Homogeneous Bands

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Homogeneity

**Definition**
A countable (first order) structure $\mathcal{M}$ is *homogeneous* if every isomorphism between finitely generated substructures extends to an automorphism of $\mathcal{M}$.

**Motivation:**
- A structure $\mathcal{M}$ is uniformly locally finite (ULF) if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every $n$-generated substructure has cardinality at most $f(n)$.
- A ULF homogeneous structure is $\aleph_0$-categorical.
Some key classifications

- (Droste, Kuske, Truss (1999)) A non-trivial homogeneous (lower) semilattice is isomorphic to either \((\mathbb{Q}, <)\), the universal semilattice or a homogeneous semilinear order.

- (Schmerl (1979)) Classified posets:
  1. \(A_n\), the antichain of \(n\) elements;
  2. \(B_n\), the union of \(n\) incomparable copies of \(\mathbb{Q}\);
  3. \(C_n = A_n \times \mathbb{Q}\) with partial order
     \[(a, p) < (b, q) \text{ if and only if } p < q \text{ in } \mathbb{Q};\]
  4. \(P\), the generic poset,

where \(n \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}\).

We can recognise \(P\) by the property: if \(A, B\) and \(C\) are pairwise disjoint finite subsets such that \(A < B\), no element of \(A\) is above an element of \(C\), and no element of \(B\) is below an element of \(C\), then there exists a point \(z\) with \(A < z < B\) and incomparable with \(C\).
Semigroup basics

- An element $e$ is an **idempotent** if $e^2 = e$. A **band** $B$ is a semigroup in which every element is an idempotent.

- We may define a partial order $\leq$ on $B$, known as the **natural order**, by

$$e \leq f \iff ef = fe = e.$$ 

- The Greens relations on a band simplify to:

$$e \mathcal{R} f \iff ef = f, fe = e;$$
$$e \mathcal{L} f \iff ef = e, fe = f;$$
$$e \mathcal{D} f \iff efe = e, fef = f.$$ 

**Motivating question:** Given a homogeneous poset $P$, does there exist a homogeneous band $B$ such that $(B, \prec)$ is isomorphic to $P$?
Homogeneous semilattices

- A **semilattice** is a commutative band.

- A **lower semilattice** \((E, <)\) is a poset in which the meet \(\wedge\) of any pair of elements exists.

- If \(Y\) is a semilattice, then \((Y, <)\) is a lower semilattice. Conversely, given a lower semilattice, we may form a semilattice \((Y, \wedge)\) by defining \(a \wedge b\) as the greatest lower bound of \(\{a, b\}\).

**Lemma (TQG)**

Let \((Y, \wedge)\) be a semilattice. Then the following are equivalent:

1) \((Y, \wedge)\) is a homogeneous semigroup;
2) \((Y, <)\) is a homogeneous lower semilattice.
Rectangular bands

- A **rectangular band** is a band $B$ satisfying 
  
  \[ efe = e \text{ for all } e, f \in B. \]

- A rectangular band with a single $\mathcal{R}$-class ($\mathcal{L}$-class) is called a **right (left) zero band**.

**Proposition**

Let $I$ and $J$ be arbitrary sets. Then $B_{I,J} = (I \times J, \cdot)$ forms a rectangular band, with operation given by

\[
(i, j) \cdot (k, \ell) = (i, \ell).
\]

Moreover every rectangular band is isomorphic to some $B_{I,J}$. The natural order on $B_{I,J}$ is an anti-chain on $|I| \cdot |J|$ elements, and the Greens relations simplify to:

\[
(i, j) \mathcal{R} (k, \ell) \iff i = k \text{ and } (i, j) \mathcal{L} (k, \ell) \iff j = \ell.
\]
Homogeneous rectangular bands

Proposition

A pair of bands \(B_{I,J}\) and \(B_{I',J'}\) are isomorphic if and only if \(|I| = |I'|\) and \(|J| = |J'|\). Moreover if \(\phi_I : I \to I'\) and \(\phi_J : J \to J'\) are a pair of bijections, then \(\phi : B_{I,J} \to B_{I',J'}\) defined by

\[
(i,j)\phi = (i\phi_I, j\phi_J)
\]

is an isomorphism, and every isomorphism from \(B_{I,J}\) to \(B_{I',J'}\) can be constructed in this way.

