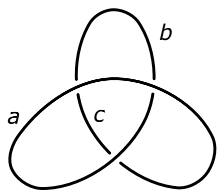
KNOT SEMIGROUPS



A semigroup is called cancellative if it satisfies two conditions: if $xz = yz$ then $x = y$, and if $xy = xz$ then $y = z$. For each given knot diagram D, we define a cancellative semigroup which we call the *knot semigroup* of D and denote by KD; the construction has been introduced and studied in [1]. By an *arc* we mean a continuous line on a knot diagram from one undercrossing to another undercrossing. For example, consider the knot diagram T on the figure; it has three arcs, denoted by a, b and c. To denote a crossing on a knot diagram we shall use notation $x \dashv y \vdash z$, where x and z are the two arcs terminating at the crossing and y is the arc passing over the crossing. For example, the crossings on diagram T are $b \dashv a \vdash c$, $b \dashv a \vdash a$ and $c \dashv a \vdash a$. To define the knot semigroup of a diagram D, assume that each arc is denoted by a letter. Then at every crossing $x \dashv y \vdash z$, 'read' two defining relations $xy = yz$ and $yx = zy$. The cancellative semigroup generated by the arc letters with these defining relations is the knot semigroup KD of D. For example, on diagram T we can read relations $ba = ac$ and $ab = ca$ at the left-top crossing, relations $ba = aa$ and $ab = aa$ at the right-top crossing and relations $ca = aa$ and $ac = aa$ at the bottom crossing. Using these relations, one can deduce equalities of words in KT. In particular, from $aa = ba = ca$, using cancellation, one can deduce $a = b = c$, that is, all generators are equal to one another; in other words, KT is an infinite cyclic semigroup.

Currently I work (with several collaborators) on several fascinating questions related to knot semigroups, including the following.

(1) We proved that a braid is trivial if and only if certain conditions are satisfied in its knot semigroup. We have nearly finished proving that a knot diagram represents a trivial knot if and only if its knot semigroup is infinite cyclic (as in the example above). If a knot diagram is a trivial knot, the proof of this fact can be presented visually as a sequence of elegant tangle diagrams.

(2) We are in the process of clarifying how the knot semigroup of a knot diagram is related to other algebras defined by knot diagrams, including the kei, the quandle and several types of groups (especially the π-orbifold group).

(3) Rational knots are an important 'nice' class of knots. We are working on describing knot semigroups of alternating diagrams of rational knots and, in particular, finding a simple algebraic proof of the flype conjecture for rational knots (note that knot semigroups are not knot invariants: a knot semigroup of a knot diagram expresses certain properties of both the knot and the specific diagram of the knot).

References

[1] Alexei Vernitski. Describing semigroups with defining relations of the form xy=yz and yx=zy and connections with knot theory. *Semigroup Forum*, pages 1–17, 2016.