is a surjective homomorphism between two subgroups of $G$.

A *partial endomorphism* $(G, \phi)$ is any surjective homomorphism $\phi$ between two subgroups of $G$.

## $G$-bisets

Let $A$ be a set on which $G$ acts both on the left and the right in such a way that the actions are compatible. Then we say that $S$ is a $G$-*biset*.

If there is an element $a \in A$ such that $A = GaG$ we say that the biset is *irreducible*.

If the righthand $G$-action is free we say that $A$ is a *covering bimodule*.

**Lemma** *Let $S$ be a left Rees monoid. Let $A$ be the set of atoms of $S$. Then $A$ is a covering bimodule which is irreducible iff $S$ is irreducible.*

The following construction is due to Nekrashevych.

Let $\phi: H \to G$ be a partial endomorphism of $G$. On the set $G \times G$ define $(g_1, h_1) \equiv (g_2, h_2)$ if and only if $g_1^{-1} g_2 \in H$ and $\phi(g_1^{-1} g_2) = h_1 h_2^{-1}$.

$\equiv$ is an equivalence relation. Denote the equivalence class containing $(g_1, h_1)$ by $[g_1, h_1]$. Write $B = \mathsf{B}(G, \phi) = (G \times G)/\sim$ and define two actions $G \times B \times G$ by $g[g_1, h_1] = [gg_1, h_1]$ and $[g_1, h_1]g = [g_1, h_1 g]$.

This gives us an irreducible covering bimodule where the righthand action is free.

**Theorem** *Every irreducible covering bimodule is constructed in this way.*

Let $S$ be an irreducible left Rees monoid with group of units $G$.

Let $s$ be an atom. Then we may construct a partial endomorphism $(G, \phi_s)$.

**Lemma** *Any two partial endomorphisms constructed as above are* essentially *the same.*

We may therefore speak of *the* partial endomorphism associated with an irreducible left Rees monoid.

## Tensor monoids

Let $M$ be an arbitrary $(G, G)$-biset. Put

$$\mathsf{T}(M) = \bigcup_{n=0}^{\infty} M^{\otimes n}$$

with the obvious multiplication. Then it is a monoid called the *tensor monoid*, associated with the $G$-biset $M$.

**Theorem**

1. *Tensor monoids are precisely the equidivisible monoids equipped with length functions.*

2. *The left cancellative tensor monoids are precisely the left Rees monoids.*

3. *From each partial endomorphism* $\phi$ *of* $G$ *we may construct an irreducible left Rees monoid whose associated partial endomorphism is essentially* $\phi$.

## Generalized HNN extensions

Given a partial endomorphism $\phi\colon A \to B$ of $G$ we have shown how to construct an irreducible left Rees monoid. We now show that there is a direct way of doing this. It is a generalization of an HNN extension.

**Theorem** *Let $(G, \phi)$ be a partial endomorphism where $\phi\colon A \to B$. Let*

$$M = \text{Mon}\langle G, x\colon ax = x\phi(a) \text{ for all } a \in A\rangle$$

*be the monoid with generators $G \cup \{x\}$ and relations all those that hold in $G$ together with the relations explicitly stated. Then $M$ is a left Rees monoid isomorphic to the one constructed from the partial endomorphism.*

Let $S$ be an irreducible left Rees monoid determined by the partial endomorphism $(G, \phi)$.

**Lemma** *$S$ is also right cancellative, and so cancellative, iff $\phi$ is injective.*

**Theorem** *The universal groups of irreducible Rees monoids are HNN extensions of their groups of units and every HNN extension arises in this way. In addition, such monoids may be embedded in their universal groups.*

Using this result, we may also prove directly the following.

**Theorem** *A left Rees monoid may be embedded in a group iff it is right cancellative.*

## 3. Further work

- The connection between irreducible Rees monoids and HNN extensions of groups raises the question of whether such monoids and their generalizations intervene in an anlogous way in the general theory of graphs of groups.

- Monoids equipped with length functions are examples of monoids in which Green's relations are determined by units; connected algebraic monoids are particularly well-known members of this class of semigroups.

- There is a connection between self-similar group actions and strong representations of the polycyclic monoids.

- Associated with a self-similar group action is a double category of squares. This seems to underlie work of Burger and Mozes and their successors.