We may thus denote \(B_{\kappa,\delta}\) to be the unique (up to isomorphism) rectangular band with \(\kappa\) \(\mathcal{R}\)-classes and \(\delta\) \(\mathcal{L}\)-classes.

Corollary

Rectangular bands are homogeneous. Moreover any homogeneous band \(B\) such that \((B, \prec) \cong A_n\) is isomorphic to some \(B_{i,j}\), where \(i \cdot j = n\).
While there exists a classification theorem for general bands, it is far too complex for use. Moreover, no general isomorphism theorem exists, so its usefulness in understanding homogeneous bands is minimal. However a weaker form of the theorem will be of use:

**Theorem**

Let $B$ be an arbitrary band. Then $Y = S/D$ is a semilattice and $B$ is a semilattice of rectangular bands $B_\alpha$ (which are the $D$-classes), that is,

$$B = \bigcup_{\alpha \in Y} B_\alpha \text{ and } B_\alpha B_\beta \subseteq B_{\alpha\beta}.$$ 

We therefore understand the *global* structure of any band, but not the local structure.
Substructure of homogeneous bands

**Lemma (TQG)**

If $B = \bigcup_{\alpha \in Y} B_{\alpha}$ is a homogeneous band, then:

i) $\text{Aut}(B)$ is transitive on $B$, that is if $e, f \in B$ then there exists $\theta \in \text{Aut}(B)$ such that $e\theta = f$;

ii) $Y$ is homogeneous;

iii) $B_{\alpha} \cong B_{\beta}$ for all $\alpha, \beta \in Y$.

- However homogeneity does not pass to all induced substructures of $B$. For example take $B$ to be the band corresponding to a homogeneous semilinear order. Then the poset $(B, <)$ is not homogeneous.

- Understanding how the rectangular bands interact in a band is thus key to homogeneity.
Suppose now that $B = \bigcup_{\alpha \in \mathcal{Y}} B_{\alpha}$ is such that $(B, \prec) \cong \mathcal{B}_n$. Then $B$ satisfies the following condition: for each $e_{\alpha}$ and $\beta \leq \alpha$, there exists a unique $e_{\beta} \in B_{\beta}$ such that $e_{\beta} < e_{\alpha}$. Indeed if $e_{\alpha} > e_{\beta}, f_{\beta}$, then $\{e_{\alpha}, e_{\beta}, f_{\beta}\}$ forms a non-linear, non-antichain, and thus is not embeddable in $\mathcal{B}_n$.

A **normal band** is a band $B$ satisfying

\[zxyz = zyxz \text{ for all } x, y, z \in B.\]

This is equivalent to $B$ satisfying the condition above.

A band $B$ is called a **left/right normal band** if it is normal and each $B_{\alpha}$ is a left/right-zero band.
Let $Y$ be a semilattice, and $\{B_\alpha : \alpha \in Y\}$ be a collection of disjoint rectangular bands. For each $\alpha \geq \beta$ in $Y$, let $\phi_{\alpha,\beta} : B_\alpha \to B_\beta$ be a morphism such that:

i) $(\forall \alpha \in Y) \phi_{\alpha,\alpha} = 1_{B_\alpha}$;

ii) for all $\alpha \geq \beta \geq \gamma$ in $Y$,

$$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}.$$

Define multiplication on $B = \bigcup_{\alpha \in Y} B_\alpha$ by the rule that, for each $e \in B_\alpha$, $f \in B_\beta$,

$$ef = (e\phi_{\alpha,\alpha\beta})(f\phi_{\beta,\alpha\beta}).$$

Then $B$ forms a band, called a strong semilattice of rectangular bands, and denoted $[Y, B_\alpha, \phi_{\alpha,\beta}]$.

