KNOT SEMIGROUPS

A semigroup is called cancellative if it satisfies two conditions: if \( xz = yz \) then \( x = y \), and if \( xy = xz \) then \( y = z \). For each given knot diagram \( D \), we define a cancellative semigroup which we call the \textit{knot semigroup of} \( D \) and denote by \( KD \); the construction has been introduced and studied in [1]. By an \textit{arc} we mean a continuous line on a knot diagram from one undercrossing to another undercrossing. For example, consider the knot diagram \( T \) on the figure; it has three arcs, denoted by \( a \), \( b \) and \( c \). To denote a crossing on a knot diagram we shall use notation \( x \leftarrow y \rightarrow z \), where \( x \) and \( z \) are the two arcs terminating at the crossing and \( y \) is the arc passing over the crossing. For example, the crossings on diagram \( T \) are \( b \leftarrow a \rightarrow c \), \( b \leftarrow a \rightarrow a \) and \( c \leftarrow a \rightarrow a \). To define the knot semigroup of a diagram \( D \), assume that each arc is denoted by a letter. Then at every crossing \( x \leftarrow y \rightarrow z \), ‘read’ two defining relations \( xy = yz \) and \( yx = zy \). The cancellative semigroup generated by the arc letters with these defining relations is the knot semigroup \( KD \) of \( D \). For example, on diagram \( T \) we can read relations \( ba = ac \) and \( ab = ca \) at the left-top crossing, relations \( ba = aa \) and \( ab = aa \) at the right-top crossing and relations \( ca = aa \) and \( ac = aa \) at the bottom crossing. Using these relations, one can deduce equalities of words in \( KT \). In particular, from \( aa = ba = ca \), using cancellation, one can deduce \( a = b = c \), that is, all generators are equal to one another; in other words, \( KT \) is an infinite cyclic semigroup.

Currently I work (with several collaborators) on several fascinating questions related to knot semigroups, including the following.

(1) We proved that a braid is trivial if and only if certain conditions are satisfied in its knot semigroup. We have nearly finished proving that a knot diagram represents a trivial knot if and only if its knot semigroup is infinite cyclic (as in the example above). If a knot diagram is a trivial knot, the proof of this fact can be presented visually as a sequence of elegant tangle diagrams.

(2) We are in the process of clarifying how the knot semigroup of a knot diagram is related to other algebras defined by knot diagrams, including the \textit{ki}, the \textit{quandle} and several types of groups (especially the \( \pi \)-\textit{orbifold group}).

(3) Rational knots are an important ‘nice’ class of knots. We are working on describing knot semigroups of alternating diagrams of rational knots and, in particular, finding a simple algebraic proof of the flype conjecture for rational knots (note that knot semigroups are not knot invariants: a knot semigroup of a knot diagram expresses certain properties of both the knot and the specific diagram of the knot).

References