**Proposition**

A band is normal if and only if it is isomorphic to a strong semilattice of rectangular bands.
Isomorphism theorem for normal bands

- Not only do normal bands have a structure theorem that allows us to understand the local structure, but vitally there exists an isomorphism theorem:

**Theorem**

Let $B = [Y, B_\alpha, \phi_{\alpha,\beta}]$ and $B' = [Z, B'_\alpha, \psi_{\alpha,\beta}]$ be normal bands. Let $\pi : Y \to Z$ be an isomorphism, and for every $\alpha \in Y$, let $\theta_\alpha : B_\alpha \to B'_{\alpha\pi}$ be an isomorphism such that for any $\alpha \geq \beta$ in $Y$, the diagram

$$
\begin{array}{ccc}
B_\alpha & \xrightarrow{\theta_\alpha} & B'_{\alpha\pi} \\
\downarrow{\phi_{\alpha,\beta}} & & \downarrow{\psi_{\alpha\pi,\beta\pi}} \\
B_\beta & \xrightarrow{\theta_\beta} & B'_{\beta\pi}
\end{array}
$$

commutes. Then $\theta = \bigcup_{\alpha \in Y} \theta_\alpha$ is an isomorphism of $B$ into $B'$, denoted $[\theta_\alpha, \pi]$. Conversely, every isomorphism of $B$ into $B'$ can be so obtained for unique $\pi$ and $\theta_\alpha$. 
Let \( B = [Y, B_\alpha, \phi_{\alpha, \beta}] \) be a normal band with each \( B_\alpha \) isomorphic to \( B_{n,m} \) for some (fixed) \( n, m \in \mathbb{N}^* \).

**Lemma (TQG)**

If \( B \) is homogeneous then each \( \phi_{\alpha, \beta} \) is surjective. Moreover, if any \( \phi_{\alpha, \beta} \) is an isomorphism, then \( B \cong Y \times B_{n,m} \).

**Lemma (TQG)**

The band \( Y \times B_{n,m} \) is homogeneous if and only if \( Y \) is homogeneous. Moreover \( (Y \times B_{n,m}, \leq) \) is isomorphic to \( nm \) incomparable copies of \( Y \).

**Corollary**

A band \( B \) is homogeneous and is such that \( (B, <) \cong \mathcal{B}_n \) if and only if \( B \cong \mathbb{Q} \times B_{i,j} \), where \( i \cdot j = n \).
To consider the case where the connecting morphisms are not injective, we turn to a method of model theory; Fraïssé’s Theorem.

Let $\mathcal{K}$ be a class of structures.

We say that $\mathcal{K}$ has the **joint embedding property** (JEP) if given $B_1, B_2 \in \mathcal{K}$, then there exists $C \in \mathcal{K}$ and embeddings $f_i : B_i \to C$.

We say that $\mathcal{K}$ has the **amalgamation property** (AP) if given $A, B_1, B_2 \in \mathcal{K}$ (where $A \neq \emptyset$) and embeddings $f_i : A \to B_i$, then there exists $D \in \mathcal{K}$ and embeddings $g_i : B_i \to D$ such that

$$f_1 \circ g_1 = f_2 \circ g_2.$$
Fraïssé’s Theorem

**Theorem (Fraïssé’s theorem)**

Let $L$ be a countable signature and let $\mathcal{K}$ be a non-empty finite or countable set of f.g. $L$-structures which is closed under induced substructures and satisfies JEP and AP.

Then there is an $L$-structure $D$, unique up to isomorphism, such that $|D| \leq \aleph_0$, $\mathcal{K}$ is the age of $D$ and $D$ is homogeneous. We call $D$ the **Fraïssé limit** of $\mathcal{K}$.

**Example 1:** The class of all finite rectangular bands, $\mathcal{K}_{RB}$, forms a Fraïssé class, with Fraïssé limit $B_{\aleph_0,\aleph_0}$.

**Example 2:** Let $\mathcal{K}$ be the class of all finite bands. Since the class of all bands forms a variety, $\mathcal{K}$ is closed under both substructure and (finite) direct product, and thus has JEP. However T. Imaoka showed in 1976 that AP does not hold.
Normal bands

**Proposition**

The classes $\mathcal{K}_N$, $\mathcal{K}_{RN}$ and $\mathcal{K}_{LN}$ of all finite normal, right normal and left normal bands respectively form Fraïssé classes. Their Fraïssé limits will be denoted $B_N$, $B_{RN}$ and $B_{LN}$, respectively.

**Lemma (TQG)**

Let $B_N = [Y, B_\alpha, \phi_{\alpha,\beta}]$ be the generic normal band. Then

i) $Y$ is the universal semilattice;

ii) $(B, <) \not\cong \mathbb{P}$;

iii) $B_\alpha \cong B_{\mathbb{N}_0, \mathbb{N}_0}$ for all $\alpha \in Y$;

iv) $\phi_{\alpha,\beta}$ is surjective but not injective for all $\alpha \in Y$;

**Proof.**

ii) Let $e_\alpha, f_\alpha \in B_\alpha$. Then $e_\alpha \perp f_\alpha$, and there does not exist $g \in B$ such that $g > e_\alpha, f_\alpha$ since $B$ is normal. Clearly this cannot hold in $\mathbb{P}$. \qed
Homogeneous bands over $\mathbb{Q}$

- Let $\rho$ be an equivalence relation on a band $B$. We say that $B$ satisfies $\rho$-**covering** if for any $e, f, g \in B$,

$$e > f \quad \text{and} \quad f \rho g \Rightarrow e > g.$$ 

- For example if $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ satisfies $\mathcal{D}$-covering then for any $\alpha > \beta$, $e \in B_\alpha$ and $f \in B_\beta$ we have $e > f$.

**Lemma (TQG)**

Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ be a band satisfying $\rho$-covering, where $\rho = \mathcal{D}, \mathcal{R}$ or $\mathcal{L}$. Then $B$ is homogeneous if and only if $B_\alpha \cong B_\beta$ for all $\alpha, \beta \in \mathbb{Q}$. Moreover, if $\rho = \mathcal{D}$ and $B$ is homogeneous then $(B, \prec)$ is isomorphic to the homogeneous poset $(\mathcal{A}_n \times \mathbb{Q}, \prec)$, where $n = |B_\alpha|$. 
Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_{\alpha}$ and $C = \bigcup_{\alpha \in \mathbb{Q}} C_{\alpha}$ be a pair of homogeneous bands satisfying $\rho$-covering, where $\rho = D, R$ or $L$. Then $B \cong C$ if and only if $B_{\alpha} \cong C_{\alpha}$.

We may thus denote $D_{n,m}, R_{n,m}$ and $L_{n,m}$ as the unique (up to isomorphism) homogeneous band with $D, R$ and $L$-covering respectively and $D$-classes isomorphic to $B_{n,m}$.

**Corollary (TQG)**

A band $B = \bigcup_{\alpha \in \mathbb{Q}} B_{\alpha}$ is homogeneous if and only if isomorphic to either

i) $B_{n,m} \times \mathbb{Q}$;

ii) $D_{n,m}, R_{n,m}$ or $L_{n,m}$,

for some $n, m \in \mathbb{N}^*$. 

Homogeneous bands over $\mathbb{Q}$
The other cases

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a homogeneous band.

**Proposition (TQG)**

*If $Y$ is a non-linear semilinear order then $B$ is normal.*

**Lemma (TQG)**

*If $Y$ is the universal semilattice and $B$ is not normal then for any $e, f \in B_\alpha$ we have*

$$eBe \setminus \{e\} = \{g \in B : g < e\} = \{g \in B : g < f\} = fBf \setminus \{f\}.$$  

*Moreover $(B, <) \not\cong \mathbb{P}$.*

**Open problem:** Are homogeneous bands over the universal semilattice necessarily normal?
Proposition

The following bands are homogeneous:

i) Generic type: $B_N, B_{RN}$ and $B_{LN}$;

ii) $Y \times B_{n,m}$ where $Y$ is a homogeneous semilattice;

iii) $D_{n,m}, R_{n,m}$ and $L_{n,m},$

for any $n, m \in \mathbb{N}^*$. Moreover if $P$ is a homogeneous poset then there exists a homogeneous band $B$ such that $(B, <) \cong P$ if and only if $P \not\cong \mathbb{P}$.

Note: Given a homogeneous poset $P \not\cong \mathbb{P}$, the existence of a homogeneous band $B$ such that $(B, <) \cong P$ is not unique in general. In fact $B$ is unique up to isomorphism if and only if $P$ is trivial or $(\mathbb{Q}, <)$. 